Matematicheskaya fizika, analiz, geometriya
2002, v. 9, No. 4, p. 648–662

Problems in the geometry of submanifolds

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Received February 7, 2002
Communicated by Yu.A. Aminov

This article grew out of several talks that the author presented at the Banach Institute and at the University of Białystok in Poland during November of 2001. It describes six problems from the geometry of submanifolds. Some of the problems come from the theory of constant curvature submanifolds in Euclidean space, as well as applications of Morse theory of the height function to the problem of relating curvature and topology of submanifolds in Euclidean space. Others come from infinite-dimensional Morse theory of minimal surfaces in Riemannian manifolds.

1. Introduction

At the beginning of the new millennium, it was fashionable to present lists of open problems within several specialities of mathematics.

A kind invitation from Professor Aminov to give a talk at the Banach Institute in Poland on "open problems in the geometry of submanifolds" gave the author an opportunity to reflect on problems which have motivated his own research over the last several years. Some of these problems are likely to be difficult, while others might yield to a simple technique or trick that has so far proven to be elusive. I believe that all of them have the potential to develop into dissertation topics for ambitious graduate students, if they are suitably modified.

The problems I want to share with you fall into the loosely defined area of geometry of submanifolds, more specifically the Riemannian geometry of submanifolds of low codimension in Euclidean space, and the theory of two-dimensional minimal surfaces in Riemannian manifolds. In the first area, we emphasize problems which are generalizations from the classical geometry of surfaces, problems which possess a certain compelling beauty. The theory of constant curvature surfaces is a starting point, which is closely related to one of the simplest of nonlinear hyperbolic partial differential equations, the sine-Gordon equation.

Mathematics Subject Classification 2000: 53A07, 53A10, 54C20.

On the other hand, the theory of minimal surfaces in Riemannian manifolds treats what is arguably the simplest global nonlinear elliptic partial differential equation by techniques from topology and global analysis (in particular, Morse theory). Our point of view is to regard minimal surfaces as conformal harmonic maps, solutions to the nonlinear version of Laplace's equation. The reader will find that the tools needed to solve problems in the Morse theory of conformal harmonic maps are quite similar to those used to understand other nonlinear partial differential equations which arise in geometry, such as the modified Cauchy–Riemann equations needed for Gromov's theory of pseudoholomorphic curves, or the celebrated Seiberg–Witten equations of four-dimensional geometry and topology. Technology developed in one context may have applications in others.

The application of topological and geometric methods to the understanding of nonlinear partial differential equations is a thread of investigation that should continue long into the twenty-first century.

2. Submanifolds of constant curvature

From the classical theory of surfaces of constant curvature in Euclidean space come three classical theorems:

Hilbert's Theorem. *There is no isometric immersion from the complete hyperbolic plane* $\mathbb{H}^2$ *into three-dimensional Euclidean space.*

Liebmann's Theorem. *There is only one isometric immersion from the constant curvature sphere* $\mathbb{S}^2$ *into three-dimensional Euclidean space, up to rigid motion.*

Theorem. *Any isometric immersion from Euclidean two-space* $\mathbb{E}^2$ *into three-dimensional Euclidean space is a cylinder.*

The first two of these were proven at the turn of the century in a celebrated paper of David Hilbert [12], the second being attributed by him to Liebmann. Hilbert's theorem was important for two reasons: First, the existence of such an immersion would have provided a very visible proof that the axioms of hyperbolic geometry are at least as consistent as those of Euclidean three-space. Indeed, it was later possible to prove that the hyperbolic plane admits an isometric immersion into Euclidean N-space, when N is sufficiently large; this is a consequence of Nash's celebrated isometric imbedding theorem [11]. Second, it was already known that there was a very large supply of noncomplete surfaces of constant negative curvature, making Hilbert's theorem one of the first *global* theorems from the Riemannian geometry of surfaces.

The third theorem was actually proven somewhat later. Indeed, it is a special case of a theorem of Hartman [13], which was the culmination of earlier efforts of Pogorelov, Hartman, Nirenberg, and O'Neill in the 50's and 60's:

Cylinder Theorem. *A smooth isometric immersion $f : \mathbb{E}^n \to \mathbb{E}^{n+k}$ with $k < n$ can be factored (after a rigid motion) into a product of isometric immersions,*

$$f = f_0 \times id : \mathbb{E}^{n-k} \times \mathbb{E}^k \longrightarrow \mathbb{E}^{n-k} \times \mathbb{E}^{2k}.$$

It was thus quite natural to explore whether the theorems of Hilbert and Liebmann could also be extended to higher dimensions, just like the cylinder theorem. Indeed, the Gauss equation suggests that one might hope to get a fairly complete understanding of the structure of n-dimensional submanifolds of constant curvature in N-dimensional Euclidean space $\mathbb{E}^N$ when $N \leq 2n-1$.

Some first steps had already been carried out by Cartan [3] in his well-known paper of 1919–20. Cartan's work showed that there were no submanifolds of constant negative curvature in $\mathbb{E}^{2n-2}$. Moreover, he used his theory of differential systems in involution [5] to show that local real analytic isometric imbeddings from open subsets of $\mathbb{H}^n$ into $\mathbb{E}^{2n-1}$ depend upon $n(n-1)$ functions of a single variable. In particular, surfaces of constant negative curvature in $\mathbb{E}^3$ depend upon two functions of a single variable, in agreement with the classical and beautiful local isometric imbedding theorem of Janet and Cartan [4].

In [18], the author showed that the existence of an isometric immersion $f : \mathbb{H}^n \to \mathbb{E}^{2n-1}$ implies the existence of global coordinates on $\mathbb{H}^n$ whose coordinate vectors are unit-length asymptotic vectors, thereby extending the main step in the standard proof of Hilbert's theorem to n dimensions. This suggested:

Conjecure 1. The n-dimensional hyperbolic space $\mathbb{H}^n$ admits no isometric immersion into $\mathbb{E}^{2n-1}$.

Tenenblat and Terng, Xavier, and Aminov have worked on this and related problems (see [31] and [33]), but the conjectured extension of Hilbert's theorem remains open.

Reviewing briefly the contents of [18], we suppose that M^n is a simply connected Riemannian manifold of constant curvature -1, not necessarily complete, and we are given an isometric immersion $f : M^n \to \mathbb{E}^{2n-1}$. If $h : \mathbb{E}^{2n-1} \to \mathbb{H}^{2n}$ is the standard isometric imbedding from $\mathbb{E}^{2n-1}$ onto a horosphere, then the composition $g = h \circ f$ is a "developable submanifold" of $\mathbb{H}^{2n}$, providing a link with the theory behind the cylinder theorem.

Suppose that TM and NM are the tangent and normal bundles of g. Then the second fundamental form $\beta : TM \times TM \to NM$ must satisfy the Gauss equation

$$\langle \beta(x,z), \beta(y,w) \rangle - \langle \beta(x,w), \beta(y,z) \rangle = 0,$$

for $x, y, z, w \in T_pM$, $\langle \cdot, \cdot \rangle$ denoting the metric on the normal bundle induced by the Riemannian metric on $\mathbb{H}^{2n}$. We say that the second fundamental form β is *flat* when it satisfies this equation. (In the terminology of É. Cartan, the components of β with respect to a normal frame are *exteriorly orthogonal* symmetric bilinear forms.)

A unit-length vector field $Z : M \to TM$ is asymptotic for f if and only if the second fundamental form α for f satisfies $\alpha(Z, Z) = 0$, or equivalently the second fundamental form β of g satisfies

$$\beta(Z, Z) = \text{the unit normal to } h(\mathbb{E}^{2n-1}).$$

It follows from the calculations in [18] that there are exactly 2^n unit-length asymptotic vector fields whose pairwise Lie brackets are zero. Their integral curves give the global asymptotic coordinate systems on M. Each unit-length asymptotic vector field Z yields a vector bundle isomorphism from the tangent bundle to the normal bundle,

$$\Phi_Z : TM \to NM \quad \text{defined by} \quad \Phi_Z(x) = \alpha(Z(p), x), \quad \text{for } x \in T_pM.$$

Moreover, the calculations in [18] show that the normal bundle is flat and the unit normal to $h(\mathbb{E}^{2n-1})$ is parallel with respect to the normal connection.

If $n = 2$, we can thus construct a coordinate system (z_1, z_2) such that $\partial/\partial z_1$ and $\partial/\partial z_2$ are unit-length coordinate vectors. As pointed out already in the early articles of Hilbert and Holmgren [14], the angle θ between the asymptotic vectors satisfies the sine-Gordon equation

$$\frac{\partial^2 \theta}{\partial z_1 \partial z_2} = \sin\theta;$$

in fact Holmgren used this equation as starting point for his proof of Hilbert's theorem. In terms of principle coordinates (y_1, y_2), related to the asymptotic coordinates by

$$z_1 = y_1 + y_2, \qquad z_2 = y_1 - y_2\,,$$

the sine-Gordon equation takes the form

$$\frac{\partial^2 \theta}{\partial y_1^2} - \frac{\partial^2 \theta}{\partial y_2^2} = \sin\theta.$$

The theory of the sine-Gordon equation has developed extensively due to its relationship with the theory of solitons.

We can pull the flat connections on the normal bundle back via the Φ_Z's to get a rich collection of flat connections on the tangent bundle. It is tempting to speculate that some combination of these connections coupled with the proof of the Allendoerfer–Weil–Chern generalization of the Gauss–Bonnet formula might yield interesting relationships between volumes of coordinate polyhedra in asymptotic coordinates, and integrals of solid angles generated by asymptotic vector fields, at least in the case where n is even. A rich supply of integral formulae can be obtained efficiently by means of Quillen's theory of superconnections [25], as extended by Mathai and Quillen. In any case, the structure of hyperbolic submanifolds remains an interesting topic for future research.

As pointed out in [21], there is a beautiful duality between submanifolds of constant negative and constant positive curvature. One manifestation of this is that simply connected nonumbilic surfaces of constant positive curvature are in one-to-one correspondence with solutions of the partial differential equation

$$\frac{\partial^2\theta}{\partial y_1^2} + \frac{\partial^2\theta}{\partial y_2^2} = -\sinh\theta$$

with $\theta > 0$, where (y_1, y_2) are suitable principle coordinates. In fact, one can prove Liebmann's theorem by establishing this equation and showing that $\theta \to \infty$ as one approaches umbilic points. In the case of isometric immersions from the sphere, if the open set of nonumbilics were nonempty, θ would have to assume a minimum value, calculation the minimum principle for Laplace's equation (see [21, § 2]). Liebmann's theorem then follows from the standard fact that if a compact has only umbilic points, it must be a standard round sphere.

We can ask for a higher-dimensional version of Liebmann's theorem for constant positive curvature. Indeed, if $M^{(2n-1,1)}$ is the hyperboloid of one sheet in the Minkowski space-time with coordinates $(x_1, \ldots x_{2n}, t)$ and metric

$$ds^2 = dx_1^2 + \cdots + dx_{2n}^2 - dt^2,$$

there is a totally umbilic isometric immersion $h : \mathbb{E}^{2n-1} \to M^{(2n-1,1)}$. Note that $M^{(2n-1,1)}$ has a metric of Lorentz signature. If we are given an isometric immersion $f : M^n \to \mathbb{E}^{2n-1}$, where M^n is an n-dimensional manifold of constant curvature one, the composition $h \circ f$ is once again a "developable submanifold" of $M^{(2n-1,1)}$. The second fundamental form β of g once again satisfies the Gauss equation

$$\langle\beta(x,z),\beta(y,w)\rangle - \langle\beta(x,w),\beta(y,z)\rangle = 0, \tag{1}$$

for $x, y, z, w \in T_pM$, but this time the inner product $\langle\cdot,\cdot\rangle$ on the normal bundle has Lorentz signature.

An analysis of (1) is presented in [21] and it leads to a classification of the possible algebraic structures for β. Points of M^n are divided into two types, nonumbilics and weak umbilics. Theorem 3 of [21] states that in the compact case, in which we have an isometric immersion $f : \mathbb{S}^n \to \mathbb{E}^{2n-1}$, all points are weak umbilics, which means that there is a unit-length section $e_{2n-1} : \mathbb{S}^n \to N\mathbb{S}^n$ such that

$$\langle\alpha(x,y), e_{2n-1}\rangle = \langle x, y\rangle,$$

where α is the second fundamental form of f. The proof is a direct generalization of the proof of the classical Liebmann theorem stated before. However, it is still not known what existence of the section e_{2n-1} implies about the structure of the isometric immersion.

Question 2. Can any smooth isometric immersion $f : \mathbb{S}^n \to \mathbb{E}^{2n-1}$ be extended to an isometric immersion $\widetilde{f} : D^{n+1} \to \mathbb{E}^{2n-1}$, where

$$D^{n+1} = \{(x_1, \dots , x_{n+1}) \in \mathbb{E}^{n+1} : (x_1)^2 + \cdots + (x_{n+1})^2 \leq 1\}?$$

This would give a simple geometric extension of Liebmann's theorem to n dimensions. The answer is known to be yes when $n = 3$ (see [22]). It seems very likely that the question has a positive answer in the case of real analytic isometric immersions.

We conclude this section by pointing out that the theory of flat symmetric bilinear forms, symmetric bilinear forms satisfying (1) has become a useful tool in studying rigidity of submanifolds. It can be used to prove Allendoerfer's well-known rigidity theorem [6] and generalizations thereof [7].

3. Morse theory and curvature of submanifolds

When we generalize from constant curvature submanifolds to submanifolds whose curvature tensor satisfies inequalities, different techniques are needed. Morse theory presents itself as a tempting tool to unravel the relationship between topology and curvature of submanifolds of low codimension in Euclidean space.

Recall that if u is a unit-length vector in Euclidean space $\mathbb{E}^N$ and $f : M^n \to \mathbb{E}^N$ is an isometric immersion of a Riemannian manifold M^n, we can define a function

$$h_u : M^n \to \mathbb{R} \qquad \text{by} \qquad h_u(p) = u \cdot p,$$

where the dot denotes the usual Euclidean dot product. A point $p \in M^n$ is a critical point for h_u if and only if u is perpendicular to M^n at p. If p is a critical point, the Hessian of h_u at p is the symmetric bilinear form

$$d^2h_u(p) : T_pM \times T_pM \longrightarrow \mathbb{R} \quad \text{given by} \quad d^2h_u(p)(x, y) = \langle \alpha(x, y), u\rangle, \qquad (2)$$

for $x, y \in T_pM$. Here α is the second fundamental form of the isometric immersion f.

With these preparations out of the way, we can present a proof schema for arguments which might relate curvature to topology: Suppose we have an isometric immersions $f : M^n \to \mathbb{E}^N$. Then conditions on the Riemann–Christoffel curvature tensor R of M^n or the curvature operator $\mathcal{R}$ can be expected to yield restrictions on the second fundamental form α via the Gauss formula

$$\langle R(x,y)w,z\rangle = \langle \mathcal{R}(x\wedge y), z\wedge w\rangle = \langle \alpha(x,z),\alpha(y,w)\rangle - \langle \alpha(x,w),\alpha(y,z)\rangle,$$

for $x,y,z,w \in T_pM$. Conditions on the second fundamental form then yield conditions on the index of the Hessian via (2). Finally, Morse theory provides restrictions on the Betti numbers of M.

This proof scheme is used in [19] to show that if M^n is a compact conformally flat submanifold of $\mathbb{E}^{n+p}$, then $H_k(M;F)=0$ for $p<k<n-p$, where F is any field. The proof is based upon the theory of flat symmetric bilinear forms, those which satisfy (1).

Here is another example: suppose that M^n is a compact n-dimensional manifold of $\mathbb{E}^{2n-1}$ with negative sectional curvatures. It follows from Bezout's theorem of algebraic geometry that there is at least one complex vector $x+iy$ such that $\alpha(x+iy,x+iy)=0$. Writing out real and imaginary parts yields

$$\alpha(x,x)=\alpha(y,y) \qquad \alpha(x,y)=0,$$

which implies by the Gauss equation

$$\langle R(x,y)y,x\rangle = \langle \alpha(x,x),\alpha(x,x)\rangle > 0 \qquad \text{unless} \qquad \alpha(x,x)=\alpha(y,y)=0.$$

Thus negative sectional curvature implies that there exists an asymptotic vector $z\in T_pM$ such that $\alpha(z,z)=0$. Negative sectional curvature also implies that $x\mapsto \alpha(z,x)$ is a vector space isomorphism. (In particular, there could not be even a local isometric immersion of such a manifold into $\mathbb{E}^{2n-2}$.)

Since M is compact, each height function h_u must have a local minimum at some point $p\in M$, and the Hessian $d^2h_u(p)$ must be semidefinite. Let $z\in T_pM$ be an asymptotic vector and choose $x\in T_pM$ so that $\alpha(z,x)=u$. Then

$$\frac{d}{dt}\langle \alpha(z+tx,z+tx),u\rangle\bigg|_{t=0} = 2,$$

showing that $\langle \alpha(y,y),u\rangle$ must assume both signs when y is close to z, contradicting definiteness. Thus we obtain the theorem of Chern, Kuiper, and Otsuki: A compact manifold of negative curvature has no isometric immersion in $\mathbb{E}^{2n-1}$. (See [15, v. II, p. 29].)

A slightly more general scheme would be to average over the unit-length vectors u. This idea was used in [21] to prove that if M^n is a compact manifold of positive sectional curvatures which admits an isometric immersion into $\mathbb{E}^{n+2}$ then M^n must be homeomorphic to a sphere, when $n\geq 3$. More generally, we could ask for a positive curvature analog of the Chern–Kuiper–Otsuki theorem:

Question 3. If M^n is an n-dimensional manifold of positive sectional curvatures which admits an isometric immersion into $\mathbb{E}^{2n-1}$, must M^n be homeomorphic to a sphere?

In particular, we can ask whether such a manifold must be simply connected. Reference [23] shows that the answer is yes if M has constant positive curvature, but the proof is surprisingly subtle. It is based upon Chern–Simons invariants, and we do not see how to generalize it even to the case of Riemannian manifolds with sectional curvatures $K(\sigma)$ satisfying the inequality

$$\delta \leq K(\sigma) \leq 1, \quad \text{for any } \delta \text{ with } 0 < \delta < 1,$$

without bounding the size of the fundamental group.

4. Morse theory of minimal surfaces in Riemannian manifolds

Of course, one of the most impressive features of Morse theory is that it applies not only to functions on finite-dimensional manifolds, but also to certain ordinary and partial differential equations. The simplest vector-valued linear second-order ordinary differential equations asks for a function

$$\gamma : \mathbb{R} \longrightarrow \mathbb{E}^N \qquad \text{such that} \qquad \gamma'(t) = 0.$$

The solutions are just affinely parametrized straight lines, $\gamma(t) = at + b$, where $a, b \in \mathbb{E}^N$.

The simplest way to make this equation nonlinear is to require that γ takes its values within a smooth submanifold M of $\mathbb{E}^N$; thus we seek a function

$$\gamma : \mathbb{R} \longrightarrow M \qquad \text{such that} \qquad (\gamma'(t))^T = 0,$$

where $(\cdot)^T$ denotes tangential component. Curves γ which solve this equation are called *geodesics*. It is especially interesting to explore the existence of periodic geodesics, or equivalently, smooth closed geodesics.

Suppose that M is compact and let

$$\Lambda^1(M) = \{\gamma : S^1 \to \mathbb{E}^N : \ f \text{ is } L^2_1 \text{ and } f(t) \in M, \text{ for all } t \in S^1\}$$

and define the *energy* $E : \Lambda^1(M) \to \mathbb{R}$ by

$$E(f) = \frac{1}{2} \int\limits_{S^1} \langle \gamma'(t), \gamma'(t) \rangle dt.$$

The critical points of this function (in the sense of the calculus of variations) are the smooth closed curves $\gamma : S^1 \to M$ which satisfy the differential equation

$(\gamma'(t))^T = 0$, that is, the smooth closed geodesics. This suggests that Morse theory might be applied to the energy function E to prove existence of smooth closed geodesics.

This viewpoint has turned out to be very successful. One can approximate $\Lambda^1(M)$ by finite-dimensional spaces of broken geodesics, the approach used by Morse, which is presented in Milnor's classic text [17]. This can be used to prove many classical results including Fet's beautiful theorem that every compact Riemannian manifold possesses at least one smooth closed geodesic. A particularly beautiful presentation of many of these results, with an emphasis on the importance of equivariant Morse theory can be found in [2].

It is undoubtedly more elegant to phrase the theory in terms of infinite-dimensional manifolds modeled on Hilbert spaces, as developed by Palais and Smale [24]. However, it seems that all known theorems in the theory of closed geodesics can be proven within the context of finite-dimensional approximations. It is only when one considers partial differential equations that the technology of infinite-dimensional manifolds appears to be indispensable.

Indeed, the development of infinite-dimensional Morse theory [24] was partially motivated by a desire to attack partial differential equations by techniques from global analysis. In this regard, Smale wrote in 1977:

> The most interesting case for more than one independent variable is minimal surfaces. In the theory of Plateau's problem, I had been intrigued by a result of Morse–Tompkins and Schiffman in 1939. Their theorem asserted that if a Jordan curve in $\mathbb{R}^3$ spans two stable minimal surfaces, then it spans a third of unstable type. This was suggestive of a Morse theory for Plateau's problem. In the sixties, I tried without success to find such a theory, or to imbed the Morse–Tompkins–Schiffman result in a general conceptual setting. Tromba and Uhlenbeck may now have succeeded in initiating a development of calculus of variations in the large for more than one independent variable ([30, p. 692–693]).

Here is how one might envision the development of a Morse theory for elliptic partial differential equations in two independent variables: We begin by observing that the simplest vector-valued linear elliptic partial differential equation in two variables is the vector-valued Laplace equation, which asks for a function

$$f : \Sigma \longrightarrow \mathbb{E}^N \qquad \text{such that} \qquad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Here Σ is a Riemann surface and (x, y) are arbitrary conformal coordinates on Σ, the Laplace equation being invariant under change of conformal coordinates. Of course Σ must be noncompact for this equation to have nonconstant solutions.

To make this equation nonlinear, we require that f takes its values within a smooth compact submanifold M of $\mathbb{E}^N$; thus we seek a function

$$f : \Sigma \longrightarrow M \qquad \text{such that} \qquad \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)^T = 0,$$

where $(\cdot)^T$ denotes tangential component, as before. Maps f which satisfy this equation are called *harmonic maps*. There is often a rich supply of harmonic maps from compact Riemann surfaces into compact Riemannian manifolds.

It turns out that the image of harmonic maps of nonzero area are minimal surfaces if and only if they are conformal. Let $\mathcal{T}$ be the Teichmüller space of (marked) conformal structures on Σ. Then a conformal harmonic map is exactly a critical point for the energy function

$$E : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \to \mathbb{R}. \tag{3}$$

Here $\mathrm{Map}(\Sigma, M)$ denotes the space of smooth maps from Σ to M and E is defined by

$$E(f) = \frac{1}{2}\int_{\Sigma} \left[\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial f}{\partial y}\right] dxdy = \frac{1}{2}\int_{T^2} |df|^2 dA,$$

where (x, y) are local conformal coordinates on Σ (the integrand being independent of the choice) and $|df|$ and dA represent the norm of df and the area element with repect to any metric on Σ within the conformal structure. By the uniformization theorem, we can normalize the metric on Σ by assuming that it has constant Gaussian curvature one in the case where Σ has genus zero, is flat with total area one when Σ has genus one, and is of constant curvature -1, when Σ has genus greater than one.

However, in order to apply critical point theory to the energy function, we would need condition C of Palais and Smale to hold. This would require, however, that we complete the space $\mathrm{Map}(\Sigma, M)$ with respect to the L_1^2-topology and this topology is too weak for the theory of Hilbert manifolds. Therefore Sacks and Uhlenbeck [26, 27] introduced a perturbed version of the energy, the α-energy

$$E_\alpha : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \to \mathbb{R}.$$

For $\omega \in \mathcal{T}$, we set $E_{\alpha,\omega}(f) = E_\alpha(f, \omega)$. Then the α-energy is defined by

$$E_{\alpha,\omega}(f) = \frac{1}{2}\int\int_{\Sigma} (1 + |df|^2)^\alpha dA + (\text{constant}),$$

for $\alpha > 1$, where the Riemannian metric appearing in the integrand is chosen to be the normalized metric described before, and the constant is chosen so that as

$\alpha \to 1$, the α-energy approaches the usual energy. The key point is that $E_{\alpha,\omega}$ can be shown to satisfy condition C when $\mathrm{Map}(\Sigma, M)$ is completed with respect to the L_1^p-topology where $p = 2\alpha$, so one can apply critical point theory on Banach manifolds.

In the case of genus zero, the Teichmüller space $\mathcal{T}$ degenerates to a point and it was possible for Sacks and Uhlenbeck to provide a version of Fet's theorem for minimal two-spheres: any simply connected compact Riemannian manifold contains at least one minimal two-sphere, which might be immersed and contain branch points. Moreover, under suitable conditions on the fundamental group, they proved existence of stable minimal surfaces of higher genus. On the other hand, it was clear from their work that a full Morse theory does not hold for the usual energy, because "bubbling" can occur as α approaches one.

It was natural to apply this theory to finding relationships between curvature and topology. Indeed, work of Schoen and Yau [28] had shown that minimal surfaces could be used with great effect to find relationships between scalar curvature and topology. What about sectional curvature? We point out that it is still unknown whether $S^2 \times S^2$ admits a metric of positive sectional curvature. In applying the second variation formula to minimal surfaces, however, it was found that modifications of the notion of sectional curvature were more natural in the context of minimal surfaces.

An integration by parts, which is carried out in [16], shows that the second variation of energy at a harmonic map f extends to the symmetric complex bilinear form

$$d^2E(f) : \Gamma(TM \otimes \mathbb{C}) \times \Gamma(TM \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

which satisfies the formula

$$d^2E(f)(W, \overline{W}) = 4 \int\limits_{T^2} \left[\|\nabla_{\partial/\partial\bar{z}} W\|^2 - \langle \mathcal{R}\left(\frac{\partial f}{\partial z} \wedge W\right), \frac{\partial f}{\partial \bar{z}} \wedge \overline{W} \rangle \right] dxdy,$$

where $z = x + iy$ is a conformal coordinate on Σ and ∇ is the complex linear extension of the Levi–Civita connection to the complexified tangent bundle $TM \otimes \mathbb{C}$. This formula suggests that *complex* sectional curvature should play a role in the theory of stability of minimal surfaces.

Recall the definition. If z and w are linearly independent elements of $T_pM \otimes \mathbb{C}$, the *complex sectional curvature* of the two-plane spanned by z and w is

$$\frac{\langle \mathcal{R}(z \wedge w), \bar{z} \wedge \bar{w} \rangle}{\langle z \wedge w, \bar{z} \wedge \bar{w} \rangle},$$

where the bar denotes complex conjugation. The complex two-plane is said to be *isotropic* if

$$\langle z, z \rangle = \langle w, w \rangle = \langle z, w \rangle = 0.$$

Finally, the Riemannian manifold is said to have *positive isotropic curvature* if the complex sectional curvature of each isotropic two-plane is positive.

In joint work with Micallef [16], the Sacks–Uhlenbeck theory of harmonic two-spheres was applied to give a new proof of the sphere theorem from Riemannian geometry: If M is a compact simply connected manifold with positive isotropic curvature of dimension at least four, it must be homeomorphic to a sphere. In particular, $S^2 \times S^2$ does not admit a metric of positive isotropic curvature.

This suggested many new questions. For example, can one replace "homeomorphic" by "diffeomorphic"? Hamilton proved this when $n = 4$.

However, our focus here is on questions that might be resolved via minimal surfaces. We recall a conjecture of Chern that the only abelian subgroups of manifolds of positive real sectional curvature are cyclic. Shankar [29] found two series of counterexamples to this conjecture with fundamental group containing $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, but the examples do not have positive isotropic curvatures. We might therefore ask whether Chern's conjecture holds for manifolds of positive isotropic curvature:

Question 4. Is it true that the only abelian subgroups of the fundamental group of a compact Riemannian manifold with positive isotropic curvature are cyclic?

Note that $S^1 \times S^3$ has positive scalar curvature and is conformally flat and must therefore have positive isotropic curvature. More generally, it is not hard to show that the boundary of an ϵ-neighborhood of a graph in $\mathbb{R}^5$ can be perturbed so that its induced metric is conformally flat and of positive scalar curvature. Thus the free product

$$\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} \quad (n \text{ times})$$

occurs as the fundamental group of a compact Riemannian manifold with positive isotropic curvature. A remarkable recent result of Fraser [9] shows that a compact Riemannian manifold with positive isotropic curvature cannot have a fundamental group which contains the abelian group $\mathbb{Z} \oplus \mathbb{Z}$.

As pointed out in [16], manifolds whose real sectional curvatures satisfy

$$\frac{1}{4}\kappa < K(\sigma) \leq \kappa, \qquad \text{where} \qquad \kappa : M \to (0, \infty)$$

have positive isotropic curvature. If κ is constant, this is exactly the condition used by Berger and Klingenberg in their earlier version of the sphere theorem. In odd dimensions, however, one might hope to weaken the inequality:

Question 5. Does there exist a δ, $0 < \delta < 1/4$, such that an *odd-dimensional* compact simply connected Riemannian manifold M whose real sectional curvatures satisfy

$$\delta\kappa < K(\sigma) \leq \kappa, \qquad \text{where} \qquad \kappa : M \to (0, \infty)$$

is a smooth function, must be homeomorphic to $\mathbb{S}^n$?

The answer is known to be yes if κ is constant by work of Abresch and Meyer [1], but we suspect that an improvement might be possible by means of minimal surfaces.

However, the key question in the Morse theory of minimal surfaces consists of determining which cohomology classes in $H^*(\mathrm{Map}(\Sigma, M))$ give rise to minimal surfaces via the minimax construction. Note that the cohomology of this space of mappings is quite rich; for example, in the case where Σ is a torus, $\mathrm{Map}(\Sigma, M)$ is an iterated free loop space and its real cohomology could be calculated via Sullivan's method of minimal models just as in [32]. Since the fibration $\Lambda^2(M) \to \Lambda^1(M)$ (the map being evaluation on a loop that is homotopically nontrivial in T^2) possesses a section, the Betti numbers $\Lambda^2(M)$ are at least as large as those of $\Lambda^1(M)$. It has been conjectured that if M is rationally hyperbolic, the sum of the first k Betti numbers of $\Lambda^1(M)$ grows exponentially with k (see [8, p. 519]).

However, for some cohomology classes, the minimax construction degenerates as one moves to the boundary of Teichmüller space in (3). In other cases, bubbling interferes with the limiting process. In yet other cases, we might expect multiple covers or branched covers of other minimal surfaces. These difficulties must be confronted in order to make progress on

Question 6. Let g be a fixed genus. Given any compact simply connected Riemannian manifold M^n, must M possess infinitely many geometrically distinct minimal surfaces of genus g?

Recall that Gromoll and Meyer used Morse theory to show that under weak topological conditions a compact Riemannian manifold must have infinitely many geometrically distinct smooth closed geodesics [10]. Vigué-Poirrier and Sullivan [32] then showed that this topological condition was satisfied unless the real cohomology of M was generated by a single element. Thus a first step towards answering Question 6 would be to establish analogs of these results within the Morse theory of minimal surfaces. One might expect the different genera ($g = 0$, $g = 1$, $g > 1$) to require somewhat different techniques.

References

[1] *U. Abresch and W. Meyer*, A sphere theorem with a pinching constant below 1/4. — *J. Dif. Geom.* (1996), v. 44, p. 214–261.

[2] *R. Bott*, Lectures on Morse theory, old and new. — *Bull. AMS* (1982), v. 7, p. 331–358.

[3] *É. Cartan*, Sur les variétés de courbure constante d'un espace euclidien ou non-euclidien. — *Bull. Soc. Math. France* (1919), v. 47, p. 125–160; (1920), v. 48, p. 132–208.

[4] *É. Cartan*, Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien. — *Bull. Soc. Pol. Math.* (1927), v. 6, p. 1–7.

[5] *É. Cartan*, Les systemes différentiels extérieurs et leurs applications géométriques. Hermann, Paris (1945).

[6] *M. Dajczer*, Submanifolds and isometric immersions. Publish or Perish, Houston TX (1990).

[7] *M. do Carmo and M. Dajczer*, Conformal rigidity. — *Amer. J. Math.* (1987), v. 109, p. 963–985.

[8] *Y. Felix, S. Halperin, and J.-C. Thomas*, Rational homotopy theory. — Springer–Verlag, New York (2000).

[9] *A. Fraser*, Fundamental groups of manifolds of positive isotropic curvature. Preprint (2001).

[10] *D. Gromoll and W. Meyer*, Periodic geodesics on compact Riemannian manifolds. — *J. Diff. Geom.* (1969), v. 3, p. 493–510.

[11] *M. Gromov*, Partial differential relations. Springer, New York (1986).

[12] *D. Hilbert*, Über Flachen von konstanter Gausser Krümmung. — *Trans. Amer. Math. Soc.* (1901), v. 2, p. 87–99.

[13] *P. Hartman*, On isometric immersions in Euclidean space of manifolds with nonnegative sectional curvatures II. — *Trans. Amer. Math. Soc.* (1970), v. 147, p. 529–540.

[14] *E. Holmgren*, Sur les surfaces à courbure constante negative. — *C. R. Acad. Sci. Paris* (1902), v. 134, p. 740–743.

[15] *S. Kobayashi and K. Nomizu*, Foundations of differential geometry (two volumes). Wiley, New York (1963–69).

[16] *M. Micallef and J. D. Moore*, Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes. — *Ann. Math.* (1988), v. 127, p. 199–227.

[17] *J. Milnor*, Morse theory. Princeton Univ. Press, Princeton, NJ (1963).

[18] *J.D. Moore*, Isometric immersions of space forms in space forms. — *Pacific J. Math.* (1972), v. 40, p. 157–166.

[19] *J.D. Moore*, Conformally flat submanifolds of Euclidean space. — *Math. Ann.* (1977), v. 225, p. 89–97.

[20] *J.D. Moore*, Submanifolds of constant positive curvature I. — *Duke Math. J.* (1977), v. 44, p. 449–484.

[21] *J.D. Moore*, Codimension two submanifolds of positive curvature. — *Proc. Amer. Math. Soc.* (1978), v. 70, p. 72–74.

[22] *J.D. Moore*, On extendability of isometric immersions of spheres. — *Duke Math. J.* (1996), v. 85, p. 685–699.

[23] *J.D. Moore*, Euler characters and submanifolds of constant positive curvature. — *Trans. Amer. Math. Soc.* (To appear).

[24] *R. Palais and S. Smale*, A generalized Morse theory. — *Bull. Amer. Math. Soc.* (1964), p. 165–172.

[25] *D. Quillen*, Superconnections and the Chern character. — *Topology* (1985), v. 24, p. 89–95.

[26] *J. Sacks and K. Uhlenbeck*, The existence of minimal immersions of 2-spheres. — *Ann. Math.* (1981), v. 113, p. 1–24.

[27] *J. Sacks and K. Uhlenbeck*, Minimal immersions of closed Riemann surfaces. — *Trans. Amer. Math. Soc.* (1982), v. 271, p. 639–652.

[28] *R. Schoen and S.T. Yau*, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with non-negative scalar curvature. — *Ann. Math.* (1979), v. 110, p. 127–142.

[29] *K. Shankar*, On the fundamental groups of positively curved manifolds. — *J. Diff. Geom.* (1998), v. 49, p. 179–182.

[30] *S. Smale*, Review of "Global variational calculus: Weierstrass integrals on a Riemannian manifold" by Marston Morse. — *Bull. Amer. Math. Soc.* (1977), v. 83, p. 683–697.

[31] *K. Tenenblat and C.L. Terng*, Bäcklund's theorem for n-dimensional submanifolds of R^{2n-1}. — *Ann. Math.* (1980), v. 111, p. 477–490.

[32] *M. Vigué-Poirrier and Dennis Sullivan*, The homology theory of the closed geodesic problem. — *J. Diff. Geom.* (1976), v. 11, p. 633–644.

[33] *F. Xavier*, A non-immersion theorem for hyperbolic manifolds. — *Comm. Math. Helv.* (1985), v. 60, p. 280-283.