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On the growth of entire generating functions of multiply positive sequences

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The paper deals with the class of entire generating functions of k-times positive (by Fecete, Pólya, Schoenberg) sequences of coefficients. For all integers k, $1 \le k < \infty$, we give the exhaustive description of the growth and trigonomertical around zero indicators of the above functions.

1. Introduction

The multiply positive sequences (also called Pólya frequency sequences) were introduced by M. Fekete in 1912 (see [1]) in connection with the problem on the exact calculation of the number of positive zeros of the real polynomial.

The class of all multiply positive sequences of order $m \in \mathbf{N} \cup \{\infty\}$ (m-times positive) is denoted by PF_m and consists of the sequences $\{c_k\}_{k=0}^{\infty}$ such that all minors of order $\le m$ (all minors if $m = \infty$) of the infinite matrix

$$\left\| \begin{array}{ccccc} c_0 & c_1 & c_2 & c_3 & \dots \\ 0 & c_0 & c_1 & c_2 & \dots \\ 0 & 0 & c_0 & c_1 & \dots \\ 0 & 0 & 0 & c_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\| \qquad (1)$$

are nonnegative. We will denote by SPF_m the class of all sequences from PF_m such that all minors of order $\le m$ of matrix (1) without vanishing rows (columns) are positive. The class of corresponding generating functions

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

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is also denoted by $PF_m(SPF_m)$.

The class PF_∞ was completely described by M. Aissen, I.J. Schoenberg, A. Whitney, and A. Edrei in [2] (see also [3, p. 412]).

Theorem ASWE. *A function $f \in PF_\infty$ iff*

$$f(z) = Cz^n e^{\gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z)/(1 - \beta_k z),$$

where $C \geq 0, n \in \mathbf{Z}, \gamma \geq 0, \alpha_k \geq 0, \beta_k \geq 0, \sum(\alpha_k + \beta_k) < \infty$.

In 1955 I.J. Schoenberg set up the problem of characterizing the classes PF_m, $m \in \mathbf{N}$. In this paper we investigate the growth of entire generating functions of multiply positive sequences. By Theorem ASWE the growth of entire functions from PF_∞ is not greater than the exponential type. The aim of this paper is to prove the following result.

Theorem 1. *Let m be any positive integer. For a given proximate order $\rho(r) \to \rho, \rho \geq 0$ there exists an entire function $f \in PF_m$ with proximate order $\rho(r)$.*

For the case $\rho < 1$ this theorem can be deduced from Theorem ASWE. Indeed, for a given proximate order $\rho(r) \to \rho$, $0 \leq \rho < 1$ there exists an entire function f with negative zeros of proximate order $\rho(r)$. We obtain Theorem 1 as a consequence of the following result in the case $\rho > 0$.

Theorem 2. *Let m be any positive integer. Let $h : \mathbf{R} \to \mathbf{R}$ satisfy the following condition:*

0) there exists $\theta_0 \in (0, \pi]$ such that

$$h(\theta) = h(0)\cos(\rho\theta), \quad |\theta| \leq \theta_0,$$

then h is the indicator of an entire function $f \in PF_m$ of the proximate order $\rho(r) \to \rho > 0$ if and only if h satisfies the following conditions:

1) h is a 2π-periodic, ρ-trigonometrically convex function;

2) h is an even function;

3) $\max\{h(\theta) : -\pi \leq \theta \leq \pi\} = h(0)$.

We believe that Theorem 2 is true without restrictional condition 0). But now we have not proved it.

The indicators corresponding to the class of entire Hermitian-positive functions of finite order and to the class of entire absolutely monotonic functions of finite order were completely characterized by A.A. Goldberg and I.V. Ostrovskii in [5, 6].

Note that the necessity of conditions 1)–3) in Theorem 2 is clear. So, it suffices to prove that for any function h, satisfying the conditions of Theorem 2, there exists an entire function $f \in PF_m$ such that $h(\theta, f) = h(\theta)$. The method of proof is similar to the method used in [5, 6]. The rest of this paper is devoted to the construction of such function $f \in PF_m$.

2. The reduction of the problem on a trigonometrical around zero indicator to a statement on the asymptotic behavior of some multiple integrals

P r o o f o f T h e o r e m 2. It follows from the well known theorems of entire functions theory [4] that there exists an entire function φ_1 of completely regular growth (c.r.g.) with respect to the proximate order $\rho(r)$ with the indicator $h/2$ and without zeros in the angle $|\arg z| < \theta_0$. The entire function $\varphi_2(z) = \bar{\varphi}_1(\bar{z})$ has the same properties. Put $g_1 = \varphi_1\varphi_2$. Then g_1 is an entire function of c.r.g. with respect to $\rho(r)$ with indicator $h(\theta, g_1) = h(\theta)$ without zeros in the angle $|\arg z| < \theta_0$ and $g_1(x) = |\varphi_1(x)|^2 > 0$. Since $g_1(x) \to \infty, x \to \infty$, we can assume $g_1(x) \geq e, x \geq 0$.

We need the following fact mentioned in [5].

Lemma 1. *Let $\rho(r) \to \rho > 0$ be a proximate order. Let $\delta(r)$ be a nonnegative function $\delta(r) \to 0,\ r \to \infty$. There exists a proximate order $\rho_1(r) \to \rho$ such that*

$$r^{\rho_1(r)} = o(r^{\rho(r)}), \quad \delta(r) r^{\rho(r)} = o(r^{\rho_1(r)}), \quad r \to \infty. \tag{2}$$

For the reader's convenience we present the proof of this lemma.

P r o o f. Set $\varepsilon(r) = \max_{t \geq r}(\delta(t) + \frac{1}{\log t})^{\frac{1}{2}}$. We will denote by $\log_k(r)$ the k-th iteration of $\log r$. We define the function

$$l(r) = \varepsilon(0), \quad 0 \leq r \leq r_1 = e^{e^e};$$

$$l(r) = l(r_1) + \log_3 r_1 - \log_3 r, \quad r_1 \leq r \leq u_1,$$

where u_1 is the smallest root of the equation

$$\varepsilon(r) = l(r_1) + \log_3 r_1 - \log_3 r$$

on $= [r_1, \infty)$. Put

$$l(r) = l(u_1), \quad u_1 \leq r \leq r_2 = u_1 + 1.$$

We continue this process and define the function $l(r)$ on $[0, \infty)$. Obviously, the function $l(r) \in C[0, \infty)$ and $l(r)$ has a continuous derivative everywhere with the exception of the points r_n and u_n. In the points r_n and u_n there are left-side and right-side derivatives and

$$l'(r) = \begin{cases} 0, & u_j < r < r_{j+1}; \\ (r \log r \log_2 r)^{-1}, & r_j < r < u_j. \end{cases}$$

With the help of small modifications we obtain $l \in C^1(0, \infty)$ and

$$l'(r) = O((r \log r \log_2 r)^{-1}), r \to \infty. \tag{3}$$

It is clear that

$$l(r) \geq \varepsilon(r) \geq \frac{1}{\log r}, \quad r \geq r_1; \quad l(r) \downarrow 0, \quad r \to \infty. \tag{4}$$

Set

$$r^{\rho_1(r)} = r^{\rho(r)} l(r).$$

With the help of (3),(4) it is not difficult to prove that

$$\rho_1(r) \to \rho, \quad r\rho_1'(r) \log r \to 0, \quad r^{\rho_1(r)} = o(r^{\rho(r)}), \quad r \to \infty.$$

By (4) we have

$$\delta(r) r^{\rho(r)} \leq \delta^{\frac{1}{2}}(r) \varepsilon(r) r^{\rho(r)} \leq \delta^{\frac{1}{2}}(r) l(r) r^{\rho(r)} = o(r^{\rho_1(r)}).$$

Lemma 1 is proved.

Let

$$\delta_1(r) = \max \left\{ \left(\tfrac{\log |g_1(re^{i\theta})|}{r^{\rho(r)}} - h(\theta) \right)^+ : 0 \leq \theta \leq 2\pi \right\};$$

$$\delta_2(r) = \left| \tfrac{\log g_1(r)}{r^{\rho(r)}} - h(0) \right|. \tag{5}$$

Put $\delta(r) = \max\{\delta_1(r), \delta_2(r)\}$. Evidently, $\delta(r) \geq 0$, $\delta(r) \to 0$, $r \to \infty$. By Lemma 1 there exists a proximate order $\rho_1(r) \to \rho$ such that (2) holds.

Without loss of generality we can assume that $\rho(r) \in C^1[0, \infty)$, $\rho_1(r) \in C^1[0, \infty)$. Let $h_1(\theta)$ satisfy conditions 0)–3) and moreover

3′) $h_1(0) > h_1(\theta), \quad \theta \in [-\pi, 0) \cup (0, \pi]$.

Reasoning as for g_1 we obtain the entire function g_2 of c.r.g. with respect to the proximate order $\rho_1(r)$, without zeros in the angle $|\arg z| < \theta_0$ and such that $h(\theta, g_2) = h_1(\theta)$, $g_2(x) \geq e$, $x \geq 0$.

Let $\varepsilon_k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}, k = 0, 1, \ldots, n-1$. Set

$$\varphi(z) = \frac{e^z + e^{\varepsilon_1 z} + \cdots + e^{\varepsilon_{n-1} z}}{n}. \tag{6}$$

Lemma 2. *Let $\psi(z) = \varphi(z^{1/n})$. Then ψ satisfies the following conditions:*
a) ψ is an entire function of order $1/n$;
b) $\psi \in SPF_\infty$;
c) ψ has only negative zeros;
d) $C(n)\exp\{x^{1/n}\} \le \psi(x) \le \exp\{x^{1/n}\}$.

P r o o f. Note that

$$\psi(z) = \sum_{j=0}^{\infty} \frac{z^j}{(nj)!}\,.$$

Obviously a) holds. Since the matrix (1) of coefficients of ψ is the submatrix of the matrix (1) of coefficients of $e^z \in SPF_\infty$, the function $\psi \in SPF_\infty$. By Theorem ASWE ψ has only negative zeros. The estimate d) is obvious. Lemma 2 is proved.

Let n be a positive integer such that $\rho > \frac{1}{n}$. Set

$$g(z) = g_1(z)g_2(z), f(z) = \varphi(pz^{1/n})g(z), \tag{7}$$

where p is a positive integer.

Theorem 2 is a corollary of the following fact.

Proposition 1. $\forall m \in \mathbf{N} \exists p_0(m) \in \mathbf{N} \forall p \ge p_0(m), p \in \mathbf{N} \Longrightarrow f \in PF_m$.

Indeed, for any $p \in \mathbf{N}$ the function f is an entire function of c.r.g. with respect to $\rho(r)$ and $h(\theta, f) = h(\theta, g_1) = h(\theta)$. So to prove Theorem 2 it is enough to prove Proposition 1.

Let

$$f_\varepsilon(z) = \varphi(\sqrt[n]{z})g(\varepsilon z) = \varphi(\sqrt[n]{z})g_1(\varepsilon z)g_2(\varepsilon z),\ (\varepsilon = \frac{1}{p^n}). \tag{8}$$

It is clear that

$$\forall \varepsilon > 0\ :\ (f(z) \in PF_m \iff f(\varepsilon z) \in PF_m).$$

Thus, we can rewrite Proposition 1 in the following equivalent form.

Proposition 1$'$. $\forall m \in \mathbf{N}\ \exists \varepsilon_0(m) > 0\ \forall \varepsilon,\ 0 < \varepsilon < \varepsilon_0(m),\ \frac{1}{\sqrt[n]{\varepsilon}} \in \mathbf{N} \implies f_\varepsilon(z) \in PF_m$.

Put

$$f_\varepsilon(z) = \sum_{k=0}^{\infty} a_k(\varepsilon) z^k \tag{9}$$

and consider the determinants ($a_k(\varepsilon) = 0$ for $k < 0$):

$$A_k^\nu(\varepsilon) = \det \|a_{k+j-i}(\varepsilon)\|_{i,j=0}^{\nu-1},\ k = 0, 1, 2, \ldots\,. \tag{10}$$

Proposition 2. $\forall \nu \in \mathbf{N} \quad \exists N(\nu) \in \mathbf{N} \quad \forall \varepsilon, \quad 0 < \varepsilon < 1, \quad \forall k \geq N(\nu) \Longrightarrow A_k^\nu(\varepsilon) > 0.$

To deduce Proposition $1'$ from Proposition 2 we need the following lemma from [7], which is similar to a lemma of Schoenberg [8].

Lemma 3. ([7]) *Let $\{a_k\}_{k=0}^\infty$ be a sequence of positive numbers such that $\sum_0^\infty a_k < \infty$. Consider the matrix with ν rows and infinitely many columns*

$$A_\nu = \|a_{j-i}\|_{i=0,\dots,\nu-1; j=0,1,\dots} \; (a_k = 0, \; for \; k < 0).$$

Suppose that for every $\nu = 1, \dots, m$ the matrix A_ν satisfies the following condition: all minors of order ν, composed of consecutive columns, are positive. Then $\{a_k\} \in PF_m$.

We deduce Proposition $1'$ from Proposition 2. We fix a positive integer $\nu \in \{1, 2, \dots, m\}$ and consider the matrix

$$A_\nu(\varepsilon) = \|a_{j-i}(\varepsilon)\|_{i=0,\dots,\nu-1; j=0,1=,\dots},$$

where $a_k(\varepsilon)$ are defined by (9) for $k \geq 0$ and $a_k = 0$ for $k < 0$. Clearly its minors of order ν, composed of consecutive columns, are the determinants $A_k^\nu(\varepsilon) > 0$ for any ε, $0 < \varepsilon < 1$, and $k \geq N(\nu)$.

Note that $f_\varepsilon(z) \to \varphi(\sqrt[n]{z})g(0)$, $\varepsilon \to 0$ uniformly on any compact set of $\mathbf{C}$-plane. By Lemma 2 $\varphi(\sqrt[n]{z}) = \psi(z) \in SPF_\infty$. Therefore,

$$\forall \nu \in \mathbf{N} \exists \varepsilon(\nu) > 0 \forall \varepsilon, 0 < \varepsilon < \varepsilon_\nu \forall k = 0, 1, \dots, N(\nu) - 1 \Longrightarrow A_k^\nu(\varepsilon) > 0.$$

So,

$$\forall \nu \in \mathbf{N} \exists \varepsilon(\nu) > 0 \forall \varepsilon, \; 0 < \varepsilon < \varepsilon_\nu \forall k = 0, 1, 2 \dots \infty \Longrightarrow A_k^\nu(\varepsilon) > 0.$$

Putting $\tilde{\varepsilon} = \min\{\varepsilon(1), \dots, \varepsilon(m)\}$, we obtain that the sequence $\{a_k(\varepsilon)\}$ satisfies the assumptions of Lemma 3 and thus $\{a_k(\varepsilon)\} \in PF_m$. We will show that Proposition 2 follows from Proposition 3 on asymptotic behavior of the multiple integrals

$$I_k^\nu(\eta, \varepsilon) = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \Re \left\{ \prod_{j=1}^{\nu} \left(e^{-ik\zeta_j} \frac{f_\varepsilon(e^{\eta + i\zeta_j})}{f_\varepsilon(e^\eta)} \right) \right\} \times \prod_{1 \leq \alpha < \beta \leq \nu} 4 \sin^2 \frac{\zeta_\alpha - \zeta_\beta}{2} d\zeta_1 \dots d\zeta_\nu, \tag{11}$$

where the function f_ε is defined by (8).

Proposition 3. $\forall \nu \in \mathbf{N} \quad \exists N(\nu) \in \mathbf{N} \quad \forall \varepsilon, 0 < \varepsilon < 1 \quad \forall k \geq N(\nu) \quad \exists \eta = \eta(k, \nu, \varepsilon) > 0 \implies I_k^\nu(\eta, \varepsilon) > 0.$

To deduce Proposition 2 from Proposition 3 we need the following lemma from [7].

Lemma 4. *Let* $f(z) = \sum_{k=0}^\infty a_k z^k$ *be an entire function and let*

$$A_k^\nu = \det \|a_{k+j-l}\|_{l=0,\dots,\nu-1;j=0,\dots,\nu-1} = \qquad (a_k = 0, \ for \ k < 0).$$

Then, for any $r > 0$

$$A_k^\nu \nu! (2\pi)^\nu r^{k\nu} = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \prod_{j=1}^{\nu} \left(e^{-ik\zeta_j} f(re^{i\zeta_j}) \right) \times \prod_{1 \leq \alpha < \beta \leq \nu} 4 \sin^2 \frac{\zeta_\alpha - \zeta_\beta}{2} d\zeta_1 \dots d\zeta_\nu. \tag{12}$$

For the reader's convenience we present the proof of Lemma 4.

P r o o f. Since

$$a_k = \frac{r^{-k}}{2\pi} \int_{-\pi}^{\pi} e^{-ik\zeta} f(re^{i\zeta}) d\zeta, \quad k \in \mathbf{Z},$$

we have

$$A_k^\nu = \frac{r^{-k\nu}}{(2\pi)^\nu} \det \| \int_{-\pi}^{\pi} f(re^{i\zeta}) e^{-i(k+j)\zeta} e^{il\zeta} \|_{l,j=0}^{\nu=-1}.$$

By virtue of the Pólya composition formula [9] for any 2 sets of functions $\psi_0, \dots, \psi_{\nu-1}$ and $\varphi_0, \dots, \varphi_{\nu-1}$, we have

$$\det \| \int_a^b \psi_\alpha(x) \varphi_\beta(x) dx \|_{\alpha,\beta=0}^{\nu-1}$$

$$= \frac{1}{\nu!} \int_a^b \dots \int_a^b \det \|\psi_\alpha(x_\beta)\|_{\alpha,\beta=0}^{\nu-1} \det \|\varphi_\alpha(x_\beta)\|_{\alpha,\beta=0}^{\nu-1} dx_0 \dots dx_{\nu-1} .$$

Applying this formula to $\psi_\alpha(\zeta) = f(re^{i\zeta})e^{-i(k+\alpha)\zeta}$ and $\varphi_\alpha(\zeta) = e^{i\alpha\zeta}$ and the formula for the Vandermonde determinant, we obtain the proof of Lemma 4.

Now let the function f_ε to be given by (8), the determinant $A_k^\nu(\varepsilon)$ by (9) and (10) and the integral $I_k^\nu(\nu,\varepsilon)$ by (11). By Lemma 4 we have

$$I_k^\nu(\eta,\varepsilon) = \Re\left(\nu! e^{\eta k\nu}(2\pi)^\nu (f_\varepsilon(e^\eta))^{-\nu} A_k^\nu(\varepsilon)\right).$$

Since $f_\varepsilon = (r) > 0$ for $r \geq 0$ and the determinants $A_k^\nu(\varepsilon)$ are real, it follows that

$$sign I_k^\nu(\eta,\varepsilon) = sign A_k^\nu(\varepsilon),$$

and we see that Proposition 2 is a corollary of Proposition 3.

Let $\zeta = (\zeta_1, \dots, \zeta_\nu) \in \mathbf{R}^\nu$. We will use the l_∞-norm. To prove Proposition 3 we write the integral (11) as

$$I_k^\nu(\eta,\varepsilon) = \left(\int\limits_{||\zeta||\leq\sigma} \dots \int + \int\limits_{\sigma\leq||\zeta||\leq\pi} \dots \int\right) \Re\left\{\prod_{j=1}^{\nu}\left(e^{-ik\zeta_j}\frac{f_\varepsilon(e^{\eta+i\zeta_j})}{f_\varepsilon(e^\eta)}\right)\right\}$$

$$\times \prod_{1\leq\alpha<\beta\leq\nu} 4\sin^2\frac{=}{\zeta_\alpha - \zeta_\beta} 2 d\zeta_1 \dots d\zeta_\nu = J_1 + J_2, \tag{13}$$

where $\sigma > 0$ and $\eta > 0$ will be chosen with the help of reasoning usually applied in the saddle-point method. We will estimate J_1 from bellow and J_2 from above.

3. The estimate from bellow of the integral J_1

Put

$$\begin{aligned} b_\varepsilon(\eta) &= \log f_\varepsilon(e^\eta) = \log\varphi(e^{\eta/n}) + \log g(\varepsilon e^\eta) \\ &= \log\varphi(e^{\eta/n}) + \log g_1(\varepsilon e^\eta) + \log g_2(\varepsilon e^\eta). \end{aligned} \tag{14}$$

Since $g_{1,2}(z) \neq 0$ when $|\arg z| < \theta_0$ and $\varphi(\sqrt[n]{z})$ has only negative zeros, we have $f_\varepsilon(e^{\eta+i\zeta}) \neq 0$ in the circle $\{\zeta \in \mathbf{C} : |\zeta| \leq \theta_0/2\}$. Thus, for $|\zeta| \leq \theta_0/2$ we have the decomposition

$$\log\left\{e^{-ik\zeta}\frac{f_\varepsilon(e^{\eta+i\zeta})}{f_\varepsilon(e^\eta)}\right\} = -ik\zeta + \sum_{j=1}^{\infty} b_\varepsilon^{(j)}(\eta)\frac{i^j\zeta^j}{j!}. \tag{15}$$

Since f is of c.r.g. and $f \neq 0$, when $|\arg z| < \theta_0$, there exists

$$\lim_{r\to\infty} r^{-\rho(r)}\log f(r) = h(0) > 0,$$

and hence,

$$\frac{1}{\eta}b_\varepsilon(\eta) = \frac{1}{\eta}\log f(\varepsilon e^\eta) \geq \frac{1}{\eta}C_1(\varepsilon e^\eta)^{\rho(\varepsilon e^\eta)} \to \infty,\ \eta \to \infty$$

(here and further we denote positive constants by C with indexes).

Thus,

$$\limsup_{\eta\to+\infty} b'_\varepsilon(\eta) = +\infty\,,$$

and the equation

$$b'_\varepsilon(\eta) = k \tag{16}$$

has solutions for any $k \geq k_0 = \max_{0\leq\varepsilon\leq 1}[b'_\varepsilon(0)] + 1$.

Let $\eta = \eta(\varepsilon, k)$ be the smallest root of (16). This choice of η lets us write (15) in the form

$$\log\left\{e^{-ik\zeta}\frac{f_\varepsilon(e^{\eta+i\zeta})}{f_\varepsilon(e^\eta)}\right\} = -\frac{1}{2}\zeta^2 b''_\varepsilon(\eta) + \tau_\varepsilon(\zeta,\eta), \tag{17}$$

where

$$\tau_\varepsilon(\zeta,\eta) = \sum_{j=3}^{\infty} b_\varepsilon^{(j)}(\eta)\frac{i^j\zeta^j}{j!}\,. \tag{18}$$

We need the following lemma.

Lemma 5. *For $k \geq k_0, k_0 \in \mathbf{N}$, and $\eta = \eta(\varepsilon, k)$ the following inequalities are valid:*

$$k = b'_\varepsilon(\eta) \leq C_3\left(e^{\frac{\eta}{n}} + (\varepsilon e^\eta)^{\rho(\varepsilon e^\eta)}\right); \tag{19}$$

$$k \geq C_4\left(e^{\frac{\eta}{n}} + (\varepsilon e^\eta)^{\rho(\varepsilon e^\eta)}\right)^{3/4}; \tag{20}$$

$$|b_\varepsilon^{(j)}(\eta)| \leq \frac{2^j j!}{\theta_0^j}C_5 k^{4/3}. \tag{21}$$

P r o o f. Applying the Schwarz formula to the function $\log f_\varepsilon(e^{\eta+z})$ in the circle $|z| \leq \theta_0/2$, differentiating with respect to z, setting $z = 0$, taking into account that $\log f_\varepsilon(e^\eta) > 0, \eta \in \mathbf{R}$, and (8), we obtain

$$|b_\varepsilon^{(j)}(\eta)| \leq \frac{2^{j+1}j!}{\theta_0^j\pi}\int_{-\pi}^{\pi}\log^+|f_\varepsilon(e^{\eta+\frac{\theta_0}{2}e^{i\tau}})|d\tau$$

$$\leq \frac{2^{j+1}j!}{\theta_0^j\pi}\left(\int\limits_{-\pi}^{\pi}\log^+|\varphi(e^{\frac{\eta}{n}+\frac{\theta_0}{2n}e^{i\tau}})|d\tau+\int\limits_{-\pi}^{\pi}\log^+|g(\varepsilon e^{\eta+\frac{\theta_0}{2}e^{i\tau}})|d\tau\right). \tag{22}$$

Since $\varphi(\sqrt[n]{z})$ has nonnegative coefficients, using d), we have

$$\log^+|\varphi(e^{\frac{\eta}{n}+\frac{\theta_0}{2n}e^{i\tau}})|\leq \log\varphi(e^{\frac{\eta}{n}+\frac{\theta_0}{2n}\cos\tau})\leq e^{\frac{\eta}{n}+\frac{\theta_0}{2n}\cos\tau}\leq C_6 e^{\frac{\eta}{n}}. \tag{23}$$

Using properties of the proximate order [4], we obtain

$$\log^+|g(xe^{\frac{\theta_0}{2}e^{i\tau}})|\leq C_7(xe^{\frac{\theta_0}{2}\cos\tau})^{\rho(xe^{\frac{\theta_0}{2}\cos\tau})}\leq C_8(x^{\rho(x)}+1),\quad x\geq x_0. \tag{24}$$

Since $\frac{\log^+|g(xe^{\frac{\theta_0}{2}e^{i\tau}})|}{x^{\rho(x)}+1}$ is uniformly continuous in x with respect to τ, the analogous inequality is true for all $x\geq 0$. Hence by (23) and (24),

$$|b_\varepsilon^{(j)}(\eta)|\leq \frac{2^j j!}{\theta_0^j}C_8\left(e^{\frac{\eta}{n}}+(\varepsilon e^{\eta})^{\rho(\varepsilon e^{\eta})}\right), \tag{25}$$

in particular, (19) holds.

Now we prove (20) and (21). Since $\eta=\eta(\varepsilon,k)$ is the smallest root of (16), we have $b'_\varepsilon(\eta)<k$ for $\eta<\eta(\varepsilon,k)$. Consequently

$$b_\varepsilon(\eta(\varepsilon,k))=\int\limits_0^{\eta(\varepsilon,k)}b'_\varepsilon(\eta)d\eta+b_\varepsilon(0)<k\eta+\max_{0\leq\varepsilon\leq 1}\log g(\varepsilon)\leq k\eta+C_{11}. \tag{26}$$

Since $\lim_{r\to\infty}\frac{\log g(r)}{r^{\rho(r)}}=h(0)>0$, and $\log g(r)\geq 2,\quad r\geq 0$, with the help of d) we obtain

$$b_\varepsilon(\eta)=\log\varphi(e^{\frac{\eta}{n}})+\log g(\varepsilon e^{\eta})\geq C_{12}\left(e^{\frac{\eta}{n}}+(\varepsilon e^{\eta})^{\rho(\varepsilon e^{\eta})}\right).$$

But

$$\eta<4ne^{\frac{\eta}{4n}}<4n\left(e^{\frac{\eta}{n}}+=(\varepsilon e^{\eta})^{\rho(\varepsilon e^{\eta})}\right)^{1/4},$$

thus by (26),

$$C_{13}\left(e^{\frac{\eta}{n}}+(\varepsilon e^{\eta})^{\rho(\varepsilon e^{\eta})}\right)\leq k\left(e^{\frac{\eta}{n}}+(\varepsilon e^{\eta})^{\rho(\varepsilon e^{\eta})}\right)^{1/4},\quad k\geq k_0,$$

whence we obtain (20). (21) follows from (20) and (25). Lemma 5 is proved.

It follows from (21) that for $\zeta\leq\frac{\theta_0}{4}$ me have

$$|\tau_\varepsilon(\zeta,\eta)|\leq C_5k^{4/3}\sum_{p=3}^{\infty}\frac{2^p|\zeta|^p}{\theta_0^p}\leq C_{17}|\zeta|^3k^{4/3}.$$

Choose

$$\sigma = \sigma(k) = \left(\frac{\pi}{3C_{17}\nu}\right)^{1/3} k^{-4/9}, \tag{27}$$

then for $|\zeta| \le \sigma$ we have

$$|\tau_\varepsilon(\zeta,\eta)| \le \frac{\pi}{3\nu}. \tag{28}$$

Applying (17), (21), we obtain

$$\Re \prod_{j=1}^{\nu} \left\{ e^{-ik\zeta_j} \frac{f_\varepsilon(e^{\eta+i\zeta_j})}{f_\varepsilon(e^\eta)} \right\} \ge \frac{1}{2} e^{-\frac{\pi}{3}} \exp\left(-C_{18}k^{4/3} \sum_{j=1}^{\nu} \zeta_j^2\right). \tag{29}$$

So, with the help of (27) we obtain

$$\begin{aligned} J_1 \ge & \left(\frac{2}{\pi}\right)^{\nu(\nu-1)} C_{19} \int_{\|\zeta\|\le\sigma} \!\!\dots\! \int \exp\left(-C_{18}k^{4/3}\sum_{j=1}^{\nu}\zeta_j^2\right) \\ & \times \prod_{1\le\alpha<\beta\le\nu} (\zeta_\alpha-\zeta_\beta)^2 d\zeta_1\dots d\zeta_\nu \\ = & \left(\frac{2}{\pi}\right)^{\nu(\nu-1)} C_{19} k^{-\frac{2\nu^2}{3}} \int_{\|u\|\le C_{20}k^{\frac{2}{9}}} \!\!\dots\! \int \exp\left(-C_{18}\sum_{j=1}^{\nu}u_j^2\right) \\ & \times \prod_{1\le\alpha<\beta\le\nu} (u_\alpha-u_\beta)^2 du_1\dots du_\nu \ge C_{21}(\nu)k^{-\frac{2\nu^2}{3}}, \end{aligned} \tag{30}$$

where $C_{21}(\nu) > 0$ depends only on ν.

4. The estimate from above of the integral J_2

The integration domain $\{\zeta : \sigma < \|\zeta\| \le \pi\}$ in J_2 is contained in the union of the domains

$$\{\zeta : |\zeta_1| \le \pi, \dots, |\zeta_{j-1}| \le \pi, \sigma < |\zeta_j| \le \pi, |\zeta_{j+1}| \le \pi, \dots, |\zeta_\nu| \le \pi\}, j = 1, \dots, \nu,$$

and the integrand has a majorant

$$2^{(\nu-1)\nu} \prod_{j=1}^{\nu} \left| \frac{f_\varepsilon(e^{\eta+i\zeta_j})}{f_\varepsilon(e^\eta)} \right|,$$

which is symmetric with respect to $\zeta_1, \dots, \zeta_\nu$. Therefore,

$$\begin{aligned}
|J_2| &\le \nu 2^{(\nu-1)\nu} \int\limits_{\sigma<|\zeta|\le\pi} \left|\frac{f_\varepsilon(e^{\eta+i\zeta})}{f_\varepsilon(e^\eta)}\right| d\zeta \left(\int\limits_{-\pi}^{\pi} \left|\frac{f_\varepsilon(e^{\eta+i\zeta})}{f_\varepsilon(e^\eta)}\right| d\zeta \right)^{\nu-1} \\
&= \nu 2^{(\nu-1)\nu} \left(\int\limits_{\sigma<|\zeta|\le\frac{\theta_0}{4}} + \int\limits_{\frac{\theta_0}{4}<|\zeta|\le\pi} \right) \left(\left|\frac{f_\varepsilon(e^{\eta+i\zeta})}{f_\varepsilon(e^\eta)}\right| d\zeta \right) \left(\int\limits_{-\pi}^{\pi} \left|\frac{f_\varepsilon(e^{\eta+i\zeta})}{f_\varepsilon(e^\eta)}\right| d\zeta \right)^{\nu-1} \\
&= C_{22}(\nu)(I_1+I_2)I_3. \qquad (31)
\end{aligned}$$

To estimate the integrals I_1 and I_3 we need the following lemma, which is similar to a lemma from [5].

Lemma 6.

$$\log|g(re^{i\theta})| - \log g(r) \le -C_{23} r^{\rho(r)}\theta^2 + C_{24}, \quad \textit{where} \quad |\theta| \le \frac{\theta_0}{4}. \qquad (32)$$

P r o o f. Since g is of c.r.g. and has no zeros in the angle $\{\theta : |\theta| \le \theta_0/2\}$, for $|\theta| \le \theta_0/2$ the following equation holds:

$$\log|g(re^{i\theta})| = h(0) r^{\rho(r)} \cos(\rho(r)\theta) + o(r^{\rho(r)}), \quad r \to \infty.$$

It is easy to prove that for $r_0/2 \le r \le 3r_0/2$ the following equation is true:

$$r^{\rho(r)} = r^{\rho(r_0)}(1+o(1)), \quad r_0 \to \infty.$$

So,

$$\log|g(re^{i\theta})| = h(0) r^{\rho(r_0)} \cos(\rho(r_0)\theta) + o(r^{\rho(r_0)}), \quad r_0 \to \infty.$$

Denote by

$$\lambda_{r_0}(re^{i\theta}) = o(r^{\rho(r_0)}), \quad r_0 \to \infty, \quad |\theta| \le \theta_0/2. \qquad (33)$$

Since $\lambda_{r_0}(re^{i\theta}) = \lambda_{r_0}(re^{-i\theta})$, we have

$$\frac{\partial}{\partial\theta} \lambda_{r_0}(re^{i\theta}) \Big|_{\theta=0} = 0,$$

whence for $|\theta| \le \theta_0/4$ it is true that

$$\lambda_{r_0}(re^{i\theta}) - \lambda_{r_0}(r) = \frac{\partial^2}{\partial\theta^2} \lambda_{r_0}(re^{i\theta}) \Big|_{\theta=\tau_0} \frac{\theta^2}{2}, \quad |\tau_0| \le \frac{\theta_0}{4}. \qquad (34)$$

We will assume that $r_0 e^{\frac{\theta_0}{4}}/2 \le r \le 3r_0 e^{\frac{-\theta_0}{4}}/2$. The function $\lambda_{r_0}(re^{i(\tau_0+z)})$ is harmonic in the circle $|z| \le \theta_0/4$. Thus by the Poissonian formula

$$\lambda_{r_0}(re^{i(\tau_0+z)}) = \frac{1}{2\pi}\int_0^{2\pi} \lambda_{r_0}(re^{i(\tau_0+\frac{\theta_0}{4}e^{i\psi})})\Re\frac{\theta_0 e^{i\psi}+4z}{\theta_0 e^{i\psi}-4z}d\psi,$$

whence by (33), we obtain

$$\begin{aligned}\left|\frac{\partial^2}{\partial\theta^2}\lambda_{r_0}(re^{i\theta})\Big|_{\theta=\tau_0}\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\lambda_{r_0}(re^{i(\tau_0+\frac{\theta_0}{4}e^{i\psi})})\Re\,\frac{64\theta_0 e^{i\psi}}{(\theta_0 e^{i\psi}-4z)^3}\Big|_{z=0}d\psi\right|\\ &= o(r^{\rho(r)}), \quad r_0 \ge x_0.\end{aligned}$$

Hence, applying (33) and (34), we obtain for $|\theta| \le \theta_0/4$

$$\begin{aligned}\log|g(re^{i\theta})| - \log g(r) &\le -2h(0)r^{\rho(r_0)}\sin^2\frac{\rho(r_0)\theta}{2} + |\lambda_{r_0}(re^{i\theta}) - \lambda_{r_0}(r)|\\ &\le -h(0)r^{\rho(r_0)}\frac{(\rho(r_0)\theta)^2}{\pi^2}, \quad r_0 \ge x_0,\end{aligned}$$

and thus

$$\log|g(re^{i\theta})| - \log g(r) \le -C_{23}r^{\rho(r_0)}\theta^2 + C_{24}$$

for any $r \ge 0$ and $|\theta| \le \theta_0/4$. Lemma 6 is proved.

By (8)

$$\begin{aligned}\log|f_\varepsilon(e^{\eta+i\zeta})| - \ log f_\varepsilon(e^\eta) &= \log|\varphi(e^{(\eta+i\zeta)/n})| - \log\varphi(e^{\eta/n})\\ &+ \ \log|g(\varepsilon e^{\eta+i\zeta})| - \log g(\varepsilon e^\eta). \end{aligned} \tag{35}$$

Since $\varphi(\sqrt[n]{z})$ has nonnegative coefficients, with the help of d) we obtain

$$\begin{aligned}\log|\varphi(e^{(\eta+i\zeta)/n})| - \log\varphi(e^{\eta/n}) &\le e^{\frac{\eta}{n}}(\cos\frac{\zeta}{n} - 1) + C_{27}\\ &\le -C_{28}e^{\eta/n}\zeta^2 + C_{27}, \quad |\zeta| \le \pi.\end{aligned} \tag{36}$$

By (32),(35) and (36)

$$\log|f_\varepsilon(e^{\eta+i\zeta})| - \log f_\varepsilon(e^\eta) \le -C_{29}\left(e^{\frac{\eta}{n}} + (\varepsilon e^\eta)^{\rho(\varepsilon e^\eta)}\right)\zeta^2 + C_{30}, \quad |\zeta| \le \theta_0/4. \tag{37}$$

Hence, for $\sigma < |\zeta| \le \theta_0/4$ by (27) and (19) we have

$$\log|f_\varepsilon(e^{\eta+i\zeta})| - \log f_\varepsilon(e^\eta) \le -C_{31}(\nu)k^{\frac{1}{9}} + C_{30}$$

and

$$I_1 \leq 2\pi e^{C_{30}} \exp(-C_{31}(\nu)k^{\frac{1}{9}}). \tag{38}$$

Now we will estimate I_2. By (5) and (7)

$$\log|g(re^{i\theta})| - \log g(r) \leq (h(\theta) - h(0))r^{\rho(r)} + \delta(r)r^{\rho(r)}$$

$$+(h_1(\theta) - h_1(0))r^{\rho_1(r)} + o(r^{\rho_1(r)}).$$

From the property 3) of the function h we obtain that $h(\theta) - h(0) \leq 0$ for all $|\theta| \leq \pi$. By property $3'$) of the function h_1 we have

$$C_{32} = \min\{(h_1(0) - h_1(\theta)), \quad \theta_0/4 \leq |\theta| \leq \pi\} > 0,$$

and applying (2), we obtain

$$\log|g(re^{i\theta})| - \log g(r) \leq -C_{32}r^{\rho_1(r)} + o(r^{\rho_1(r)}) \leq -C_{32}r^{\frac{\rho(r)}{2}} + C_{33}, \quad r \geq 0,$$

whence by (35) and (36)

$$\log|f_\varepsilon(e^{\eta+i\zeta})| - \log f_\varepsilon(e^\eta) \leq -C_{34}\left(e^{\frac{\eta}{n}} + (\varepsilon e^\eta)^{\rho(\varepsilon e^\eta)}\right)^{1/2} + C_{33}, \quad \theta_0/4 \leq |\zeta| \leq \pi. \tag{39}$$

Thus by (19),

$$I_2 \leq 2\pi e^{C_{33}} \exp(-C_{34}k^{\frac{1}{2}}). \tag{40}$$

By (39) and (37)

$$\left|\frac{f_\varepsilon(e^{\eta+i\zeta})}{f_\varepsilon = (e^\eta)}\right| \leq e^{\max(C_{33}, C_{30})}, \quad |\zeta| \leq \pi.$$

So

$$I_3 \leq C_{35}. \tag{41}$$

From (30), (31), (38), (40) and (41) we obtain Proposition 3.

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