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On conductivity of composite material composed from thin random layers

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We study conductive properties of a composite material, consisting of alternating layers with finite and infinite specific conductivity. On the basis of assumptions that separation boundaries of layers are stochastic functions and the layers thickness ε (in an oder of magnitude) is infinitely descending we have derived the formula for effective electrical conductivity of the relevant sample for the total current flowing in the direction being normal to a layer surface.

1. Physical formulation of a problem

Over last years the significant interest is observed to the study of composite materials properties. Effective characteristics of composites (such as electrical and thermal conductivity, etc.), as a rule, essentially differ as compared with the relevant characteristics of constituent components and depend on their configuration and properties. In superconductivity theory, for instance, the study of composite conductive properties is of great interest when one of components is in a superconductive state (i.e. is characterized by infinite conductivity) but all others are in a normal state. Thus we study the composites with strictly contrast properties of components. Some of these composites can be met in nature and the others are prepared only artificially.

Relying on their internal structure the composite materials can be subdivided into several different groups. One of such types is a layered composite. The presented work is devoted to the study of a layer composite conductive properties if layers are alternatingly arranged with finite and infinite conductivity and have

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random layer boundaries. The layers are supposed to be of a thickness ε in an order of magnitude. We consider the problem of finding an effective conductivity of the material described above in the direction perpendicular to the layers under the condition of the small ε.

We suppose that our problem is of a plane character such that the composite properties are not change along the axis x_3. For the sample of the thickness L and the height H, consisting of $2N_\varepsilon+1$ layers (the thickness is a value of an order of magnitude $\varepsilon = \dfrac{H}{2N_\varepsilon+1}$), we specify rectangular coordinates, as it is shown in Fig. 1. The potential of an electrical field is supposed to take the value 0 and 1 at the surfaces $x_2 = 0$ and $x_2 = H$, respectively. As a result, the effective composite conductivity is equal to a total current that flows through the composite in the direction of the axis x_2.

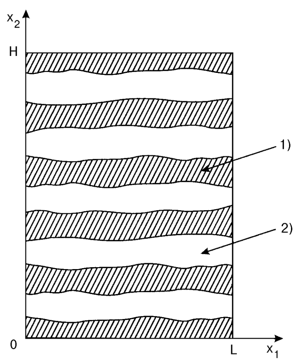

Fig. 1. 1) Layers $D_{i\varepsilon}$ with the ideal conductivity;
2) layers $G_{i\varepsilon}$ with the finite conductivity σ

We consider, as a simple example, that particular case when the layer separation boundaries are straight lines. In an each layer with the ideal conductivity the potential u_ε is constant, i.e., satisfies the following condition:

$$u_\varepsilon = c_i, \quad i = 0, \dots, N_\varepsilon;$$

here c_i are constants dependent on ε such that $c_0 = 0$ and $c_{N_\varepsilon} = 1$. If to denote as σ the specific conductivity of a material in the layers with a nonideal conductivity, then the resistance of such layer in the direction of the axis x_2 will be equal to $\dfrac{L\sigma}{\varepsilon}$. On the basis of the Ohm law for the i-th layer the equality

$$c_i - c_{i-1} = \frac{I_\varepsilon \varepsilon}{L\sigma}, \quad i = 1, \dots, N_\varepsilon \tag{1.1}$$

is correct, here I_ε is the total current. Summing up (1.1) over i, we get that

$$1 = \sum_{i=1}^{N_\varepsilon} (c_i - c_{i-1}) = \frac{I_\varepsilon H N_\varepsilon}{(2N_\varepsilon + 1)L\sigma}.$$

Thus, the current $I = \lim_{\varepsilon \to 0} I_\varepsilon$ (that the same as the effective electrical conductivity C) of the composite under our study in the case of the small ε can be reduced to the form

$$C = \frac{2L\sigma}{H}.$$

In a more general problem the separation boundaries between layers are stochastic functions and the layer thickness as a function of the variable x_1 is supposed to be determined by the formula $q_\varepsilon(x_1) = \varepsilon q(\varepsilon^{-\theta} x_1, \omega)$; here $0 < \theta < \dfrac{1}{2}$ and $q(x_1)$ is a strictly stationary, metrically transitive stochastic process, which will be precisely described in the Section 2. The main result of this work is to prove that the effective electrical conductivity C_ε of the composite under the condition $\varepsilon \to 0$ has a tendency to reach the nonrandom limit

$$C = \frac{2L\sigma}{H} \langle q^{-1}(0) \rangle,$$

with $\langle q^{-1}(0) \rangle$ being the average meaning of the stochastic process $q^{-1}(0)$, whose rigorous description and also the probability space rigorous description are represented in the Section 2.

2. Rigorous description of the problem and the main result

Let us consider the probability space $(\Omega, \mathcal{F}, P)$. The object under study is the one-dimensional, stochastic process $r(x_1, \omega), (x_1 \in \mathbb{R}_1, \omega \in \Omega)$ determined on this space. We suppose the process $r(x_1, \omega)$ to be strictly stationary and metrically transitive, and, therefore, we obtain with the probability 1 there exists the limit [2]:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T r(x_1, \omega) dx = \langle r(0) \rangle; \tag{a}$$

here $\langle\rangle$ are marks of expectation in the space $(\Omega, \mathcal{F}, P)$.

It is also supposed that

$$\langle r(0)\rangle = 0, \tag{b}$$

and with the probability 1 the inequalities

$$|r(x_1,\omega)| < \frac{1}{2}, \tag{c}$$

$$\left|\frac{\partial^k r(x_1,\omega)}{\partial {x_1}^k}\right| < A, \quad k = 1,2, \tag{d}$$

are correct; here A is independ on $\omega \in \Omega$ and $x_1 \in \mathbb{R}_1$, and derivatives of the stochastic process $r(x_1,\omega)$ exist with the probability 1.

Stochastic functions $\Gamma_{j\varepsilon} = \varepsilon(r(\varepsilon^{-\theta}x_1,\omega)+j), j = 0,\dots,2N_\varepsilon+1, \omega = \omega_j \in \Omega_j = \Omega$ describe the layer boundaries of the considered composite. Then the j - layer thickness is also a random value and is described as the continuous stochastic process $q_\varepsilon(x_1,\omega_j^2)$, determined in the probability space $(\Omega^2 = \Omega\times\Omega, \mathcal{F}^2, P^2)$ by the equalities

$$\begin{gathered} q_\varepsilon(x_1,\omega_j^2) = \varepsilon q_(\varepsilon^{-\theta}x_1,\omega_j^2), \\ q(x_1,\omega_j^2) = 1 + r(x_1,\omega_{j+1}) - r(x_1,\omega_j); \end{gathered} \tag{2.1}$$

with $\omega_j^2 = (\omega_j,\omega_{j+1})$ being the point of the event space Ω^2 such that its coordinates are the points $\omega_j = \omega \in \Omega$, $\mathcal{F}^2 = \mathcal{F}_i \times \mathcal{F}_{i+1}$ being the σ-field of measurable sets and the measure P^2 is determined on the sets of the σ-field as follows

$$P^2(F^2) = P(F_i)P(F_{i+1}),$$

where $F_i \in \mathcal{F}_i$. Realizations $r(x_1,\omega_j), j = 1,\dots,2N_\varepsilon$ of the process $r(x_1,\omega)$ are selected independently.

Lemma 2.1. *Let the random process $q_\varepsilon(x_1,\omega^2)$ be determined in the probability space $(\Omega^2,\mathcal{F}^2,P^2)$ by the equalities (2.1). If the process $r(x_1,\omega)$ satisfies the conditions (a)–(c), then $\dfrac{1}{q_\varepsilon(x_1,\omega^2)}$ is strictly stationary and metrically transitive in $(\Omega^2,\mathcal{F}^2,P^2)$.*

$\triangleright$ The proof follows from definitions of the strict stationarity and the metric transitivity [2] of a stochastic process. ■

Taking into account, that the number of the layers $G_{i\varepsilon}$ with the finite conductivity σ is equal to N_ε, and the number of the boundaries $\Gamma_{i\varepsilon}$, separating these layers from ideally conducting ones, is twice more, we involve into consideration the event space

$$\Omega^{N_\varepsilon} = \underbrace{\Omega \times \Omega \times ... \times \Omega}_{2N_\varepsilon}$$

and the corresponding probability space

$$(\Omega^{N_\varepsilon}, \mathcal{F}^{N_\varepsilon}, P^{N_\varepsilon}) = \prod_{i=1}^{2N_\varepsilon} (\Omega, \mathcal{F}, P).$$

Besides, we define the probability space $(\Omega^\infty, \mathcal{F}^\infty, P^\infty)$ using a scheme similar to the one in the Kolmogorov theorem [3] proof. Namely, we consider the space of events Ω^∞ as a projective limit of a filtered set of the finite products Ω_J^n; here $\{J\}$ (i.e., $\Omega^\infty = \varprojlim \Omega_J^n$) are arbitrary finite sets from $\mathbb{N}$. In the every Ω_J^n we construct the σ-field of the measurable sets $\mathcal{F}_J^n = \prod_{i \in J} \mathcal{F}_i$. Then we introduce the σ-field of measurable sets $\mathcal{F}^\infty$ in Ω^∞, which contains all measurable cylinders in Ω^∞, namely, sets of the kind

$$F_J^n \times \Omega^{\infty \setminus J},$$

with $F_J^n \in \mathcal{F}_J^n$. The measure P_J^n is defined on Ω_J^n as a product of measures P_i on Ω_i, i.e., $P_J^n = \prod_{i \in J} P_i$. Measures P_J^n are supposed to be consistent with each other in the sense that for any K, J such that $J \subset K$ the measure P_J^n on the space Ω_J^n is a projection of the measure P_K^{n+m}, determined on the space Ω_K^{n+m}, on the subspace Ω_J^n. Then P^∞ is defined as follows. The projection P^∞ on Ω_J^n coincides with P_J^n, that is

$$P_J^n(F_J^n) = P^\infty(F_J^n \times \Omega^{\infty \setminus J}),$$

where $F_J^n \in \mathcal{F}_J^n$. And then we prove an existence of the projective limit

$$\varprojlim (\Omega_J^n, \mathcal{F}_J^n, P_J^n) = (\Omega^\infty, \mathcal{F}^\infty, P^\infty).$$

The layers $G_{i\varepsilon}$, $i = 1, \ldots, N_\varepsilon$, are labeled as the layers with the finite conductivity σ; $D_{i\varepsilon}$, $i = 0, \ldots, N_\varepsilon$ are the ideally conducting layers (Fig. 1), and γ_{j0}, γ_{jl}, $j = 0, \ldots, 2N_\varepsilon$, are the left and the right boundaries of the jth layer (Fig. 2). The potential u_ε of an electrical field in $\Gamma_{N_\varepsilon+1,\varepsilon}$ and $\Gamma_{0,\varepsilon}$ takes the values 1 and 0 and also is the solution of the following problem:

$$\triangle u_\varepsilon(x) = 0, \ x = (x_1, x_2) \in G = \bigcup_{i}^{N_\varepsilon} G_{i\varepsilon}, \tag{2.2}$$

$$u_\varepsilon(x) = c_i, \ i = 0, \ldots, N_\varepsilon, \ x \in D_{i\varepsilon}\,, \tag{2.3}$$

$$\int_{\Gamma_{i,\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\Gamma = \int_{\Gamma_{i+1,\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\Gamma, \; i = 0, 2, \ldots, 2N_\varepsilon + 1, \tag{2.4}$$

$$\frac{\partial u_\varepsilon(x)}{\partial n} = 0, \; x \in \gamma_0 = \bigcup_{j=0}^{2N_\varepsilon} \gamma_{j0}, \; \gamma_L = \bigcup_{j=0}^{2N_\varepsilon} \gamma_{jL}. \tag{2.5}$$

Here c_i are some constant values, depending on ε, such that $c_0 = 0$ and $c_{N_\varepsilon} = 1$. The conditions (2.4) imply that the total current flowing through the opposite boundaries of ideally conducting layers has the same value. The condition (2.5) means that the lateral composite boundaries are isolated.

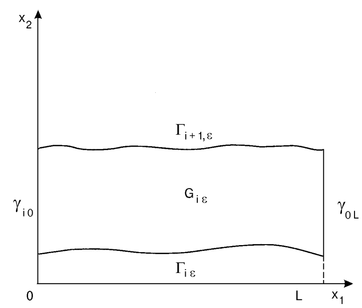

Fig. 2. Layers $G_{i\varepsilon}$ with the finite conductivity σ

As a potential difference of an electrical field on upper and lower boundaries of the composite is equal 1, its electrical conductivity C_ε by definition is equal to the current I_ε, flowing through the composite in the direction of the axes x_2, i.e. $C_\varepsilon = I_\varepsilon$, and also is determined by the formula

$$C_\varepsilon = I_\varepsilon = \sigma \int_{\Gamma_{i,\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\Gamma, \tag{2.6}$$

where $\Gamma_{i,\varepsilon}$ being an arbitrary boundary.

One more expression for electrical conductivity is intended to be obtained next. The potential $\widetilde{u}_{i\varepsilon}$ in the every layer $G_{i\varepsilon}$ is the solution of the following boundary value problem:

$$\begin{cases} \triangle\widetilde{u}_{i\varepsilon}(x) = 0, & x \in G_{i\varepsilon}, \\ \widetilde{u}_{i\varepsilon}(x) = c_i, & x \in \Gamma_{i,\varepsilon}, \\ \widetilde{u}_{i\varepsilon}(x) = c_{i+1}, & x \in \Gamma_{i+1,\varepsilon}, \\ \dfrac{\partial\widetilde{u}_{i\varepsilon}(x)}{\partial n} = 0, & x \in \gamma_{i0}, \gamma_{iL}. \end{cases} \tag{2.7}$$

By using Green's formula in the region $G_{i\varepsilon}$ and, taking into account (2.2)–(2.5), (2.6), we write the simple sequence of equalities

$$\sigma \sum_{i=1}^{N_\varepsilon} \int\limits_{G_{i\varepsilon}} |\nabla\widetilde{u}_{i\varepsilon}|^2 dx = \sigma \sum_{i=1}^{N_\varepsilon} (c_i - c_{i-1}) \int\limits_{\Gamma_{i,\varepsilon}} \frac{\partial\widetilde{u}_{i\varepsilon}}{\partial n} d\Gamma = \sigma \int\limits_{\Gamma_{i,\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\Gamma = C_\varepsilon.$$

Thus, we get

$$C_\varepsilon = \sigma \sum_{i=1}^{N_\varepsilon} \int\limits_{G_{i\varepsilon}} |\nabla\widetilde{u}_{i\varepsilon}|^2 dx. \tag{2.8}$$

On the other hand, the problem about minimization of the functional

$$C_\varepsilon(u_\varepsilon) = \sigma \int\limits_G |\nabla u_\varepsilon|^2 dx \tag{2.9}$$

over the class of functions $H_\varepsilon = \{w \in W_2^1(G_{i\varepsilon}), w(x) = c_i, i = 0..., N_\varepsilon, x \in D_{i\varepsilon}\}$ corresponds to the problem (2.2)–(2.5). Indeed, let us consider the increment $u_\varepsilon + h\delta_\varepsilon$ of functions u_ε, here δ_ε is an arbitrary function from the class H_ε, substitute it in the functional (2.9)

$$C_\varepsilon(u_\varepsilon + h\delta_\varepsilon) = \sigma \int\limits_G |\nabla u_\varepsilon|^2 dx + 2\sigma h \int\limits_G (\nabla u_\varepsilon, \nabla\delta_\varepsilon) dx + h^2\sigma \int\limits_G |\nabla\delta_\varepsilon|^2 dx$$

and equate to zero a linear part of the increment $C_\varepsilon(u_\varepsilon + h\delta_\varepsilon)$. We obtain

$$0 = \int\limits_G (\nabla u_\varepsilon, \nabla\delta_\varepsilon) dx = \sum_{i=1}^{N_\varepsilon} (-\int\limits_{G_{i\varepsilon}} \triangle\widetilde{u}_{i\varepsilon}\delta_\varepsilon dx + \int\limits_{\partial G_{i\varepsilon}} \frac{\partial\widetilde{u}_{i\varepsilon}}{\partial n}\delta_\varepsilon d\Gamma).$$

From an arbitrariness of δ_ε over the class H_ε follows that the function u_ε, minimizing the functional (2.9), is the solution of the problem (2.2)–(2.5).

Hereafter we shall derive the formula for the electrical conductivity of the composite, using the functional (2.8). We notice also that as the thickness of layers $G_{i\varepsilon}$ is the stochastic process, then the conductivity defined by the formula (2.8) is a random variable in the space $(\Omega^\infty, \mathcal{F}^\infty, P^\infty)$.

The main result of the present work is following

Theorem 2.1. *Let $\Gamma_{i,\varepsilon}$ be boundaries of the layers determined by the equalities*

$$\Gamma_{i,\varepsilon} = \varepsilon(r(\varepsilon^{-\theta}x_1, \omega) + i), i = 0, \dots, 2N_\varepsilon;$$

here the random process $r(x_1, \omega)$ satisfies the conditions (a)–(d). Then $C_\varepsilon(\omega^\infty)$ under the condition $\varepsilon \to 0$ converges with the probability 1 in the space $(\Omega^\infty, \mathcal{F}^\infty, P^\infty)$ to the nonrandom limit

$$C = \frac{2L\sigma}{H}\langle q^{-1}(0)\rangle,$$

where $\langle q^{-1}(0)\rangle$ being an average meaning of the random variable $q^{-1}(0)$.

R e m a r k. It follows from the Theorem 2.1 that the random variable $C_\varepsilon(\omega^{N_\varepsilon})$ under the condition that $\varepsilon \to 0$ converges in probability in the space $(\Omega^{N_\varepsilon}, \mathcal{F}^{N_\varepsilon}, P^{N_\varepsilon})$ to the nonrandom limit

$$C = \frac{2L\sigma}{H}\langle q^{-1}(0)\rangle,$$

namely, for any $\delta > 0$

$$\lim_{\varepsilon\to 0} P^{N_\varepsilon}\{\omega^{N_\varepsilon} : |C_\varepsilon(\omega^{N_\varepsilon}) - C| < \delta\} = 1. \tag{2.10}$$

3. Construction of the solution in the layer

We consider the domain $G_{i\varepsilon}$ in $\mathbb{R}_2$, bounded by the lines γ_{i0}, γ_{iL} and $\Gamma_{i,\varepsilon} = \varepsilon r_i(\varepsilon^{-\theta}x_1) = \varepsilon(r(\varepsilon^{-\theta}x_1, \omega_i) + i)$, $\Gamma_{i+1,\varepsilon} = \varepsilon r_{i+1}(\varepsilon^{-\theta}x_1) = \varepsilon(r(\varepsilon^{-\theta}x_1, \omega_{i+1}) + i + 1)$, $0 < \theta < \dfrac{1}{2}$; here the indices $i+1$ and i correspond to the upper and the lower boundaries of $G_{i\varepsilon}$, respectively (Fig.2). We presume that $r_i(\varepsilon^{-\theta}x_1), r_{i+1}(\varepsilon^{-\theta}x_1)$ are two realizations of the stochastic process $r(\varepsilon^{-\theta}x_1, \omega)$. Henceforth, wherever if it will not cause misunderstanding, we omit ω in the expression $r(\varepsilon^{-\theta}x_1, \omega)$. In accordance with (2.1)

$$q_\varepsilon(x_1) = \varepsilon q(\varepsilon^{-\theta}x_1, \omega_i^2) = \varepsilon(1 + r_{i+1}(\varepsilon^{-\theta}x_1) - r_i(\varepsilon^{-\theta}x_1)),$$

and $q_\varepsilon(x_1, \omega_i^2)$ is one of realizations of the stochastic process $q_\varepsilon(x_1, \omega^2)$.

The current flows through the domain $G_{i\varepsilon}$ in the direction of the axes x_2, therefore the electrical field potential $u_{i\varepsilon}(x_1) = u_{i\varepsilon}(x_1, \omega^2)$ of the layer $G_{i\varepsilon}$ is assumed to be equal 1 and 0 on $\Gamma_{i+1,\varepsilon}$ and $\Gamma_{i,\varepsilon}$, respectively. Then the potential $u_{i\varepsilon}(x_1)$ is the solution of the problem

$$\begin{cases} \triangle u_{i\varepsilon}(x) = 0, & x \in G_{i\varepsilon}, \\ u_{i\varepsilon}(x) = 0, & x \in \Gamma_{i,\varepsilon}, \\ u_{i\varepsilon}(x) = 1, & x \in \Gamma_{i+1,\varepsilon}, \\ \dfrac{\partial u_{i\varepsilon}(x)}{\partial x_1} = 0, & x \in \gamma_{i0}, \gamma_{iL}. \end{cases} \tag{3.1}$$

Since, the function $u_{i\varepsilon}$ is harmonic in the simply connected domain $G_{i\varepsilon}$, then the potential function $v_{i\varepsilon}$ conjugated with $u_{i\varepsilon}$ exists and is connected with $u_{i\varepsilon}$ by the Cauchy–Riemann conditions

$$\frac{\partial u_{i\varepsilon}}{\partial x_1} = -\frac{\partial v_{i\varepsilon}}{\partial x_2}, \quad \frac{\partial v_{i\varepsilon}}{\partial x_1} = \frac{\partial u_{i\varepsilon}}{\partial x_2}.$$

Since $\dfrac{\partial u_{i\varepsilon}}{\partial x_1} = \dfrac{\partial v_{i\varepsilon}}{\partial x_2} = 0$ for $x \in \gamma_{i0}, \gamma_{iL}$, then $v_{i\varepsilon}$ is a constant along the lines γ_{i0}, γ_{iL}. And as this function is determined to within an additive constant, we assume that $v_{i\varepsilon}(0, x_2) = 0$. Then $v_{i\varepsilon}(L, x_2) = A_\varepsilon$, where $A_\varepsilon = \int\limits_{\Gamma_{i+1,\varepsilon}} \dfrac{\partial v_{i\varepsilon}}{\partial \tau} d\Gamma = \int\limits_{\Gamma_{i+1,\varepsilon}} \dfrac{\partial u_{i\varepsilon}}{\partial n} d\Gamma$. Hence everywhere by τ and n we designate tangent and normal directions to a some curve. From boundary conditions for the function $u_{i\varepsilon}$ it is obtained, that $\int\limits_{\gamma_{iL}} \dfrac{\partial u_{i\varepsilon}}{\partial x_2} dx_2 = \int\limits_{\gamma_{iL}} \dfrac{\partial v_{i\varepsilon}}{\partial x_1} dx_2 = 1$. Thus, $v_{i\varepsilon}$ is the solution of the problem

$$\begin{cases} \triangle v_{i\varepsilon} = 0, & x \in G_{i\varepsilon}, \\ v_{i\varepsilon}(0, x_2) = 0, & \\ v_{i\varepsilon}(L, x_2) = A_\varepsilon, & \\ \dfrac{\partial v_{i\varepsilon}}{\partial n} = 0, & x \in \Gamma_{i,\varepsilon}, \Gamma_{i+1,\varepsilon}, \\ \displaystyle\int\limits_{\gamma_{iL}} \dfrac{\partial v_{i\varepsilon}}{\partial x_1} dx_2 = 1 & \end{cases} \tag{3.2}$$

and there is a one-to-one relation between the problems (3.1) and (3.2). The existence and the uniqueness of the solution of the problem (3.1) are well known, whence it follows the existence and the uniqueness of the problem (3.2) solution.

We introduce in $G_{i\varepsilon}$ the curvilinear orthogonal coordinate system $\{s, t : -\frac{\varepsilon}{2} \le s \le \frac{\varepsilon}{2}, -\varepsilon < t < L + \varepsilon\}$ such that the boundaries of the layer $\Gamma_{i+1,\varepsilon}$ and $\Gamma_{i\varepsilon}$ have coincided with the coordinate lines $s = \pm\frac{\varepsilon}{2}$, respectively, and lines being orthogonal to them correspond to fixed values of the parameter t. Such coordinates can be described with the help of the equation system

$$\left\{ \begin{aligned} F_1^\varepsilon(x_1, x_2, s, t) = x_2 - r_{i+1}(\varepsilon^{-\theta}x_1)(\tfrac{\varepsilon}{2} + s) - r_i(\varepsilon^{-\theta}x_1)(\tfrac{\varepsilon}{2} - s) = 0, \\ F_2^\varepsilon(x_1, x_2, s, t) = x_1 - \varepsilon^\theta g(\varepsilon^{-1}x_2, \varepsilon^{-\theta}t, \varepsilon) = 0, \end{aligned} \right. \tag{3.3}$$

where g being an unknown function. We assume that $\eta = \varepsilon^{-1}x_2$, $\xi = \varepsilon^{-\theta}t$.

Next we use the condition of an orthogonality of the coordinate system $\{s, t\}$, and obtain from (3.3) the differential equation for the function $g(\eta, \xi, \varepsilon)$:

$$\frac{\partial g}{\partial \eta} = \varepsilon^{2-2\theta}\left((r_i(g))'\frac{\eta - r_{i+1}(g)}{r_{i+1}(g) - r_i(g)} - (r_{i+1}(g))'\frac{\eta - r_i(g)}{r_{i+1}(g) - r_i(g)}\right), \tag{3.4}$$

and add to it the initial condition

$$g(0, \xi, \varepsilon) = \xi. \tag{3.5}$$

The problem (3.4), (3.5) is uniquely solvable for every $\eta \in \mathbb{R}_1$, since, by virtue of the function $r(x_1)$ properties (c), (d) the right part of the equation (3.4) is bounded and satisfies the Lipschitz condition uniformly on $g \in \mathbb{R}_1$ and on η from any finite interval $(-N, N)$.

Lemma 3.1. *Let N be any positive number. Then uniformly on $\eta \in (-N, N)$ and $\xi \in \mathbb{R}_1$ for the solution $g(\eta, \xi, \varepsilon)$ of the problem (3.4), (3.5) for $\varepsilon \to 0$ following estimates take place*

$$\frac{\partial^k g}{\partial \xi^k} = \xi^{1-k} + O(\varepsilon^{2-2\theta}), k = 0, 1,$$

$$\{\frac{\partial g}{\partial \eta}, \frac{\partial^2 g}{\partial \xi^2}, \frac{\partial^2 g}{\partial \eta \partial \xi}\} = O(\varepsilon^{2-2\theta}).$$

Lemma 3.2. *The system (3.3) is uniquely solvable with respect to the variables x_1 and x_2 in $G_{i\varepsilon}$, i.e., there exist the functions R_1 and R_2 such that*

$$\left\{ \begin{aligned} x_1 = R_1(s, t), \\ x_2 = R_2(s, t). \end{aligned} \right.$$

Herewith for $R_1(s, t), R_2(s, t)$ under the condition of the small ε the following estimates

$$\begin{gathered} R_{1s} = O(\varepsilon^{1-\theta}), \quad R_{2s} = q(\varepsilon^{-\theta}t) + O(\varepsilon^{2-2\theta}), \\ R_{1t} = 1 + O(\varepsilon^{2-2\theta}), \quad R_{2t} = O(\varepsilon^{1-\theta}), \\ \frac{\partial}{\partial t}|\frac{\partial R}{\partial s}| = \varepsilon^{-\theta}q'(\varepsilon^{-\theta}t) + O(\varepsilon^{2-3\theta}), \quad \frac{\partial}{\partial t}|\frac{\partial R}{\partial t}| = O(\varepsilon^{2-3\theta}) \end{gathered} \tag{3.6}$$

are correct. The system (3.3) is uniquely solvable also with respect to the variables s *and* t*, and for* $\varepsilon \to 0$ *the evaluations*

$$\frac{\partial t}{\partial x_1} = 1 + O(\varepsilon^{2-2\theta}), \frac{\partial t}{\partial x_2} = O(\varepsilon^{1-\theta}) \tag{3.7}$$

are valid.

The proofs of Lemmas 3.1, 3.2 are similar to the ones from [1].

Let us consider the function $\widetilde{v}_{i\varepsilon}(t), -\varepsilon < t < L + \varepsilon$, which is the solution of the problem

$$\begin{cases} \frac{d}{dt}(q(\varepsilon^{-\theta}t)\frac{d\widetilde{v}_{i\varepsilon}}{dt}) = 0, \\ \widetilde{v}_{i\varepsilon}(0) = 0, \\ \frac{d\widetilde{v}_{i\varepsilon}}{dt}(\varepsilon^{-\theta}L) = \frac{b_\varepsilon}{q(\varepsilon^{-\theta}L)}, \end{cases} \tag{3.8}$$

where b_ε being an unknown constant dependent on ε, which we obtain as follows. We define in the domain $G_{i\varepsilon}$ the function $\widetilde{v}_{i\varepsilon} = \widetilde{v}_{i\varepsilon}(x_1, x_2)$ and extend the solution $\widetilde{v}_{i\varepsilon}(t)$ of the problem (3.8) for s so that in variables (t, s) $\widetilde{v}_{i\varepsilon}$ does not depend on s for fixed values of t. Then we select the constant b_ε so that the condition $\int\limits_{\gamma_{iL}} \frac{\partial \widetilde{v}_{i\varepsilon}}{\partial x_1} dx_2 = 1$ is satisfied. Hence, we show that $b_\varepsilon = \frac{1}{\varepsilon} + O(\varepsilon^{2-2\theta})$. With the help of the equation $L = R_1(s,t)$ we can express points t, lying on the boundary γ_{iL}, as a function dependent on the variable s, i.e., $t = \varphi(s)$ (it can be made since $R_{1t} = 1 + O(\varepsilon^{2-2\theta}) \neq 0$). Then substituting the expression for t in the equation $x_2 = R_2(s,t)$ and taking into account that $\varphi'(s) = -\frac{R_{1s}}{R_{1t}}$, we obtain that $dx_2 = R_{2s}ds + R_{2t}\varphi'(s)ds = (R_{2s} - \frac{R_{2t}R_{1s}}{R_{1t}})ds$. From the orthogonality condition of the coordinate system $\{t, s\}$ follows, that $\frac{R_{2t}R_{2s}}{R_{1t}} = -R_{1s}$, whence $dx_2 = \frac{|R_s|^2}{R_{2s}} ds$. Using the estimates of the lemma 3.2, we write the following sequence of the equalities:

$$1 = \int\limits_{\gamma_{iL}} \frac{\partial \widetilde{v}_{i\varepsilon}}{\partial x_1} dx_2 = \int\limits_{\gamma_{iL}} \frac{\partial \widetilde{v}_{i\varepsilon}}{\partial t}\frac{\partial t}{\partial x_1} dx_2 = \int\limits_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} b_\varepsilon (1 + O(\varepsilon^{2-2\theta}))ds = b_\varepsilon(\varepsilon + O(\varepsilon^{3-2\theta})),$$

from which we obtain the unknown constant $b_\varepsilon = \frac{1}{\varepsilon} + O(\varepsilon^{2-2\theta})$. Thus, the last condition of the problem (3.8) can be written as

$$\frac{d\widetilde{v}_{i\varepsilon}}{dt}(\varepsilon^{-\theta}L) = \frac{1}{\varepsilon q(\varepsilon^{-\theta}L)} + O(\varepsilon^{2-2\theta}).$$

We represent the solution of the problem (3.2) as the following

$$v_{i\varepsilon} = \widetilde{v}_{i\varepsilon} + \delta_{i\varepsilon}, \tag{3.9}$$

here $\widetilde{v}_{i\varepsilon}$ is the solution of the problem (3.8), extended on s as described above. From (3.2) and (3.8) it follows, that $\delta_{i\varepsilon}$ is the solution of the problem

$$\begin{cases} \triangle\delta_{i\varepsilon}(x) = -\triangle\tilde{v}_{i\varepsilon}(x), x \in G_{i\varepsilon}, \\ \delta_{i\varepsilon}(0, x_2) = -\tilde{v}_{i\varepsilon}(0, x_2), \\ \delta_{i\varepsilon}(L, x_2) = M_\varepsilon - \tilde{v}_{i\varepsilon}(L, x_2), \\ \dfrac{\partial\delta_{i\varepsilon}(x)}{\partial n} = 0, x \in \Gamma_{i,\varepsilon}, \Gamma_{i+1,\varepsilon}, \\ \displaystyle\int\limits_{\gamma_{iL}} \frac{\partial\delta_{i\varepsilon}(x)}{\partial x_1} dx_2 = 0, \end{cases} \tag{3.10}$$

where M_ε being an arbitrary unknown constant value.

Hereafter we will need an estimate for the magnitude $\int\limits_{G_{i\varepsilon}} |\nabla\delta_{i\varepsilon}|^2 dx$. For this purpose we solve the following problem about minimization of the functional

$$F(v_{i\varepsilon}) = \int\limits_{G_{i\varepsilon}} |\nabla v_{i\varepsilon}|^2 dx - \frac{2}{|\gamma_{iL}|} \int\limits_{\gamma_{iL}} v_{i\varepsilon} dx_2 \to inf, \tag{3.11}$$

in the class

$$V = \{v_{i\varepsilon}(x_1, x_2) \in (W_2^1(G_{i\varepsilon})) : v_{i\varepsilon}(0, x_2) = 0, v_{i\varepsilon}(L, x_2) = const\},$$

here $|\gamma_{iL}|$ is the length of the curve γ_{iL}.

The function, on which the minimum of the functional (3.11) in the class V is reached, is the solution of the problem (3.2). Indeed, let us consider the increment $F(v_{i\varepsilon} + h\varphi_{i\varepsilon})$ of the functional F, where $\varphi_{i\varepsilon}$ is an arbitrary function from V,

$$F(v_{i\varepsilon} + h\varphi_{i\varepsilon}) = \int\limits_{G_{i\varepsilon}} |\nabla v_{i\varepsilon}|^2 dx + 2h \int\limits_{G_{i\varepsilon}} (\nabla v_{i\varepsilon}, \nabla\varphi_{i\varepsilon}) dx + h^2 \int\limits_{G_{i\varepsilon}} |\nabla\varphi_{i\varepsilon}|^2 dx$$
$$- \frac{2}{|\gamma_{iL}|} \int\limits_{\gamma_{iL}} v_{i\varepsilon} dx_2 - \frac{2h}{|\gamma_{iL}|} \int\limits_{\gamma_{iL}} \varphi_{i\varepsilon} dx_2$$

and equate to zero the linear part of $F(v_{i\varepsilon} + h\varphi_{i\varepsilon})$

$$0 = \int\limits_{G_{i\varepsilon}} (\nabla v_{i\varepsilon}, \nabla\varphi_{i\varepsilon}) dx - \frac{1}{|\gamma_{iL}|} \int\limits_{\gamma_{iL}} \varphi_{i\varepsilon} dx_2 = -\int\limits_{G_{i\varepsilon}} \triangle v_{i\varepsilon}\varphi_{i\varepsilon} dx + \int\limits_{\Gamma_{i+1,\varepsilon}} \frac{\partial v_{i\varepsilon}}{\partial n}\varphi_{i\varepsilon} d\Gamma$$
$$+ \int\limits_{\Gamma_{i,\varepsilon}} \frac{\partial v_{i\varepsilon}}{\partial n}\varphi_{i\varepsilon} d\Gamma + \int\limits_{\gamma_{iL}} (\frac{\partial v_{i\varepsilon}}{\partial x_1} - \frac{1}{|\gamma_{iL}|})\varphi_{i\varepsilon} dx_2.$$

From the arbitrariness of $\varphi_{i\varepsilon}$ we get the connection between the variational problem for the functional (3.11) and the boundary problem (3.2). With the help of the variational formulation (3.11) in the class V of the problem (3.2) we obtain the variational formulation of the problem (3.10). For this purpose we substitute in the functional (3.11) the representation (3.9) of the problem (3.2) solution

$$F(v_{i\varepsilon}) = \int\limits_{G_{i\varepsilon}} |\nabla \widetilde{v}_{i\varepsilon}|^2 dx + 2 \int\limits_{G_{i\varepsilon}} (\nabla \widetilde{v}_{i\varepsilon}, \nabla \delta_{i\varepsilon}) dx + \int\limits_{G_{i\varepsilon}} |\nabla \delta_{i\varepsilon}|^2 dx$$
$$- \frac{2}{|\gamma_{iL}|} \int\limits_{\gamma_{iL}} (\widetilde{v}_{i\varepsilon} + \delta_{i\varepsilon}) dx_2 \to inf.$$

We minimize the functional $F(v_{i\varepsilon})$ on the function $\delta_{i\varepsilon}$, as $\widetilde{v}_{i\varepsilon}$ is known. Moving away items which are not containing the function $\delta_{i\varepsilon}$, we obtain that $\delta_{i\varepsilon}$ minimizes the functional

$$\widetilde{F}(\delta_{i\varepsilon}) = 2 \int\limits_{G_{i\varepsilon}} (\nabla \widetilde{v}_{i\varepsilon}, \nabla \delta_{I\varepsilon}) dx + \int\limits_{G_{i\varepsilon}} |\nabla \delta_{i\varepsilon}|^2 dx - \frac{2}{|\gamma_{iL}|} \int\limits_{\gamma_{iL}} \delta_{i\varepsilon} dx_2 \to inf \qquad (3.12)$$

in the class of functions

$$\widetilde{V} = \{w(x_1, x_2) \in (W_2^1(G_{i\varepsilon})) : w(0, x_2) = -\widetilde{v}(0, x_2), w(L, x_2) = M_\varepsilon - \widetilde{v}(L, x_2)\},$$

where M_ε being an arbitrary constant value. Let us consider the function $\widetilde{w}_\varepsilon = w_{\varepsilon 0} + w_{\varepsilon L}$, where $w_{\varepsilon 0}(x_1, x_2) = -\widetilde{v}(x_1, x_2)\varphi_{\varepsilon 0}(x_1)$, $w_{\varepsilon L}(x_1, x_2) = (\widetilde{v}(L, x_2) - \widetilde{v}(x_1, x_2))\varphi_{\varepsilon L}(x_1)$, $\varphi_{\varepsilon 0}(x_1) = \varphi(\frac{x_1}{\varepsilon})$, $\varphi_{\varepsilon L}(x_1) = \varphi(\frac{L - x_1}{\varepsilon})$, and $\varphi(x_1)$ being "cutting function" defined by the formula

$$\varphi(x_1) = \begin{cases} 1, x_1 < 1, \\ exp(1 - \dfrac{1}{1 - (x_1 - 1)^2}), x_1 \in [1, 2], \\ 0, x_1 > 2. \end{cases}$$

It is easy to prove, that $\widetilde{w}_\varepsilon \in \widetilde{V}$. We evaluate $\widetilde{F}(\widetilde{w}_\varepsilon)$ and show, that $\widetilde{F}(\widetilde{w}_\varepsilon) = O(1)$.

Using the estimates of the Lemma 3.2 and the fact, that $\varphi'_{\varepsilon 0} = O(\frac{1}{\varepsilon})$, $\varphi'_{\varepsilon L} = O(\frac{1}{\varepsilon})$, we can write the following estimates

$$\frac{\partial \widetilde{w}_\varepsilon}{\partial x_1} = O(\frac{1}{\varepsilon}), \quad \frac{\partial \widetilde{w}_\varepsilon}{\partial x_2} = O(\frac{1}{\varepsilon^\theta}),$$
$$\frac{\partial \widetilde{v}_{i\varepsilon}}{\partial x_1} = O(\frac{1}{\varepsilon}), \quad \frac{\partial \widetilde{v}_{i\varepsilon}}{\partial x_2} = O(\frac{1}{\varepsilon^\theta}),$$

$$\widetilde{w}_\varepsilon(L,x_2)=O(\varepsilon^{3-2\theta}).$$

Whence it follows, that $\widetilde{F}(\widetilde{w}_\varepsilon)=O(1)$.

As the function $\delta_{i\varepsilon}$ minimizes the functional $\widetilde{F}$ in the class $\widetilde{V}$, and $\widetilde{w}_\varepsilon$ belongs to this class, then

$$\widetilde{F}(\delta_{i\varepsilon})\le\widetilde{F}(\widetilde{w}_\varepsilon). \tag{3.13}$$

Using (3.13) and (3.12), we estimate $\int\limits_{G_{i\varepsilon}}|\nabla\delta_{i\varepsilon}|^2dx$. Making an integration by parts in the first item of (3.12), we obtain the following expression with the help of the Cauchy–Schwarz–Bunyakovskii inequality

$$\begin{aligned}\int\limits_{G_{i\varepsilon}}|\nabla\delta_{i\varepsilon}|^2dx \;\le\; & C\Big(\underbrace{\sqrt{\int\limits_{G_{i\varepsilon}}(\triangle\widetilde{v}_{i\varepsilon})^2dx}\sqrt{\int\limits_{G_{i\varepsilon}}\delta^2_{i\varepsilon}dx}}_{I_1}+\underbrace{\sqrt{\int\limits_{\gamma_{i0}}(\frac{\partial\widetilde{v}_{i\varepsilon}}{\partial x_1})^2dx_2}\sqrt{\int\limits_{\gamma_{i0}}\delta^2_{i\varepsilon}dx_2}}_{I_2}\\ &+\underbrace{\Big|\int\limits_{\gamma_{iL}}(\frac{\partial\widetilde{v}_{i\varepsilon}}{\partial x_1}-\frac{1}{|\gamma_{iL}|})\delta_{i\varepsilon}dx_2\Big|}_{I_3}+1\Big)=C(I_1+I_2+I_3+1).\end{aligned} \tag{3.14}$$

Let us evaluate the values I_1, I_2 and I_3 in the last inequality. It is easy to obtain the following estimates for the expression the expression $\int\limits_{G_{i\varepsilon}}(\triangle\widetilde{v}_{i\varepsilon})^2dx$ taking into account the problem (3.8) conditions:

$$\begin{aligned}\frac{d\widetilde{v}_{i\varepsilon}}{dt}&=\frac{1}{\varepsilon q(\varepsilon^{-\theta}t)}+O(\varepsilon^{2-2\theta}),\\ \frac{d^2\widetilde{v}_{i\varepsilon}}{dt^2}&=-\frac{q'(\varepsilon^{-\theta}t)}{q^2(\varepsilon^{-\theta}t)\varepsilon^{1+\theta}}+O(\varepsilon^{2-3\theta}).\end{aligned} \tag{3.15}$$

With the help of the value $\triangle\widetilde{v}_{i\varepsilon}$ representation in the coordinate system $\{s,t\}$ we write the sequence of the equalities

$$\begin{aligned}\triangle\widetilde{v}_{i\varepsilon}&=\frac{1}{|R_t||R_s|}\frac{d}{dt}\Big(\frac{|R_s|}{|R_t|}\frac{d\widetilde{v}_{i\varepsilon}}{dt}\Big)=\frac{1}{q(\varepsilon^{-\theta}t)}\frac{d}{dt}\Big(q(\varepsilon^{-\theta}t)\frac{d\widetilde{v}_{i\varepsilon}}{dt}\Big)\\ &+\frac{d\widetilde{v}_{i\varepsilon}}{dt}\Big(\frac{|R_s|_t|R_t|-|R_t|_t|R_s|}{|R_t|^3|R_s|}-\frac{\varepsilon^{-\theta}q'(\varepsilon^{-\theta}t)}{q(\varepsilon^{-\theta}t)}\Big)+\frac{d^2\widetilde{v}_{i\varepsilon}}{dt^2}\Big(\frac{1}{|R_t|^2}-1\Big).\end{aligned}$$

Hence, using the estimates of the Lemma 3.2 and taking into account the formulae (3.15), we get

$$\triangle\widetilde{v}_{i\varepsilon}=O(\varepsilon^{1-3\theta}),$$

and, therefore,

$$\int\limits_{G_{i\varepsilon}} (\triangle\widetilde{v}_{i\varepsilon})^2 dx = O(\varepsilon^{3-6\theta}). \tag{3.16}$$

We represent the function $\delta_{i\varepsilon}(t,s)$ in the form

$$\delta_{i\varepsilon}(t,s) = \int\limits_{\varphi(s)}^{t} \frac{\partial\delta_{i\varepsilon}(\tau,s)}{\partial\tau} d\tau - \widetilde{v}_{i\varepsilon}(0,x_2), \tag{3.17}$$

where $\varphi(s) = \tau$ being an arbitrary point on the boundary γ_{i0}.

It is easy to confirm that

$$\widetilde{v}_{i\varepsilon}(0,x_2) = O(\varepsilon^{3-2\theta}). \tag{3.18}$$

Hence, taking into account the Lemma 3.2, the expression (3.17), the estimate (3.18) and the Schwarz inequality, we obtain

$$\delta^2_{i\varepsilon}(t,s) \leq C \int_0^L |\nabla\delta_{i\varepsilon}|^2 dx.$$

Using the inequality $2I_1 \leq 1 + I_1^2$, we get the estimate

$$I_1 = O(1) + O(\varepsilon^{3-6\theta}) \int_{G_{i\varepsilon}} |\nabla\delta_{i\varepsilon}|^2 dx. \tag{3.19}$$

It is easy to prove that

$$I_2 = O(\varepsilon^{3-2\theta}) \tag{3.20}$$

with the help of the Lemma 3.2, the problem (3.8) conditions and (3.18).

In order to estimate the item I_3 we represent $\delta_{i\varepsilon}$ on the γ_{iL} as follows

$$\delta_{i\varepsilon}(L,x_2) = M_\varepsilon - \widetilde{v}_{i\varepsilon}(L) + \widetilde{v}_{i\varepsilon}(L) - \widetilde{v}_{i\varepsilon}(L,x_2) = O(\varepsilon^{3-2\theta}).$$

As $\dfrac{\partial\widetilde{v}_{i\varepsilon}}{\partial x_1} - \dfrac{1}{|\gamma_{iL}|} = 0$, then

$$I_3 = 0. \tag{3.21}$$

Thus, from (3.14), (3.19)–(3.21) we have obtained the estimate for $\varepsilon \to 0$

$$\int\limits_{G_{i\varepsilon}} |\nabla\delta_{i\varepsilon}|^2 dx = O(1). \tag{3.22}$$

Evaluating the values $\int\limits_{G_{i\varepsilon}} (\nabla\delta_{i\varepsilon}, \nabla\widetilde{v}_{i\varepsilon})dx$, $\int\limits_{G_{i\varepsilon}} |\nabla\widetilde{v}_{i\varepsilon}|^2 dx$ with the help of (3.12), (3.13), (3.22) and the Lemma 3.2, we obtain for $\varepsilon \to 0$

$$\int\limits_{G_{i\varepsilon}} (\nabla\delta_{i\varepsilon}, \nabla\widetilde{v}_{i\varepsilon})dx = O(1), \tag{3.23}$$

$$\int\limits_{G_{i\varepsilon}} |\nabla\widetilde{v}_{i\varepsilon}|^2 dx = \frac{1}{\varepsilon}\int\limits_0^L \frac{dt}{q(\varepsilon^{-\theta}t)} + O(\varepsilon^{1-2\theta}). \tag{3.24}$$

We introduce the notation $T_\varepsilon = \varepsilon^{-\theta}L$. After the substitution of the variable $\tau = \varepsilon^{-\theta}t$ the estimate (3.24) takes the form

$$\int\limits_{G_{i\varepsilon}} |\nabla\widetilde{v}_{i\varepsilon}|^2 dx = \frac{L\sigma}{\varepsilon T_\varepsilon}\int\limits_0^{T_\varepsilon} \frac{d\tau}{q(\tau)} + O(\varepsilon^{1-2\theta}). \tag{3.25}$$

Because of the Cauchy–Riemann conditions for the solutions $u_{i\varepsilon}$ and $v_{i\varepsilon}$ of the problems (3.1), (3.2), respectively, the equality $|\nabla u_{i\varepsilon}|^2 = |\nabla v_{i\varepsilon}|^2$ is valid. We calculate the electrical conductivity of the layer $G_{i\varepsilon}$ (or that the same the current I_ε). For this purpose we use Green's formula, conditions of the problem (3.1) and the Cauchy–Riemann conditions for the functions $u_{i\varepsilon}$ and $v_{i\varepsilon}$. Then we obtain

$$C_{i\varepsilon} = I_{i\varepsilon} = \sigma\int\limits_{\Gamma_{i+1,\varepsilon}} \frac{\partial u_{i\varepsilon}}{\partial n} = \sigma\int\limits_{G_{i\varepsilon}} |\nabla u_{i\varepsilon}|^2 dx = \sigma\int\limits_{G_{i\varepsilon}} |\nabla v_{i\varepsilon}|^2 dx. \tag{3.26}$$

On the basis of (3.9), (3.22),(3.23),(3.25) and (3.26) we can formulate the following

Theorem 3.1. *For the conductivity $C_{i\varepsilon}$ of the thin layer $G_{i\varepsilon}$ for $\varepsilon \to 0$ the asymptotic formula*

$$C_{i\varepsilon} = \sigma\int\limits_{G_{i\varepsilon}} |\nabla v_{i\varepsilon}|^2 dx = \frac{L\sigma}{\varepsilon T_\varepsilon}\int\limits_0^{T_\varepsilon} \frac{dt}{q(t)} + O(1) \tag{3.27}$$

is valid. Here $T_\varepsilon = \varepsilon^{-\theta}L$, $0 < \theta < \dfrac{1}{2}$.

4. Conductivity of the layered structure

In the Section 2 it was shown, that the electrical conductivity C_ε of the sample, consisting of alternating layers with finite and infinite conductivities, coincides with the minimum of the functional (2.8) in the class of functions H_ε. We obtain the formula for the electrical conductivity C_ε, using the functional (2.8).

The solution of the problem (2.7) $\widetilde{u}_{i\varepsilon}$ can be represented as

$$\widetilde{u}_{i\varepsilon} = (c_i - c_{i-1})u_{i\varepsilon} + c_{i-1}, i = 1..., N_\varepsilon,$$

where $u_{i\varepsilon}$ is the solution of the problem (3.1).

The formula (2.8) can be transformed into the following problem of the minimum of the function with N_ε variables $\triangle_i$:

$$C_\varepsilon = \sum_{i=1}^{N_\varepsilon} \triangle_i^2 C_{i\varepsilon}, \tag{4.1}$$

with the restriction

$$\sum_{i=1}^{N_\varepsilon} \triangle_i = 1, \tag{4.2}$$

here $\triangle_i := c_i - c_{i-1}$, $i = 1, \ldots, N_\varepsilon$. The values $C_{i\varepsilon}$ are known from the Theorem 3.1. Next we find the minimum of the function (4.1) with the auxiliary condition (4.2) and using the method of factors of the Lagrange factors. We write the Lagrange function for the problem (4.1), (4.2)

$$L(\triangle_i, \lambda) = \sigma \sum_{i=1}^{N_\varepsilon} \triangle_i^2 I_{i\varepsilon} - \lambda(\sum_{i=1}^{N_\varepsilon} \triangle_i - 1).$$

Equating zero derivatives with respect to variables $\triangle_i$ of the Lagrange function and using the auxiliary condition of the problem (4.1), we obtain

$$\triangle_i = \frac{1}{I_{i\varepsilon} \sum_{j=1}^{N_\varepsilon} \frac{1}{I_{j\varepsilon}}}. \tag{4.3}$$

Substituting the coefficients (4.3) in (4.1), we obtain

$$C_\varepsilon = \frac{1}{\sum_{j=1}^{N_\varepsilon} \frac{1}{C_{j\varepsilon}}}. \tag{4.4}$$

By using the Theorem 3.1, we pass to the limit under the condition of $\varepsilon \to 0$ in the expression (4.4). Hence with the probability 1 in the probability space $(\Omega^\infty, \mathcal{F}^\infty, P^\infty)$

$$\lim_{\varepsilon\to 0} C_\varepsilon(\omega^\infty) = \lim_{\varepsilon\to 0}\left(\frac{1}{\sigma L}\sum_{i=1}^{N_\varepsilon}\frac{\varepsilon}{\frac{1}{T_\varepsilon}\int\limits_0^{T_\varepsilon}\frac{d\tau}{q(\tau,\omega_i^2)}+O(\varepsilon)}\right)^{-1}$$

$$= \left(\frac{H}{2\sigma L}\frac{1}{\langle q^{-1}(0)\rangle}\right)^{-1} = \frac{2\sigma L}{H}\langle q^{-1}(0)\rangle, \tag{4.5}$$

that proves the Theorem 2.1.

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References

[1] *I.E. Egorova and E.Ya. Khruslov*, An asymptotical behavior of solutions of the second boundary problem in areas with random thin gaps. — *Teor. funcktsii, funktsion. anal. i ikh prilozhen.* (1989), v. 52, p. 91–103. (Russian)

[2] *G. Kramer and M. Lidbetter*, Stationary stochastic processes. Mir, Moscow (1969). (Russian)

[3] *B.V. Gnedenko and A.N. Kolmogorov*, Limit distributions for the sums of independent random variables. Gostehizdat, Moscow, Leningrad (1948). (Russian)