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# Characterization of condensations of $\sigma$-compact locally compact groups and applications to invariant measures

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We obtain necessary and sufficient conditions for existence on a topological group of a stronger $\sigma$-compact locally compact topology. An application of the obtained result for measurable topological group with an invariant measure are given.

The concept of the condensation or the $i$-isomorphism is a very important one in the category of topological groups (see [6]). A mapping $p : X \to G$ is said to be a condensation ($i$-isomorphism) if $p$ is algebraically an isomorphism and $p$ is continuous. A group $G$ is called a condensation of group $X$ (or we said $X$ condensates onto $G$) if there exists an $i$-isomorphism $p$ from $X$ onto $G$.

The main goal of this paper is to find conditions for a topological group $G$ which guarantees existence of a stronger $\sigma$-compact locally compact topology on $G$. In fact, we give an answer to the following question: When a topological group $G$ is a condensation of a $\sigma$-compact locally compact group $X$?

Evidently such group $G$ is $\sigma$-compact. Thus our problem is just to characterize the smallest class $\mathcal{A}$ of topological group such that $\mathcal{A}$ is closed with respect to condensation and $\mathcal{A} \supset A$, where $A$ is the class of $\sigma$-compact locally compact groups.

As a conclusion of the paper we consider an application of the main result to the theory of invariant measures on groups.

Let $F$ be a subset of $G$. Define $F^{(n)}$ by induction: $F^{(1)} = F, F^{(n+1)} = F \cdot F^{(n)}$, $\forall n \in \mathbb{N}$. Denote by $\omega(X)$ the weight of a space $X$. Denote also by $\mathcal{B}_X$ the Borel $\sigma$-algebra of $X$. In this paper all groups are $T_0$-group.

The following theorem is the main result of this paper.

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**Theorem 1.** *A group $G$ is a condensation of a $\sigma$-compact locally compact group $X$ iff $G$ is a $\sigma$-compact and there exists a compact set $F \subset G$ such that:*

*(i) there exist $y_1, y_2, \dots \in G$ such that $G = \cup_{k=1}^{\infty} y_k \cdot F$ ;*

*(ii) $F$ is symmetric;*

*(iii) there exist $x_1, \dots, x_n$ such that $F \cdot F \subset \cup_{i=1}^{n} x_i \cdot F$.*

*Assume, in addition, that $p$ is a condensation of $X$ onto $G$. Then $p$ is a Borel isomorphism, $\omega(X) = \omega(G)$, and if a group $Y \in A$ condensates onto $G$, then $X$ and $Y$ are topologically isomorphic.*

P r o o f. *Necessity.* Let $U$ be a symmetric neighbourhood of $e$ in $X$ with the compact closure such that $U$ generates an open subgroup $X$. It is obvious that $F = p(\bar{U})$ satisfies conditions (i)–(iii).

*Sufficiency.* Let $[F]$ be the subgroup generated by $F$. By condition (i) the collection of left cosets is at most countable. Therefore it is sufficient to prove the following theorem.

**Theorem 2.** *Let $G$ be a topological group. The following conditions are equivalent:*

1. *There exists an i-isomorphism $p$ of a $\sigma$-compact locally compact compactly generated group $X$ onto $G$.*
2. *There exists a symmetric compact set $F \subset G$ such that $F$ generates $G$ and there exist $x_1, \dots, x_n$ such that $F \cdot F \subset \cup_{i=1}^{n} x_i \cdot F$.*
3. *There exists a symmetric compact set $F \subset G$ such that $F$ generates $G$ and the collection of sets $V_F = \{V :$ there exists $n_0$ such that $\forall n \geq n_0$ there is a neighbourhood $U_n$ of the identity $e$ in $G$ with $V = U_n \cap F^{(n)}\}$ is not empty.*

*Moreover, $p$ is a Borel isomorphism, $\omega(X) = \omega(G)$ and if a group $Y \in A$ condensates onto $G$, then $X$ and $Y$ are topologically isomorphic.*

Condition 2 connects topological properties of $G$ (i.e., Condition 1) with algebraic properties of $F$ (i.e., a compact subset which generates $G$). Hence this result we can consider as an answer (in our case) on question III [6].

P r o o f. 1. $\Rightarrow$ 2. Let $U$ be a symmetric neighbourhood of $e$ in $X$ with the compact closure such that $U$ generates $X$. It is obvious that $F = p(\bar{U})$ satisfies Condition 2.

2. $\Rightarrow$ 3. Without loss of generality we can assume that $e \in F$. (Let $F_1 = F \cup \{e\}$. Then we have $F_1 \cdot F_1 = F \cdot F \cup F \cup \{e\} \subset \cup_{i=1}^{n} x_i \cdot F_1 \cup F_1$). Therefore $F^{(l)} \subset F^{(l+1)}$. Let us show that $V_F$ is not empty.

First, we prove that for all natural $l \geq 2$ there exists a neighbourhood $U_l$ of $e$ such that
$$F^{(2)} \cap U_l = F^{(l)} \cap U_l, \quad l \geq 2. \tag{1}$$
In fact, an induction argument based on Condition 2 shows that $F^{(l)} \subset \cup_{i=1}^{m_l} z_{li} \cdot F$. If $z_{li} \notin F$, then $e \notin z_{li} \cdot F$. Let $W$ be a neighbourhood of $e$ such that $W \cap z_{li} \cdot F = \emptyset$ for all indices $i$ such that $z_{li} \notin F$. Let $J = \{i : \ z_{li} \in F\}$. Then
$$F^{(l)} \cap W \subset (\cup_{i \in J} z_{li} \cdot F) \cap W \subset (F \cdot F) \cap W = F^{(2)} \cap W \subset F^{(l)} \cap W.$$
Hence, $F^{(2)} \cap W = F^{(l)} \cap W$. Set up $U_l = W$ to get (1).

Let $\{U_k^\alpha\}_{\alpha \in J_k}$ be a set of all neighbourhoods of $e$ such that $U_k^\alpha$ satisfies equation (1) for $l = k$. Set up $W_k = \cup_{\alpha \in J_k} U_k^\alpha$ a symmetric neighbourhood of $e$ satisfying equation (1) for $l = k$. Set up $V_k = F^{(k)} \cap W_k$. To prove Condition 3 (i.e., $V_F \neq \emptyset$), it suffices to show that $V_k = V_{k+1}$ for all $k \geq 4$. Since
$$F^{(2)} \cap W_{k+1} \subset F^{(k)} \cap W_{k+1} \subset F^{(k+1)} \cap W_{k+1} = F^{(2)} \cap W_{k+1}\,,$$
then $W_{k+1} \subset W_k$ and $V_{k+1} \subset V_k$, for all $k \geq 2$.

Prove the inverse inclusion. Let $x \in V_k = F^{(k)} \cap W_k = F^{(2)} \cap W_k$. If $k \geq 4$ we choose a symmetric neighbourhood $W_x$ of $e$ such that $F^{(k+3)} \cap W_x = F^{(2)} \cap W_x$ and $x \cdot W_x \subset W_k$. Then
$$(x \cdot W_x) \cap F^{(k)} = (x \cdot W_x) \cap F^{(k+1)}. \tag{2}$$
In fact, an element $z \in (x \cdot W_x) \cap F^{(k+1)}$ may be represented in the form $z = x \cdot y \in F^{(k+1)}$, where
$$y = x^{-1} \cdot z \in W_x \cap (F^{(2)} \cdot F^{(k+1)}) = W_x \cap F^{(k+3)} = W_x \cap F^{(2)} \subset F^{(2)}.$$
Therefore $z = x \cdot y \in (x \cdot W_x) \cap (F^{(2)} \cdot F^{(2)}) = (x \cdot W_x) \cap F^{(4)} \subset (x \cdot W_x) \cap F^{(k)}$.

Put $W_k^* = \cup_{x \in V_k} (x \cdot W_x) \subset W_k$. Then $W_k^* \cap F^{(k)} = V_k = W_k^* \cap F^{(2)}$, and we have
$$V_k = W_k^* \cap F^{(k)} = (\cup_{x \in V_k} (x \cdot W_x)) \cap F^{(k)} = \cup_{x \in V_k} (x \cdot W_x \cap F^{(k)})$$
$$= \cup_{x \in V_k} (x \cdot W_x \cap F^{(k+1)}) = (\cup_{x \in V_k} (x \cdot W_x)) \cap F^{(k+1)} = W_k^* \cap F^{(k+1)} = W_k^* \cap F^{(2)}.$$
Hence, $W_k^* \subset W_{k+1}$ and $V_k = W_k^* \cap F^{(k)} \subset W_{k+1} \cap F^{(k+1)} = V_{k+1}$.

3. $\Rightarrow$ 1. It is easy to prove that the system of sets $\{x \cdot V\}$ and $\{V \cdot x\}$, where $x \in G, V \in V_F$, is an open base of a topology $\tau$ on $G$.

For this we verify five conditions (i)–(v) of Theorem 4.5 [1] . Note that if $V \in V_F$ and $U$ is a neighbourhood of $e$ then $V \cap U = (U_n \cap U) \cap F^{(n)} \in V_F$.

(i) Let $U$ be a neighbourhood of $e$ in $G$ such that $U^{(2)} \subset U_{2n_0}$. Let $W = V \cap U \in V_F$. Then

$$W^{(2)} = (V \cap U) \cdot (V \cap U) = [(U_{n_0} \cap U) \cap F^{(n_0)}] \cdot [(U_{n_0} \cap U) \cap F^{(n_0)}]$$

$$\subset (U_{n_0} \cap U)^{(2)} \cap [F^{(n_0)} \cdot F^{(n_0)}] \subset U^{(2)} \cap F^{(2n_0)} \subset U_{2n_0} \cap F^{(2n_0)} = V.$$

(ii) Let $P = P^{-1}$ be a neighbourhood of $e$ in $G$ such that $P \subset U_{n_0}$ ([1, Theorem 4.6]). Then $W = P \cap V = P^{-1} \cap V \in V_F$ and

$$W^{-1} = P^{-1} \cap V^{-1} = P^{-1} \cap U_{n_0}^{-1} \cap F^{(n_0)} \subset P^{-1} \cap F^{(n_0)} \subset U_{n_0} \cap F^{(n_0)} = V.$$

(iii) Let $V = U_n \cap F^{(n)}$, $n \geq n_0$, $x \in V$ and $U$ be a neighbourhood of $e$ in $G$ such that $x \cdot U \subset U_{n_0} \cap U_{2n_0}$. Set up $W = V \cap U \in V_F$. Since $x \in V$, then $x \in F^{(n_0)}$, thus $x \cdot F^{(n_0)} \subset F^{(2n_0)}$. Then

$$x \cdot W = x \cdot V \cap x \cdot U = x \cdot U_{n_0} \cap x \cdot U \cap x \cdot F^{(n_0)} \subset U_{2n_0} \cap F^{(2n_0)} = V.$$

(iv) Let $V = U_n \cap F^{(n)}, n \geq n_0$ and $x \in G$ , then $x \in F^{(\bar{n})}$ for some $\bar{n}$. Let $U$ be a neighbourhood of $e$ in $G$ such that $x \cdot U \cdot x^{-1} \subset U_{n_0+2\bar{n}}$. Set up $W = U \cap V$, then

$$x \cdot W \cdot x^{-1} = (x \cdot U \cdot x^{-1}) \cap (x \cdot V \cdot x^{-1})$$

$$= (x \cdot U \cdot x^{-1}) \cap (x \cdot U_{n_0} \cdot x^{-1}) \cap (x \cdot F^{(n_0)} \cdot x^{-1}) \subset U_{n_0+2\bar{n}} \cap F^{(n_0+2\bar{n})} = V.$$

(v) Let $V_1, V_2 \in V_F, V_i = U_n^i \cap F^{(n)}, n \geq n_0^i, i = 1, 2$. Set up $n_0 = \max(n_0^1, n_0^2)$. Then for all $n \geq n_0$ we get

$$V_1 \cap V_2 = (U_n^1 \cap U_n^2) \cap F^{(n)} \in V_F.$$

Let us show that $\tau$ is locally compact. It will be proved if we show that $F$ is compact in $\tau$. Let $\{A_\alpha\}_{\alpha \in I}$ be an open covering of $F$. We can assume without loss of generality that $A_\alpha = x \cdot V_x$, where $V_x \in V_F, x \in F$. Then $F \subset \cup_{x \in F}(x \cdot U_{n,x} \cap x \cdot F^{(n_x)}), n_x \geq 2$. In particular, $F \subset \cup_{x \in F} x \cdot U_{n,x}$. Thus there exist $x_1, \ldots, x_m$ such that $F \subset \cup_{i=1}^m x_i \cdot U_{n,x_i}$. Put $V_{x_i} = U_{n,x_i} \cap F^{(n_{x_i})}$. Since $F \subset x_i \cdot F^{(2)} \subset x_i \cdot F^{(n_{x_i})}$, then

$$\begin{aligned} \cup_{i=1}^m x_i \cdot V_{n,x_i} &= \cup_{i=1}^m [(x_i \cdot U_{n,x_i}) \cap (x_i \cdot F^{(n_{x_i})})] \supset \cup_{i=1}^m [(x_i \cdot U_{n,x_i}) \cap F] \\ &= F \cap \cup_{i=1}^m (x_i \cdot U_{n,x_i}) = F. \end{aligned}$$

If $x_1, x_2 \in G$ then there exists a neighbourhood $U$ of $e$ such that $x_2 \notin x_1 \cdot U$. Let $V \in V_F$. Then $V \cap U \in V_F \subset \tau$ and $x_2 \notin x_1 \cdot (U \cap V)$. Therefore the topology $\tau$ has the property $T_0$ and hense is Hausdorff ([2, 4.8]).

Let $X$ be $G$ as an abstract, equipped with the topology $\tau$. Then $X$ is a compactly generated locally compact group ($F$ generates $G$). Let $p$ be the identity map from $X$ onto $G$. Then $p$ is continuous. In fact, let $U$ be a neighbourhood of $e$ in $G$ and $V \in V_F$. Then $V \cap U \in V_F$ is a neighbourhood of $e$ in $X$ and $p(V \cap U) = V \cap U \subset U$. Condition 1 is proved.

Let us prove the rest of the statements of the theorem. First, we prove that $p(\mathcal{B}_X) = \mathcal{B}_G$. It is sufficient to prove that $p(\mathcal{B}_X) \subset \mathcal{B}_G$ because $p$ is continuous. Since $p$ is bijection, $X$ is generated compactly and $\mathcal{B}_X$ is the smallest $\sigma$-algebra containing all open sets, it is sufficient to prove that $B_k = A \cap F^{(k)} \in \mathcal{B}_G$, where $A \in \tau, k \geq n_0$. Moreover, we prove that

$$B_k = A \cap F^{(k)} = C_k \cap F^{(k)}, \tag{3}$$

where $C_k$ is open in the original topology.

For all $x \in B_k$ there exists $V_x \in V_F$ such that $x \cdot V_x \subset A$ and $V_x = U_{n,x} \cap F^{(n_x)}, n_x \geq 2k$. Since $x \in F^{(k)}$ and $n_x \geq 2k$ then $F^{(k)} \subset x \cdot F^{(n_x)}$. Hence,

$$B_k = F^{(k)} \cap [\cup_{x \in B_k} x \cdot (U_{n,x} \cap F^{(n_x)})] = F^{(k)} \cap [\cup_{x \in B_k} (x \cdot U_{n,x} \cap x \cdot F^{(n_x)})]$$

$$= \cup_{x \in B_k} (x \cdot U_{n,x} \cap x \cdot F^{(n_x)} \cap F^{(k)}) = \cup_{x \in B_k} (x \cdot U_{n,x} \cap F^{(k)}) = F^{(k)} \cap (\cup_{x \in B_k} (x \cdot U_{n,x})).$$

Denote by $C_k$ the open set $\cup_{x \in B_k} (x \cdot U_{n,x})$. Thus equality (3) is proved.

Let us show that $\omega(X) = \omega(G)$. Let $\{U_\alpha\}_{\alpha \in I}$ be a base of the topology in $G$ and let $x \cdot V$ be an element of the base of the topology on $X, x \in F^{(k)}, V \subset F^{(n)}$. Then $F^{(n+k)}$ contains the open set $x \cdot V$. According to (3), there exists an open set $U$ in $G$ such that $x \cdot V = F^{(n+k)} \cap U$. Let $U = \cup_{\beta \in J} U_\beta$. Then $x \cdot V = \cup_{\beta \in J} [F^{(n+k)} \cap U_\beta]$. Moreover, $F^{(n+k)} \cap U_\beta = (F^{(n+k)} \cap U) \cap U_\beta = U_\beta \cap x \cdot V$ is open in $X$. Thus some subcollection $M$ of the set $E = \{U_\alpha \cap F^{(l)}, \alpha \in I, l = 1, 2, \dots\}$ forms the topology in $X$. But $Card(M) \leq Card(E) \leq Card(I \times \mathbb{N}) = Card(I)$. Hence, $\omega(X) \leq \omega(G)$. The inverse inequality follows from Theorem 3.1.22 [1].

Let $G = t(Y)$, where $t$ is a condensation of a $\sigma$-compact locally compact group $Y$ onto $G$. Let $\tau'$ be the topology on $G$ induced by the topology on $Y$ ($U \in \tau'$ iff $t^{-1}(U)$ is open in $Y$). Since the original topology is weaker then $\tau$ and $\tau'$ therefore $\tau = \tau'$ ([2, 6.19]). Hence $X$ and $Y$ are topologically isomorphic. The theorem is proved. ■

R e m a r k. It may happen that conditions (ii) and (iii) of Theorem 1 are redundant.

Now we give an application of Theorem 1 to A. Weil's Theorem [3].

**Definition.** *A group $G$ is called a measurable topological group with a non-trivial $\sigma$-finite (left) invariant measure $\mu$ if:*

*1) $G$ is a topological group;*

*2) $\mu$ is regular on $\mathcal{B}_G$;*

3) *there exists a $\sigma$-algebra $\mathcal{B} \subset \mathcal{B}_G$ such that $(G, \mathcal{B})$ is a measurable group (i.e., conditions (M) and (M') are hold, (see [3, Appendix 1])).*

The definitions of a locally limited topology and a limited neighbourhood are represented in [3].

**Theorem 3.** *Assume that $G$ is a measurable topological group with a nontrivial $\sigma$-finite (left) invariant measure $\mu$. Assume also that there exists a compact subset $F \subset G$ such that $\mu(F) > 0$. Then $G = p(X)$, where $p$ is a condensation of a $\sigma$-compact locally compact group $X$ and $\mu = p(m_X)$, where $m_X$ is the Haar measure on $X$.*

P r o o f. A. Weil has shown [3, Appendix 1] that on $G$ we can impose a locally limited topology $\tau^*$. Let F be a compact subset such that $\mu_F > 0$. Then there exist symmetric limited neighbourhoods $U$ and $V$ of $e$ in $\tau^*$ such that $V \cdot V \subset U \subset F \cdot F^{-1}$ [3]. Hence, there exist $x_1, \dots, x_n$ such that $V \cdot V \subset \cup_{i=1}^n x_i \cdot V$. Let $\bar{V}$ be a closure of $V$ in $\tau$. Then $\bar{V}$ is a symmetric compact and

$$\bar{V} \cdot \bar{V} \subset \overline{V \cdot V} \subset \overline{\cup_{i=1}^n x_i \cdot V} = \cup_{i=1}^n x_i \cdot \bar{V}.$$

Let $[\bar{V}]$ be the subgroup generated by $\bar{V}$. Since $\mu$ is $\sigma$-finite, the collection of (left) cosets is at most countable. Thus $\bar{V}$ satisfies the conditions of Theorem 1 on $F$. It follows from uniqueness of the Haar measure that the last assertion is valid. ∎

We remark that this result is well known for a standard group $G$ (see [4]). In a more general situation, an analogue of Theorem 3 (with a quasi-invariant measure) is known for groups of the second category only [5].

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