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On the union of sets of semisimplicity

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We introduce the notion of a set of semisimplicity, or S_3-set, as a set Λ such that if T is a representation of a LCA group G with $Sp(T) \subset \Lambda$, then T generates a semisimple Banach algebra. We prove that the union of S_3-sets is a S_3-set, provided their intersection is countable. In particular, the union of a countable set and a Helson S-set is a S_3-set.

1. Introduction

In this paper, we introduce the notion of sets of semisimplicity, or S_3-sets, and investigate their properties. Let G be a locally compact abelian group, $\Gamma := \widehat{G}$ the dual group; a closed subset Λ of Γ is called S_3-set, if for every representation T of G by bounded linear operators on a Banach space such that $Sp(T) \subset \Lambda$, the Banach algebra $\mathcal{A}(T)$, generated by "functions" of T, is semisimple, i.e., the radical $\mathcal{R}(\mathcal{A}(\mathcal{T})) = \{0\}$. Following an argument in [F, S], it is not difficult to see that any S_3-set is a set of spectral synthesis (or S-set), and that any Helson set of spectral synthesis is a S_3-set. The results of [F, S, M-V] imply that any scattered set is a S_3-set. Moreover, every S_3-set is a set of spectral resolution in the sense of Malliavin (see [B_1, p. 174]) and, therefore, not every S-set is a S_3-set. We introduce the notion of archipelago of closed sets, and show that any archipelago of S_3-sets is a S_3-set; in particular, this implies that the union of a S_3-set and a scattered set is a S_3-set. Moreover, we prove that the union of two S_3-sets is a S_3-set provided that their intersection is scattered (answering a question of G.M. Feldman).

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2. S_3-sets

Let G be a Hausdorff locally compact abelian group, written additively, with Haar measure m and dual group Γ. By $L^1(G)$ we denote the usual group algebra, and by $A(\Gamma)$ the corresponding algebra of Fourier transforms of elements of $L^1(G)$.

Let T be a bounded strongly continuous representation of G by bounded linear operators on a Banach space X ($X \neq \{0\}$), i.e., $\{T(t) : t \in G\}$ is a family of bounded linear operators on X satisfying the following conditions:

(i) $T(e) = I$, where e is the unit in G;
(ii) $T(t_1 + t_2) = T(t_1)T(t_2)$ for all t_1, t_2 in G;
(iii) the mapping $t \mapsto T(t)x$ is continuous for every $x \in X$;
(iv) $sup_{t\in G}\|T(t)\| < \infty$.

By introducing an equivalent norm on X

$$|||x||| := \sup_{t\in G} \|T(t)x\|, \ \forall x \in X,$$

one can assume that T is an isometric representation. For each function $f \in L^1(G)$, let

$$\hat{f}(\chi) = \int_G f(t)\chi(t)dt,$$

and

$$\hat{f}(T) = \int_G f(t)T(t)dt.$$

The spectrum of the representation T is defined by

$$Sp(T) = \{\chi \in \Gamma : \hat{f}(\chi) = 0 \text{ whenever } \hat{f}(T) = 0\}.$$

Let $\mathcal{A}(\mathcal{T})$ be the Banach algebra generated by $\hat{f}(T)$, $f \in L^1(G)$. The spectrum of T, $Sp(T)$, can be identified with the Gelfand space of $\mathcal{A}(\mathcal{T})$ via the formula

$$\phi_\chi(\hat{f}(T)) = \hat{f}(\chi) \text{ (see [A, L-M-F, B-V]).}$$

Definition 1. *A closed subset $\Lambda \subset \Gamma$ is called a* **set of semisimplicity** *or S_3-set, if for every isometric representation $T : G \to L(X)$ such that $Sp(T) \subset \Lambda$, the algebra $\mathcal{A}(\mathcal{T})$ is semisimple.*

As mentioned above, every scattered set as well as every Helson S-set is a S_3-set. Recall that for every closed subset E of Γ, there associate two closed ideals, $I(E)$, consisting of functions $\varphi \in A(\Gamma)$ such that $\varphi|E = 0$, and $J(E)$, consisting of functions which can be approximated by functions vanishing on a neighborhood of E. Clearly, $J(E) \subset I(E)$. A set E is called a *set of spectral synthesis*, or *S-set*, if $I(E) = J(E)$. A compact subset $E \subset \Gamma$ is called a *Helson set*, if every continuous function on E is the restriction of a function from $A(\Gamma)$. It is well known that there are Helson sets which are not S-sets (conditions for Helson sets to be S-sets are given in $[B]_2$). There are countable sets (scattered sets) which are not Helson sets, as well as Helson S-sets which are not scattered (see [B$_1$, H-R]).

Proposition 1. *Every S_3-set is a S-set.*

P r o o f. Assume that E is a closed subset of Γ which is not a S-set. Consider the quotient algebra $A(\Gamma)/J(E)$. If $\varphi \in A(\Gamma)$, then the image of φ under this homomorphism is denoted by $\hat{\varphi}$. Since E is not a S-set, the quotient algebra $A(\Gamma)/J(E)$ is not semisimple: indeed, any element $\varphi \in I(E) \setminus J(E)$ under the natural homomorphism $A(\Gamma) \to A(\Gamma)/J(E)$ will be mapped into a non-zero topological nilpotent element. Consider the representation $V : G \to L(A(\Gamma))$ defined by

$$(V(t)\varphi)(\chi) = \chi(t)\varphi(\chi),$$

and let $T : G \to L(A(\Gamma)/J(E))$ be defined by $T(g)\widehat{\varphi} = \widehat{(V(g)\varphi)}$. Then the algebra $\mathcal{A}(\mathcal{T})$ is isometrically isomorphic to $A(\Gamma)/J(E)$, hence is not semisimple. ∎

Definition 2. *A family of closed subsets of Γ, $\{E_\alpha\}_{\alpha \in F}$ is called an archipelago if:*

(i) for every $\alpha_1, \alpha_2 \in F, \alpha_1 \neq \alpha_2$, we have $E_{\alpha_1} \cap E_{\alpha_2} = \emptyset$, and

(ii) for every $F_0 \subset F$ there exists an open set $V \subset \Gamma$ and there exists $\alpha_0 \in F_0$ such that $E_{\alpha_0} \subset V$ and $V \cap E_{\alpha_j} = \emptyset$ for all $\alpha_j \in F_0, \alpha_j \neq \alpha_0$.

Proposition 2. *If $Sp(T) = \Lambda_1 \cup \Lambda_2$, where Λ_1, Λ_2 are nonempty closed subsets such that one of them is compact and $\Lambda_1 \cap \Lambda_2 = \emptyset$, then there is a projection $P \in \mathcal{A}(\mathcal{T})$ such that $Sp(T|PX) = \Lambda_1, Sp(T|(I-P)X) = \Lambda_2$.*

P r o o f. Assume, for definiteness, that Λ_1 is compact. Let $\Lambda_\alpha \subset \Lambda_2$ be a compact set, $Q_\alpha := \Lambda_1 \cup \Lambda_\alpha$, and consider the spectral subspace $X_\alpha := X(Q_\alpha)$. Let $T_\alpha(g) := T(g)|X_\alpha$. Then $Sp(T_\alpha)$ is compact, hence T_α is uniformly continuous and the algebra $\mathcal{A}(\mathcal{T}_\alpha)$ has unit. By Silov's Idempotent Theorem, there is an idempotent element $P_\alpha \in \mathcal{A}(\mathcal{T}_\alpha)$ such that $Sp(T_\alpha|P_\alpha X_\alpha) = \Lambda_1, Sp(T_\alpha|(I - P_\alpha)X_\alpha) = \Lambda_\alpha$. It is easy to see that the family of projections P_α is uniformly bounded is can be extended to a projection P on X such that $Sp(T|PX) = \Lambda_1, Sp(T|(I-P)X) = \Lambda_2$. ∎

It follows from Proposition 1 that if E_1 and E_2 are two compact S_3-sets such that $A_1 \cap A_2 = \emptyset$, then $E_1 \cup E_2$ is a S_3-set. A more general fact is proved in the next theorem.

Theorem 1. *Let $\{E_\alpha\}_{\alpha \in F}$ be an archipelago of compact S_3-sets, and let $E := \cup_{\alpha \in F} E_\alpha$. Then E is a S_3-set.*

P r o o f. Let T be a representation of G on $L(X)$ such that $Sp(T) \subset E$. Let $a \in R(\mathcal{A}(\mathcal{T}))$. Define

$$F_a := \{\alpha \in F : E_\alpha \cap Sp(a) \neq \emptyset\}.$$

There exists an open set V in Γ and an α_i such that $E_{\alpha_i} \subset V$ and $V \cap E_{\alpha_j} = \emptyset$ for all $\alpha_j \in F_a, \alpha_j \neq \alpha_i$. Take an element $f \in L^1(G)$ such that $\hat{f}(\gamma) = 1$ for all $\gamma \in A_{\alpha_i}$ and $\hat{f}(\gamma) = 0$ for all $\gamma \notin V$.

Let $X_1 := \overline{\{\hat{f}(T)ax : x \in X\}}$ and $\tilde{T}(t) := T(t)|X_1$. Since $Sp(\hat{f}(T)a) \subset E_{\alpha_i} \cap Sp(a)$, and since $\hat{f}(T)a$ is in the radical of $\mathcal{A}(\tilde{\mathcal{T}})$, it follows $\hat{f}(T)a = 0$, Hence, $Sp(a) \cap E_{\alpha_i} = \emptyset$, which is a contradiction. ∎

Proposition 3. *If E is a closed set and B is scattered, then $F := \{E, x \in B \setminus E\}$ is an archipelago.*

P r o o f. Let $F_0 \subset F$. There are three possibilities:

(i) $F_0 = \{E\}$. Then we can take as V any open set containing E.

(ii) $F_0 \subset \{x : x \in B \setminus E\}$. Since B is scattered, F_0 contains an isolated point $x_0 \in B \setminus E$, i.e. there is an open set V, $x_0 \in V$, $[V \setminus \{x_0\}] \cap B \setminus E = \emptyset$, so that the definition is fulfilled.

(iii) F_0 contains E and elements in $B \setminus E$. Since $F_0 \setminus \{E\}$ contains an isolated point, say x_0, there exists an open set V, such that $x_0 \in V$ and $V \cap [F_0 \setminus E] = \emptyset$. Choose $W = V \cap E^c$ (where $E^c = \Gamma \setminus E$), then $x_0 \in W$ and $W_0 \cap E = \emptyset$, hence the definition is fulfilled. ∎

Proposition 3 and Theorem 1 imply the following corollary.

Corollary 1. *If E is a S_3-set and B is scattered, then $E \cup B$ is S_3-set. In particular, the union of a Helson S-set and a scattered set is a S_3-set.*

Now we consider the general question of when is the union of S_3-sets a S_3-set. Let a be an element in $\mathcal{A}(\mathcal{T})$. We define

$$I_a := \{f \in L^1(G) : \hat{f}(T)a = 0\},$$

and let

$$Sp(a) := \{\chi \in \Gamma : \hat{f}(\chi) = 0 \ \forall f \in I_a\}.$$

It is not difficult to see that $Sp(a) = Sp(T|\overline{aX})$.

Lemma 1. *Assume that Λ_1, Λ_2 are S_3-sets, $T : G \to L(X)$ is a strongly continuous isometric representation such that $Sp(T) \subset \Lambda_1 \cup \Lambda_2$. If $a \in \mathcal{R}(\mathcal{A}(\mathcal{T}))$, then $Sp(a) \subset \Lambda_1 \cap \Lambda_2$.*

P r o o f. We show that $Sp(a) \subset \Lambda_2$. Let U_2 be an open set, $\Lambda_2 \subset U_2$. We show that $Sp(a) \subset \overline{U_2}$. Assume, on the contrary, that there exists $\chi \in Sp(a)$, such that $\chi \notin \overline{U_2}$. Take an element $f \in L^1(G)$ such that $\hat{f}|U_2 = 0, \hat{f}(\chi) = 1$.

Since $Sp(\hat{f}(T)a) \subset supp(\hat{f}) \cap Sp(a) \subset [\Gamma \setminus U] \cap Sp(a) \subset \Lambda_1$, and since Λ_1 is a S_3-set and $\hat{f}(T)a$ is a topological nilpotent element, it follows that $\hat{f}(T)a = 0$, i.e., $f \in I_a$. Therefore, $\hat{f}(\chi) = 0$, a contradiction. ■

We also need the following lemma (see [M–V, Proposition 6]).

Lemma 2. *If $a \in \mathcal{R}(\mathcal{A}(\mathcal{T}))$ and $a \neq 0$, then $Sp(a)$ has no isolated point.*

Theorem 2. *If Λ_1 and Λ_2 are S_3-sets and $\Lambda_1 \cap \Lambda_2$ is scattered, then $\Lambda_1 \cup \Lambda_2$ is a S_3-set.*

P r o o f. Let T be an isometric representation of G on $L(X)$ such that $Sp(T) \subset \Lambda_1 \cup \Lambda_2$. Assume that there exists an element $a \in \mathcal{R}(\mathcal{A}(\mathcal{T}))$ such that $a \neq 0$. By Lemma 1, $Sp(a) \subset \Lambda_1 \cap \Lambda_2$, hence $Sp(a)$ contains an isolated point, which is impossible by Lemma 2. ■

Theorems 1, 2 and Corollary 1 are, of course, analogous to the corresponding results concerning S-sets (see [B_1, p. 172, 187]). It is not known whether finite unions of S_3-sets are always S_3-sets. If Λ_1 and Λ_2 are S_3-sets, $Sp(T) \subset \Lambda_1 \cup \Lambda_2$ and $a \in \mathcal{R}(\mathcal{A}(\mathcal{T}))$, then Lemma 1 implies only that $a^2 = 0$.

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