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On a relation between the coefficients and the sum of the generalized Taylor series

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Let $f \in C^\infty[-1,1]$ and $\exists \rho \in [1,2)$ such that $\forall k = 0,1,2,\dots$ $||f^{(k)}||_{C[-1,1]} \le c(f)\rho^k 2^{\frac{k(k+1)}{2}}$. Then it expands in the generalized Taylor series, which was introduced by V.A. Rvachov in 1982. In this paper it is shown that if the restrictions $||f^{(n)}|| = o(2^{\frac{n(n+1)}{2}})$, $n \to \infty$ are imposed on the sum of this series, and stronger restrictions $|f^{(n)}(x_{n,k})| \le CA(n)$, $\frac{A(n+1)}{A(n)} \le 2^{n+\frac{1}{2}}$ hold for its coefficients, then these stronger restrictions will hold for the sum of the series too. As a consequence the conditions of belonging to Gevrey class and of real analyticity for the above-mentioned functions are obtained.

1. Introduction and statement of results

There exist several generalizations of the common Taylor series for the analytic functions (J. Delsarte, V.A. Marchenko, G.V. Badalyan.) In this paper we consider the generalized Taylor series which was introduced in 1982 by V.A. Rvachov in [1] (detailed exposition see in [2]). Let us put $||g|| = ||g||_{C[-1,1]}$; $d(k) = k(k+1)/2$. It was shown in [2] that if f belongs to the class $C^\infty[-1,1]$ and

$$\exists \rho \in [1;2) : ||f^{(k)}|| \le c(f)\rho^k 2^{d(k)}, \quad k = 0,1,2\dots , \tag{1}$$

then f expands in the generalized Taylor series, which is uniformly convergent on $[-1,1]$:

$$f(x) = \sum_{n=0}^{\infty} \sum_{k \in N_n} f^{(n)}(x_{n,k})\varphi_{n,k}(x),$$

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where

$$\varphi_{n,k}(x) = \sum_l c_l^{(n,k)} up(x - l2^{-n}), \tag{2}$$

$N_n = \{-2^{n-1}, \dots, 2^{n-1}\}$, $n \neq 0$; $N_0 = \{-1, 0, 1\}$; $x_{n,k} = \frac{k}{2^{n-1}}$, $n \neq 0$, $k \in N_n$; $x_{0,k} = k$, $k \in N_0$.

In the generalized Taylor series the functions $\varphi_{n,k}(x)$, which are the finite linear combinations of translates of the function $up(x)$, are similar to the functions x^n in common Taylor series.

The function

$$up(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \prod_{k=1}^{\infty} \frac{\sin t2^{-k}}{t2^{-k}} dt$$

is a solution with a compact support of the equation

$$y'(x) = 2y(2x + 1) - 2y(2x - 1). \tag{3}$$

The support of the function $up(x)$ is $[-1, 1]$. It is an even function and positive for $-1 < x < 1$, and $||up^{(n)}|| = 2^{d(n)}$. For any fixed n, k coefficients $c_l^{(n,k)}$ can either be found directly from the finite linear algebraic system of equations: $\varphi_{n,k}^{(m)}(x_{m,s}) = \delta_n^m \delta_k^s$, $m = 0, 1, 2, \dots, n$, $s \in N_m$, or derived by the recurrent formulae (2.4), (2.5) in [1]. (The values of $c_l^{(n,k)}$ are not needed for the proof of the result of present paper.)

Consider a function f belonging to the class $C^\infty[-1, 1]$ and satisfying the condition $\exists r > 0 : \forall n \in N \; ||f^{(n)}|| \leq r^n n^n$. Then f is an analytic function on $[-1; 1]$. In virtue of the analyticity the condition $|f^{(n)}(0)| \leq C \;\; \forall n \in N$ implies $||f^{(n)}|| \leq Ce \;\; \forall n \in N$.

In this paper it will be shown that the function which expands in the generalized Taylor series behaves similarly: if norms of all the derivatives of the function f satisfy some weak restrictions and some stronger restrictions are imposed on $f^{(n)}(x_{n,k})$, then these stronger restrictions hold for norms of all the derivatives. Still in this case it's not sufficient to take condition (1) as weak restrictions.

In 1986 the following problem was stated ([3, p. 57, problem 15]):

Prove that if $\varphi \in C^\infty[-1, 1]$ satisfies (1) and $|\varphi^{(n)}(x_{n,k})| < Cn^n \;\; \forall n = 0, 1, 2, \dots$, $\forall k \in N_n$, then either φ is an analytic function, or $\overline{\lim_{n\to\infty}} ||\varphi^{(n)}|| 2^{-d(n)} > 0$.

The result of the present paper is the following theorem which solves even more general problem:

Theorem. *For a function* f *let the following conditions be valid:*
1) $|f^{(n)}(x_{n,k})| \leq CA(n)$, $\forall n = 0, 1, 2, \dots, \forall k \in N_n$,
where $\frac{A(n+1)}{A(n)} \leq 2^{n+\frac{1}{2}}$;

2) $||f^{(n)}|| = o(2^{d(n)}),\ n \to \infty.$
Then
$||f^{(n)}|| \le \widetilde{C}A(n),\ \forall n = 0, 1, 2, \dots .$

Preliminary results were announced in [4].

R e m a r k. This result is in a sense the best possible. If in condition 1) instead of the set $\{x_{n,k}\}$ we take its proper subset $\{x_{n-1,k}\}$ then the conclusion of the Theorem will not hold. The function $up(x/2)$ is a counterexample. If on the other hand we drop condition 2) of the Theorem then the function $up(x)$ is a counterexample.

Corollary 1. *Let condition 2) of the theorem hold for the function* f *and* $\exists r > 0 : |f^{(n)}(x_{n,k})| \le Cr^n n^n$, $\forall n = 0, 1, 2, \dots , \forall k \in N_n$.
Then f *is analytic on* $[-1, 1]$.

Let us remind that the Gevrey class of the order a (see [5, p. 335]) is the class of C^∞-functions g such that

$$||g^{(n)}|| \le C(g)(r(g))^n n^{an},\quad n = 0, 1, 2, \dots .$$

Corollary 2. *Let a function* f *satisfy condition 2) of the theorem and there exist* $r > 0$ *and* $a > 1$ *such that* $|f^{(n)}(x_{n,k})| \le Cr^n n^{an}\ \forall n = 0, 1, 2, \dots .$
Then f *belongs to the Gevrey class of the order* a *on* $[-1, 1]$.

2. Proof of the theorem

For the proof we shall need the following result:

Lemma (Lemma 6 in [2]; see also [6]). *Let*

$$\varphi(x) = \sum_{i \in N_k} c_i \varphi_{k,i}(x),\ \text{and } |c_i| \le M.$$

Then

$$||\varphi^{(n)}|| \le \begin{cases} CM2^{d(n)-d(k)}, & n \ge k; \\ CM2^{d(n)-d(k)+n-k}, & n < k. \end{cases}$$

The function f can be represented in the form $f(x) = \sum\limits_{k=0}^{\infty} P_k(x)$, where

$$P_k(x) = \sum_{i \in N_k} f^{(k)}(x_{k,i}) \varphi_{k,i}(x). \tag{4}$$

Hence, $f^{(n)}(x) = \sum\limits_{k=0}^{\infty} P_k^{(n)}(x)$. It follows from the lemma and condition 1) of the theorem that

$$||P_k^{(n)}|| \le \begin{cases} CA(k)2^{d(n)-d(k)}, & n > k; \\ CA(k)2^{d(n)-d(k)+n-k}, & n \le k. \end{cases} \tag{5}$$

Let us choose any point $x_0 \in [-1, 1]$, which is not dyadic rational. We have

$$|f^{(n)}(x_0)| = |\sum_{k=0}^{n-1} P_k^{(n)}(x_0) + \sum_{k=n}^{\infty} P_k^{(n)}(x_0)| = |\Phi_1(x_0) + \Phi_2(x_0)|.$$

Let us estimate $\Phi_2(x_0)$, using (5):

$$|\Phi_2(x_0)| \le CA(n) \sum_{k=n}^{\infty} \frac{A(k)}{A(n)} 2^{d(n)-d(k)+n-k}$$

$$= CA(n) \sum_{k=n}^{\infty} \frac{A(k)}{A(k-1)} \cdot \ldots \cdot \frac{A(n+1)}{A(n)}\, 2^{d(n)-d(k)+n-k}$$

$$\le CA(n) \sum_{k=n}^{\infty} 2^{(k^2-n^2)/2+d(n)-d(k)+n-k} = CA(n) \sum_{l=0}^{\infty} 2^{-\frac{3}{2}l} = C_1 A(n). \tag{6}$$

Now we estimate $\Phi_1(x_0) = \sum\limits_{k=0}^{n-1} P_k^{(n)}(x_0)$. We have from (2) and (4)

$$P_k(x) = \sum_s v_s up(x - s2^{-k}), \ v_s \in R.$$

Derivating this equality and using (3), we obtain

$$P_k^{(k+1)}(x) = \sum_m \lambda_m up(2^{k+1}x + \xi_m),$$

where ξ_m are odd numbers.

Thus, $P_k^{(k+1)}(x)$ is a linear combination of functions with compact supports $up(2^{k+1}x + \xi_m)$, the interiors of the supports of which do not intersect: the length of the support of the function $up(x)$ is equal to 2, hence, the length of the supports of the functions $up(2^{k+1}x+\xi_m)$ equals $\frac{1}{2^k}$; the shift along the x-axis of the function $up(2^{k+1}x+2m+1)$ with respect to the function $up(2^{k+1}x+2l+1)$ equals $(2l+1-(2m+1))2^{-(k+1)} = (l-m)2^{-k}$, i.e., an integer multiple of the lengths of the supports of the functions $up(2^{k+1}x + \xi_m)$. Let us choose and fix any $k < n$. Let $x_0 \in [0, 1]$ and let the dyadic representation of x_0 be $x_0 = 0, p_1p_2p_3\ldots$. Then x_0 belongs to some interval $(-(l+1)2^{-(k+1)}; (1-l)2^{-(k+1)}), -2^{k+1}+1 \le l \le -1,$

and l is an odd number. (The case $x \in [-1, 0]$ is dealt with similarly.) Then $P_k^{(k+1)}(x_0) = h_k^l up(2^{k+1}x_0 + l)$. Derivating and using (3), we obtain

$$P_k^{(k+2)}(x_0) = (-1)^{p_{k+1}} h_k^l 2^{k+2} up(2^{k+2}x_0 + z),$$

where z is an odd number.

$$P_k^{(n)}(x_0) = (-1)^{p_{k+1}+p_{k+2}+\ldots+p_{n-1}} h_k^l 2^{(k+2)+(k+3)+\ldots+n} up(2^n x_0 + \widehat{z}),$$

where $\widehat{z}$ is odd. Therefore,

$$\Phi_1(x_0) = 2^{d(n)}\gamma_n(x_0) \sum_{k=0}^{n-1} 2^{-d(k)}\alpha_k(x_0), \tag{7}$$

where

$$\gamma_n(x_0) = (-1)^{p_1+p_2+\ldots+p_{n-1}} up(2^n x_0 + \widehat{z}); \quad \alpha_k(x_0) = (-1)^{p_1+p_2+\ldots+p_k} h_k^l 2^{-(k+1)}.$$

Using (5), we have

$$|\alpha_k(x_0)| = 2^{-(k+1)}|h_k^l| = 2^{-(k+1)}|P_k^{(k+1)}(-\frac{l}{2^{k+1}})|$$

$$\leq 2^{-(k+1)}CA(k)2^{d(k+1)-d(k)} = CA(k),$$

therefore, the series

$$\sum_{k=0}^{\infty} 2^{-d(k)}\alpha_k(x_0)$$

converges as $A(k) \leq A(k-1)2^{(k-1)+\frac{1}{2}} \leq \ldots \leq A(1)2^{(k^2-1)/2}$. By condition 2) of the theorem $|\Phi_1(x_0) + \Phi_2(x_0)| = o(2^{d(n)})$, $n \to \infty$. It follows from (6) that $\Phi_2(x_0) = o(2^{d(n)})$, $n \to \infty$.

Therefore, from (7) we have $\Phi_1(x_0) = o(2^{d(n)})$, $n \to \infty$.

We will show that $\sum\limits_{k=0}^{\infty} 2^{-d(k)}\alpha_k(x_0) = 0$, i.e., that $\lim\limits_{n\to\infty} S_n = 0$, where $S_n = \sum\limits_{k=0}^{n-1} 2^{-d(k)}\alpha_k(x_0)$. As x_0 is not dyadic rational, in its dyadic representation $x_0 = 0, p_1 p_2 \ldots$ zero is infinite number of times followed by 1 and vice versa.

Let us choose the sequence $\{n_i \in N\}_{i=1}^{\infty}$, $n_i \to \infty$ such that $p_{n_i} = 0$ and $p_{n_i+1} = 1$ or $p_{n_i} = 1$ and $p_{n_i+1} = 0$. Then $\forall i \in N$

$$|\gamma_{n_i}(x_0)| = up(2^{n_i}x_0 + z_{n_i}) > up\left(\frac{1}{2}\right) = \frac{1}{2}.$$

This is due to the fact, that, for instance, in situation $p_{n_i} = 1$ and $p_{n_i+1} = 0$ the point x_0 belongs to the subinterval $(-z_{n_i}2^{-n_i}; (1-2z_{n_i})2^{-(n_i+1)})$ of the interval $[-(z_{n_i}+1)2^{-n_i}; (1- \ z_{n_i})2^{-n_i}]$, hence, $up(2^{n_i}x_0 + z_{n_i}) > up\left(2^{n_i}\frac{1-2z_{n_i}}{2^{n_i+1}} + z_{n_i}\right) = up\left(\frac{1}{2}\right) = \frac{1}{2}$. We have the similar situation in the case $p_{n_i} = 0$ and $p_{n_i+1} = 1$. Therefore, the sequence $\{S_{n_i}\}$ tends to 0

$$\lim_{i\to\infty} S_{n_i} = \lim_{i\to\infty} \sum_{k=0}^{n_i-1} 2^{-d(k)}\alpha_k(x_0) = 0.$$

As the sequence $\{S_n\}$ converges to the sum of the series $\sum\limits_{k=0}^{\infty} 2^{-d(k)}\alpha_k(x_0)$, then $\{S_{n_i}\}$, being its subsequence, converges to the sum of the series. Thus,

$$\sum_{k=0}^{\infty} 2^{-d(k)}\alpha_k(x_0) = 0.$$

So we have

$$\sum_{k=0}^{n-1} 2^{-d(k)}\alpha_k(x_0) = -\sum_{k=n}^{\infty} 2^{-d(k)}\alpha_k(x_0).$$

Using this equality, we estimate $|\Phi_1(x_0)|$.

$$\begin{aligned}
|\Phi_1(x_0)| &\leq 2^{d(n)}|\sum_{k=n}^{\infty} 2^{-d(k)}\alpha_k(x_0)| \leq C\sum_{k=n}^{\infty} A(k)2^{d(n)-d(k)} \\
&= CA(n)\sum_{k=n}^{\infty}\frac{A(k)}{A(n)}2^{d(n)-d(k)} \leq CA(n)\sum_{k=n}^{\infty} 2^{(k^2-n^2)/2+d(n)-d(k)} \\
&= CA(n)\sum_{k=n}^{\infty} 2^{(n-k)/2} = CA(n)\sum_{l=0}^{\infty} 2^{-\frac{l}{2}} = C_2A(n).
\end{aligned}$$

Therefore,

$$|\Phi_1(x_0) + \Phi_2(x_0)| \leq |\Phi_1(x_0)| + |\Phi_2(x_0)| \leq CA(n).$$

Since the complement of the set of dyadic rational points is dense in $[-1, 1]$, the theorem is proved.

The statements of the corollaries immediately follow from the theorem.

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