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Jacobi operator with step-like asymptotically periodic coefficients

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The direct/inverse scattering problem is considered for the Jacobi operator with different asymptotically periodic coefficients on the half axes. It is supposed that the backgrounds on the half-axes have the period 2 and the perturbation has the second finite moment. The problem is studied by means of the generalized Marchenko approach ([8, 2]).

1. Introduction. Statement of results

For the Jacobi operator

$$(Ly)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \tag{1.1}$$

the direct/inverse scattering problem on the constant background is well studied problem (for the operator with fast stabilized coefficients see [5] and references therein, for step-like fast-stabilized coefficients see [10, 9, 4]). On the periodic background the scattering problem was completely solved for the Schrödinger operator ([2]). For the Jacobi operator, the Jost solutions and the spectral measure on the half axis are studied in [7], and similar questions on the whole axis — in [5]. The transformation operator on the periodic background is constructed in [3]. But the inverse scattering problem in the prescribed class of perturbation of

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periodic coefficients is not studied properly. We consider the scattering problem for the step-like Jacobi operator with asymptotically periodic coefficients that are close to different periodic backgrounds of period 2 on the half-axes. Namely, let $L_\pm$ be periodic operators with the coefficients $a^\pm_{n+2} = a^\pm_n > 0$, $\quad b^\pm_{n+2} = b^\pm_n \in \mathbb{R}$ correspondingly*. We suppose that these operators have non-empty gaps in their spectra and that they have one coinciding spectral band. More precisely, we suppose that

(1) the spectra of operators L_+ and L_- have the following mutual location:

$$\sigma(L_+) = [\mu_+, \nu_+] \cup [\mu, \nu],\ \sigma(L_-) = [\mu, \nu] \cup [\mu_-, \nu_-],\ \mu_+ < \nu_+ < \mu < \nu < \mu_- < \nu_-, \tag{1.2}$$

(2)** the points of auxiliary spectra δ_+ and δ_- do not coincide with the ends of gaps and their centers:

$$\delta_+ \in (\nu_+, \mu),\ \delta_- \in (\nu, \mu_-),\ \delta_\pm \neq \gamma_\pm, \tag{1.3}$$

where $\gamma_+ = (\mu_+ + \nu)/2$, $\gamma_- = (\mu + \nu_-)/2$.

Let L be the Jacobi operator, defined by formula (1.1) with coefficients that are asymptotically close to the coefficients of the operators $L_\pm$ with the following rate of convergence:

$$\sum_{n=n_0}^{\pm\infty} n^2 \{|a_n - a^\pm_n| + |b_n - b^\pm_n|\} < \infty. \tag{1.4}$$

Remind now the well-known facts of the spectral theory of the periodic Jacobi operator of period 2. Consider the Floquet–Weyl solution $\hat\psi^\pm_n(\lambda)$ of the equation

$$(L_\pm y)_n = \lambda y_n, \quad n \in \mathbb{Z}, \quad \lambda \in \mathbb{C}, \tag{1.5}$$

such that $\hat\psi^\pm_n(\lambda) \in \ell^2(\mathbb{Z}^\pm)$, $\quad \lambda \in \mathbb{C} \setminus \sigma(L_\pm)$. This solution is defined up to a multiplicative constant, depending on λ. Following [2] and [3], we choose it in such a way, that $|\hat\psi^\pm_{-1}|^2 + |\hat\psi^\pm_0|^2 = 2$, $\forall \lambda \in \sigma(L_\pm)$. Let $\theta_\pm(\lambda)$ be the Floquet momentum of the operator $L_\pm$. As known [5], the function $\omega_\pm(\lambda) = e^{i\theta_\pm(\lambda)}$ gives a conform mapping of the upper half plane $\mathbb{C}' = \mathbb{C} \bigcap \{\lambda : \mathrm{Im}\lambda > 0\}$ on the lower unit half disk with a cut $\mathbb{D}'_\pm = \mathbb{D} \bigcap \{\omega : \mathrm{Im}\omega < 0\} \setminus \beta'(\pm)$, $\beta'(\pm) = \{\omega : \arg\omega = -\frac{\pi}{2},\ e^{-h_\pm} \leq |\omega| \leq 1\}$. Here $\mathbb{D} = \{\omega : |\omega| < 1\}$, $\mathbb{T} = \{\omega : |\omega| = 1\}$, $h_+ = \sup\{|\mathrm{Im}\theta^+(\lambda)|, \lambda \in [\nu^+, \mu]\}$ and $h_- = \sup\{|\mathrm{Im}\theta^-(\lambda)|, \lambda \in [\nu, \mu^-]\}$. The

* Everywhere in this paper the sign "+" corresponds to the data on the right half-axis and the sign "−" corresponds to the left half-axis.

**Condition (1.3) as far as condition (1.10) below are unessential and are introduced to avoid complement technical difficulties.

function $\omega_\pm(\lambda)$ maps also conformally the lower half plane $\mathbb{C}'' = \mathbb{C}\bigcap\{\lambda : \mathrm{Im}\lambda < 0\}$ on the upper half disk $\mathbb{D}''_\pm = \mathbb{D}\bigcap\{\omega : \mathrm{Im}\omega > 0\}\backslash\beta''(\pm)$, with a cut $\beta''(\pm) = \{\omega : \arg\omega = \frac{\pi}{2},\, e^{-h_\pm} \le |\omega| \le 1\}$. Under the mapping $\omega_\pm(\lambda)$ the spectrum $\sigma(L_\pm)$, considered as boundary values from the half planes $\mathbb{C}'$ and $\mathbb{C}''$, has two images $\mathbb{T}'_\pm$ and $\mathbb{T}''_\pm$, where $\mathbb{T}'_\pm \bigcup \mathbb{T}''_\pm = \mathbb{T}_\pm$ is the unit circle. Put $\mathbb{W}_\pm = \mathbb{D}'_\pm \bigcup \mathbb{D}''_\pm \bigcup(-1,1)$. The inverse map $\lambda = \lambda(\omega_\pm)$ is a single valued function as $\omega_\pm \in \mathbb{W}_\pm\backslash\{0\}$ up to the boundary $\partial\mathbb{W}_\pm$. Therefore the function $\hat{\psi}_n^\pm(\lambda)$ can be considered as a function of the variable $\omega_\pm$, $\psi_n^\pm(\omega_\pm) = \hat{\psi}_n^\pm(\lambda(\omega_\pm))$. The Weyl solution $\psi_n^\pm(\omega_\pm)$ possess the properties

$$\textbf{(a)} \qquad \psi_n^\pm(\omega_\pm^{-1}) = \overline{\psi_n^\pm(\omega_\pm)},\ |\psi_n^\pm|^2 + |\psi_{n+1}^\pm|^2 = 2,\ \omega_\pm \in \mathbb{T}_\pm,\ \forall n \in \mathbb{Z}. \qquad (1.6)$$

(b) Function $\psi_n^\pm(\omega_\pm)$ takes real values on the boundaries of cuts $\beta'(\pm)$ and $\beta''(\pm)$. It takes equal values in the points of these cuts symmetrical with respect to the real axis. Function $\psi_n^\pm(\omega_\pm)$ takes also real values as $\omega_\pm \in [-1,0)\bigcup(0,+1]$.

In fact, property **(a)** imply ([3]) that $\{\psi_n^\pm(e^{i\theta}\}_{n\in\mathbb{Z}}$ is an orthogonal system of functions on $\mathbb{T}_\pm$ with respect to measure $d\theta/\pi$. Note also, that the functions $\hat{\psi}_n^\pm(\lambda)$ are continuous with respect to λ in the domain $\mathbb{C}\backslash(\sigma(L_\pm)\bigcup\delta_\pm\bigcup\gamma_\pm$, where $\gamma_\pm$ is the zero of function* $\dot{\theta}_\pm(\lambda)$.

Let now L be an operator (1.1),(1.4). As shown in [7] and [5], the equation

$$(Ly)_n = \lambda y_n \qquad (1.7)$$

has so-called Jost solutions with the asymptotic behavior $\hat{f}_n^\pm(\lambda) - \hat{\psi}_n^\pm(\lambda) \to 0$, $n \to \pm\infty$, $\lambda \in \mathbb{C}\backslash\{\gamma_\pm, \delta_\pm\}$. These solutions also can be considered as the functions of variables ω_+ and ω_-, that is $f_n^\pm(\omega_\pm) = \hat{f}_n^\pm(\lambda(\omega_\pm))$. For the Jost solutions the representation holds

$$f_n^\pm(\omega_\pm) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_m^\pm(\omega_\pm), \quad |\omega_\pm| = 1, \qquad (1.8)$$

where K_+ and K_- are the transformation operators attached to $+\infty$ and $-\infty$ correspondingly. These operators are constructed in [3]. They possess the property $LK_\pm = K_\pm L_\pm$, from which the coefficients of operator L can be related to the coefficients of the transformation operator. Besides, under condition (1.4) the estimate is valid

$$|K_\pm(n,m)| \le C \sum_{l=[\frac{n+m\mp1}{2}]}^{\pm\infty} \{|a_l - a_l^\pm| + |b_l - b_l^\pm|\}, \quad \pm m > \pm n. \qquad (1.9)$$

* Everywhere in this paper we denote by point the derivative with respect to the spectral parameter λ.

This estimate is analogous to those estimate for the Schrödinger operator ([8], Lemma 3.1.1), that allows to solve the inverse problem in the class of perturbation with the first finite moment.

The spectrum $\sigma(L)$ of the operator L consists of two intervals $[\mu^+,\nu^+]$ and $[\mu^-,\nu^-]$, of the spectrum of multiplicity one and the interval $[\mu,\nu]$ of spectrum of multiplicity two. It has also a finite number of eigenvalues $\lambda_1,\dots,\lambda_p \in \mathbb{R}\backslash(\sigma(L^+)\bigcup\sigma(L^-))$. The finiteness of the discrete spectrum of operator L can be proved in the same way as in [8] and [5]. To simplify our considerations we suppose that

$$\gamma_+,\gamma_-,\delta_+,\delta_- \notin \{\lambda_1,\dots,\lambda_p\}. \tag{1.10}$$

Under the mapping $\omega_+(\lambda)$ (resp.,$\omega_-(\lambda)$) the band of the spectrum of multiplicity two $[\mu,\nu]$ maps on the arc of the unit circle Δ_2^+ (resp., Δ_2^-), the band $[\mu_+,\nu_+]$ - on the arc Δ_1^+ (resp., interval Δ_3^-), the band $[\mu_-,\nu_-]$ - on the interval Δ_3^+ (resp., the arc Δ_1^-). We enumerate the eigenvalues of operator L in such a way that $\{\lambda_1,\dots,\lambda_s\}\subset(\nu_+,\mu)\bigcup(\nu,\mu_-)$ and $\{\lambda_{s+1},\dots,\lambda_p\} \subset \mathbb{R}\setminus[\mu_+,\nu_-]$.The first s eigenvalues of the operator L have two images on the sides of cuts $\beta'(\pm)$ and $\beta''(\pm)$ in the symmetric with respect to the real axis points. We denote them as $w'_{\pm,k}$ and $w''_{\pm,k}$ correspondingly. The set $\{\lambda_{s+1},...,\lambda_p\}$ is mapped into the intervals $((-1,0)\bigcup(0,1))\backslash\Delta_3^\pm$, we denote their images as $w_{\pm,k}$.

Consider the scattering relations

$$T_\pm f_n^\mp = R_\pm f_n^\pm + \overline{f_n^\pm}, \quad |\omega_\pm| = 1. \tag{1.11}$$

We prove that the transmission coefficients $T_\pm$ and the reflection coefficients $R_\pm$ possess the following properties:

A. *The functions $R_\pm(\omega_\pm)$, and $T_\pm(\omega_\pm)$ are bounded on the unit circle $\mathbb{T}_\pm$ and continuous on the set $\mathbb{T}_\pm\setminus\{w:\omega^4=1\}$. Function $(w_\pm^4-1)T_\pm^{-1}(\omega_\pm)$ is continuous and bounded on the set $\mathbb{T}_\pm\setminus\{w:\omega^4=1\}$. Besides, $R_\pm(\omega_\pm^{-1}) = \overline{R_\pm(\omega_\pm)}$, $T_\pm(\omega_\pm^{-1}) = \overline{T_\pm(\omega_\pm)}$.*

B. *The equalities hold*

$$T_\pm(\overline{T_\pm})^{-1} = R_\pm, \quad \omega_\pm\in\Delta_1^\pm, \tag{1.12}$$

$$\overline{R_+}(\overline{T_+}\dot\theta_+)^{-1} = -R_-(T_-\dot\theta_-)^{-1}, \quad \omega_\pm\in\Delta_2^\pm, \tag{1.13}$$

$$\dot\theta_\pm(\dot\theta_\mp)^{-1}|T_\pm|^2 = 1-|R_\pm|^2, \quad \omega_\pm\in\Delta_2^\pm, \tag{1.14}$$

$$\lim_{w\to\exp(i\pi k/2)}(w^4-1)T_\pm^{-1}(w)(R_\pm(w)+1) = 0,\ k=0,1,2,3. \tag{1.15}$$

Note, that the function $\dot\theta_\pm(\dot\theta_\mp)^{-1}$ is bounded and positive as $\omega_\pm \in \Delta_2^\pm$. Introduce the function

$$\Psi_\pm(w_\pm) = \hat\Psi(\lambda(w_\pm)) = \left(\frac{\lambda(w_\pm) - \gamma_+}{\lambda(w_\pm) - \delta_+}\,\frac{\lambda(w_\pm) - \gamma_-}{\lambda(w_\pm) - \delta_-}\right)^{1/2}, \tag{1.16}$$

where a branch of the square root is considered that takes positive values as $\lambda > \nu_-$. Put $\mathbb{P}_\pm = \mathbb{W}_\pm \setminus \left(\Delta_3^\pm \bigcup \{w_{\pm,s+1}, \dots, w_{\pm,p}\}\right)$.

C. *The function* $\Psi_\pm T_\pm \dot\theta_\pm$ *can be continued as meromorphic function into the domain* $\mathbb{P}_\pm$ *with the simple poles in the points* $w_{\pm,s+1}, \dots, w_{\pm,p}$. *It also have pole in the point* $w_\pm(\delta_\mp)$ *iff* $\delta_\mp$ *is the pole of corresponding Weyl function* $m_\mp$. *Being considered as the functions of variable* λ *in the domain* $\mathbb{C}\setminus(\sigma(L_+) \bigcup \sigma(L_-) \bigcup \{\lambda_1, \dots, \lambda_p\} \bigcup \gamma_\pm \bigcup \delta_\pm)$ *the functions* $T_\pm \dot\theta_\pm$ *satisfy the equality*

$$T_+\dot\theta_+ = T_-\dot\theta_- = (iW(\lambda))^{-1}. \tag{1.17}$$

Here $W = a_{n-1}(\hat f_{n-1}^- \hat f_n^+ - \hat f_n^- \hat f_{n-1}^+)$ is the Wronskian of the Jost solutions (1.8).

Denote as $(\alpha_k^\pm)^{-2} = \sum_{n=-\infty}^{+\infty} |f_n^\pm(\lambda_k)|^2$ the right and the left norms of eigenfunctions, corresponding to the eigenvalue λ_k. Due to (1.10) these norms are well defined.

D. *The equality holds* $(\dot W)^2|_{\lambda=\lambda_k} = (\alpha_k^+\alpha_k^-)^{-2}$, *where* $W(\lambda)$ *is defined as (1.17).*

E. *The transmission coefficients* $T_\pm$ *are bounded functions as* $\lambda \to \infty$ (i.e., $\omega_\pm \to 0$) *and*

$$T_\pm(\omega_\pm) = (K_+(-1,-1)K_-(-1,-1))^{-1} + O(\omega_\pm), \quad \omega_\pm \to 0 \tag{1.18}$$

Following the standard scheme of investigation of the inverse scattering problem, we derive the Marchenko equations on the half axes.

Lemma 1.1. *The transformation operators kernels satisfy the discrete integral equations:*

$$\frac{\delta(n,m)}{K_\pm(n,n)} = K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l)F_\pm(l,m), \tag{1.19}$$

where $\delta(n,m)$ *is the Kronneker symbol and*

$$F_\pm(n,m) = \sum_{k=1}^{p} (\alpha_k^\pm)^2 \hat\psi_n^\pm(\lambda_k)\hat\psi_m^\pm(\lambda_k)$$

$$+\frac{1}{\pi}\int\limits_{\Delta_3^{\pm}} h_{\pm}(\omega)\psi_n^{\pm}(\omega)\psi_m^{\pm}(\omega)d\omega+\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} R^{\pm}(e^{i\theta})\psi_n^{\pm}(e^{i\theta})\psi_m^{\pm}(e^{i\theta})d\theta, \tag{1.20}$$

$$h_{\pm}(\omega)=i(|T_{\pm}|^2\dot{\theta}_{\pm}(2\omega\dot{\theta}_{\mp})^{-1})(\omega+i0)>0, \quad \omega\in\Delta_3^{\pm}. \tag{1.21}$$

By the notation $\dot{\theta}_{\pm}(\dot{\theta}_{\mp})^{-1}(\omega+i0)$, as $\omega\in\Delta_3^{\pm}$ we mean values of the function $\dot{\theta}_{\pm}(\dot{\theta}_{\mp})^{-1}$ when λ goes to the points of band $[\mu_{\mp},\nu_{\mp}]$ from the lower half plane. The kernel (1.20) of equation (1.19) consists of 3 summands: the first one corresponds to the discrete spectrum of the operator L, the second one corresponds to the part of the spectrum of multiplicity one with the image $\Delta_3^{\pm}=\omega_{\pm}([\mu_{\mp},\nu_{\mp}])$. The third summand is the generalized Fourier transform of the reflection coefficients $R_{\pm}$ with respect to the eigenfunctions of operator $L_{\pm}$:

$$\widetilde{R_{\pm}}(n,m)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} R_{\pm}(e^{i\theta})\psi_n^{\pm}(e^{i\theta})\psi_m^{\pm}(e^{i\theta})d\theta. \tag{1.22}$$

The finiteness of the second moment of perturbation (1.4) implies the following condition

F. *Let* $P_{\pm}(n,m)=\widetilde{R_{\pm}}(n,m)$. *Then*

$$|P_{\pm}(n,\cdot)||\cdot+n|\in\ell^1(\mathbb{Z}_{\pm}), \pm n\geq n_0,$$
$$n^2(a_n^+P_+(n,n+1)-a_{n-1}^+P_+(n-1,n))\in\ell^1(\mathbb{Z}_+),$$
$$n^2(a_{n-1}^-P_-(n,n-1)-a_n^-P_-(n+1,n))\in\ell^1(\mathbb{Z}_-),$$

$$n^2(P_{\pm}(n,n)-P_{\pm}(n\pm1,n\pm1))\in\ell^1(\mathbb{Z}_{\pm}). \tag{1.23}$$

The conditions **A**–**F** are characteristic for our scattering problem. Namely, let L_+ and L_- be two arbitrary periodic Jacobi operators of period 2 with coefficients $a_{n+2}^{\pm}=a_n^{\pm}$, $b_{n+2}^{\pm}=b_n^{\pm}$ and mutual location of their spectra as in (1.2), (1.3). Let θ_+ and θ_- be their Floquet momenta and let $\mathbb{T}_+$ and $\mathbb{T}_-$ be the circles corresponding to points $\left|e^{i\theta(\lambda)}\right|=1$. Denote as

$$\Lambda=\{T_+(w_+),R_+(w_+),T_-(w_-),R_-(w_-),\lambda_1,\dots,\lambda_p,\alpha_1^+,\dots,\alpha_p^+,w_{\pm}\in\mathbb{T}_{\pm}\} \tag{1.24}$$

a set of functions $T_{\pm}$, $R_{\pm}$, satisfying conditions **A**–**B**, and a set $\{\lambda_1,\dots,\lambda_p\}$ of arbitrary different points on the set $\mathbb{R}\backslash\bigcup_{\pm}(\sigma(L_{\pm})\bigcup\delta_{\pm}\bigcup\gamma_{\pm})$. Here $\alpha_k^+>0, k=1,\dots,p$ are arbitrary positive numbers. Let the functions T_+ and T_- satisfy condition **C**. We select α_k^- from condition **D**, where $W(\lambda)$ is the function, defined by the condition **C**. With respect to the data $T_+, R_+, T_-, R_-, \lambda_1, \dots, \lambda_p, \alpha_1^\pm, \dots, \alpha_p^\pm$ we construct the functions (1.20) and (1.21), where $\psi_n^\pm(\omega_\pm) = \hat{\psi}_n^\pm(\lambda)$ are the Floquet–Weyl solutions of equation (1.5), normalized with respect to conditions (1.6) and **b**. Now suppose that condition **F** holds. Consider the Marchenko equations (1.19) for the transformation operator kernels. We show that under the conditions **A**–**D, F** these equations are uniquely solvable. In particular, we obtain values $K_+(-1,-1)$ and $K_-(-1,-1)$. Suppose now that condition **E** is also valid. By formulas

$$\tilde{a}_n^+ = \frac{a_n^+ K_+(n+1,n+1)}{K_+(n,n)}, \qquad \tilde{a}_n^- = \frac{a_n^- K_-(n,n)}{K_-(n+1,n+1)}, \quad n \in \mathbb{Z}, \tag{1.25}$$

$$\tilde{b}_n^+ = b_n^+ + \frac{a_n^+ K_+(n,n+1)}{K_+(n,n)} - \frac{a_{n-1}^+ K_+(n-1,n)}{K_+(n-1,n-1)}, \quad n \in \mathbb{Z},$$

$$\tilde{b}_n^- = b_n^- + \frac{a_{n-1}^- K_-(n,n-1)}{K_-(n,n)} - \frac{a_n^- K_-(n+1,n)}{K_-(n+1,n+1)}, \quad n \in \mathbb{Z}, \tag{1.26}$$

we reconstruct two operators $\tilde{L}_+$ and $\tilde{L}_-$. From equation (1.19) and condition **F** we see that $\tilde{a}_n^\pm - a_n^\pm \to 0$, $\tilde{b}_n^\pm - b_n^\pm \to 0$ as $n \to \pm\infty$. Moreover, as it is shown in [3] condition **F** implies

$$\sum_{n=n_0}^{\pm\infty} n^2 \{ |\tilde{a}_n^\pm - a_n^\pm| + |\tilde{b}_n^\pm - b_n^\pm| \} < \infty. \tag{1.27}$$

In Lemma 3.4 (Section 3) it is proved, that these operators coincide, $\tilde{L}_+ = \tilde{L}_- = L$. Condition (1.27) and this Lemma imply condition (1.4). Thus we solve the direct/inverse scattering problem and prove the following

Theorem 1.1. *Let L_+ and L_- be 2-periodic Jacobi operators, satisfying (1.2) and (1.3). Conditions* **A-F** *are necessary and sufficient for a set(1.24) to be scattering data for the Jacobi operator (1.1), satisfying conditions (1.4) and (1.10).*

2. The direct scattering problem

Introduce some necessary notations for the periodic operator $L_\pm$ with the coefficients $a_{n+2}^\pm = a_n^\pm$, $b_{n+2}^\pm = b_n^\pm$ and the spectrum as in (1.2). Let $s_n^\pm(\lambda)$ and $c_n^\pm(\lambda)$ be the solutions of equation (1.5) with the initial conditions $s_0^\pm = c_{-1}^\pm = 1, s_{-1}^\pm = c_0^\pm = 0$. Let $u_\pm(\lambda) = (s_2^\pm(\lambda) + c_1^\pm(\lambda))/2$ be the Hill discriminant (the Lyapunov function) of the operator $L_\pm$ and let $\theta_\pm(\lambda)$ be the Floquet momentum.

It is related to the function $u_\pm(\lambda)$ by the equality $u_\pm(\lambda) = \cos(2\theta_\pm(\lambda))$. We denote by $\gamma_\pm$ the zero of its derivative with respect to λ

$$\dot\theta_\pm = -\frac{1}{2}\dot u_\pm(1-u_\pm^2)^{-1/2}. \tag{2.1}$$

Recall, that in our case the auxiliary spectrum of operator $L_\pm$ consists of the only point $\delta_\pm = b_0^\pm$, being the zero of polynomial $s_1^\pm(\lambda)$. Put $w_\pm(\lambda) = \exp(i\theta_\pm(\lambda))$. Let $m_\pm(\lambda)$ be those Weyl function of the operator $L_\pm$, that correspond to the Weyl solution

$$\tilde\psi_n^\pm(\lambda) = c_n^\pm(\lambda) + m_\pm(\lambda)s_n^\pm(\lambda) \tag{2.2}$$

from the space $\ell^2(\mathbb{Z}^\pm)$ as $\lambda \in \mathbb{C}\backslash\sigma(L_\pm)$. Then [5]

$$m_\pm(\lambda) = (s_2^\pm(\lambda) - c_1^\pm(\lambda) \mp 2(u_\pm^2(\lambda)-1)^{1/2})(2s_1^\pm(\lambda))^{-1}. \tag{2.3}$$

As is shown in [3], the function

$$\hat\psi_n^\pm(\lambda) = \tilde\psi_n^\pm(\lambda)\left(\frac{s_1^\pm(\lambda)}{a_{-1}^\pm\dot u_\pm(\lambda)}\right)^{1/2} \tag{2.4}$$

possess properties (1.6) and **b** with $\psi_n^\pm(\omega_\pm) = \hat\psi_n^\pm(\lambda(\omega_\pm))$. Besides, the functions $\psi_n^\pm(\omega_\pm)$ have the following asymptotic behavior as $\omega_\pm \to 0$:

$$\psi_n^\pm(\omega_\pm) = \left(a_{-1}^\pm \dots a_{n-1}^\pm \mathcal{K}_\pm^{-n-1}\omega_\pm^{n+1}\right)^{\pm 1}(1+O(\omega_\pm)), \tag{2.5}$$

where $\mathcal{K}_\pm = (a_0^\pm a_1^\pm)^{1/2}$.

Consider now the asymptotically periodic operator L defined by formulas (1.1),(1.4). The Jost solutions of equation (1.7) have the following properties

Lemma 2.1. *([7, 5]) The functions $f_n^\pm(\omega_\pm)$ are holomorphic in the domain $\mathbb{W}^\pm\backslash\{0\}$, and continuous up to the boundary $\partial\mathbb{W}$, except of points $w_\pm(\gamma_\pm)$,$w_\pm(\delta_\pm)$ on the sides of the cuts $\beta'(\pm)$ and $\beta''(\pm)$. The estimates hold*

$$|f_n^+(\omega_+) - \prod_{m=n}^{+\infty}\frac{a_m^+}{a_m}\psi_n^+(\omega_+)| \le C\,\frac{|\omega_+|^{n+1}}{(\dot u(\lambda(\omega_+))s_1(\lambda(\omega_+)))^{1/2}}\,q_n^+, \tag{2.6}$$

$$|f_n^-(\omega_-) - \prod_{-\infty}^{m=n-1}\frac{a_m^-}{a_m}\psi_n^-(\omega_-)| \le C\,\frac{|\omega_-|^{3-n}}{(\dot u(\lambda(\omega_-))s_1(\lambda(\omega_-)))^{1/2}}\,q_n^-, \tag{2.7}$$

where $q_n^\pm = \sum_{m=n}^{\pm\infty}|m|\{|a_m - a_m^\pm| + |b_m - b_m^\pm|\}$ and C is a constant depending on $L_\pm$ and r.-h.s. of (1.4). The functions $f_n^\pm(\omega_\pm)$ take equal real values in the symmetric with respect to the real axis points of cuts $\beta'(\pm)$ and $\beta''(\pm)$. These functions also possess the symmetry property $f_n^\pm(\overline{\omega_\pm}) = \overline{f_n^\pm(\omega_\pm)}$, $\omega_\pm \in \mathbb{T}_\pm$ and take real values as $\omega_\pm \in (-1,1)$.

The solutions $f_n^\pm$ and $\overline{f_n^\pm}$ are linearly independent as $\omega_\pm \in \mathbb{T}_\pm$. Indeed, their Wronskian is independent of n and by virtue of (2.6), (2.7), (2.3) and (2.4) we have as $n \to \pm\infty$

$$< f^\pm, \overline{f^\pm} >= \lim_{n\to\pm\infty} a_{n-1}^\pm(\psi_{n-1}^\pm\overline{\psi_n^\pm} - \psi_n^\pm\overline{\psi_{n-1}^\pm}) = a_{-1}^\pm(\psi_{-1}^\pm\overline{\psi_0^\pm} - \psi_0^\pm\overline{\psi_{-1}^\pm})$$

$$= s_1^\pm(\overline{m_\pm} - m_\pm)(\dot{u}_\pm)^{-1} = \pm 2(u_\pm^2 - 1)^{1/2}(\dot{u}_\pm)^{-1} = \pm 2i(1 - u_\pm^2)^{1/2}(\dot{u}_\pm)^{-1}.$$

Formula (2.1) then imply

$$< f^\pm, \overline{f^\pm} >= \pm i(\dot{\theta}_\pm)^{-1}, \quad \omega_\pm \in \mathbb{T}_\pm. \tag{2.8}$$

From scattering relations (1.11) we have

$$T_\pm = \frac{< \overline{f_n^\pm}, f_n^\pm >}{< f_n^\mp, f_n^\pm >} = \pm\frac{i}{\dot{\theta}_\pm < f_n^\mp, f_n^\pm >}, \quad R_\pm = -\frac{< f_n^\mp, \overline{f_n^\pm} >}{< f_n^\mp, f_n^\pm >}. \tag{2.9}$$

These equalities and Lemma 2.1 imply the property **A** of the scattering matrix. Further, when $\omega_\pm \in \Delta_1^\pm$ then $\lambda(\omega_\pm) \in (\mu_\pm, \nu_\pm)$, and therefore $\omega_\mp \in \Delta_3^\mp \subset (-1,1)$. From Lemma 2.1 it follows that the functions $f_n^\mp$ take on this set real values and by (2.9) equality (1.12) holds. Formula (2.9) implies that

$$\overline{R_+}(\overline{T_+})^{-1} < f^+, \overline{f^+} >=< f^+, \overline{f^-} >= -R_-(T_-)^{-1} < \overline{f^-}, f^- > .$$

This formulas and (2.8) prove (1.13). Note that asymmetric form of representation (1.13) is not essential since $\dot{\theta}_+$ and $\dot{\theta}_-$ are real on the coinciding band $[\mu, \nu]$. Formula (1.14) follows from the identity $< f^\pm, \overline{f^\pm} >< f^\mp, \overline{f^\mp} >^{-1} |T_\mp|^2 = |R_\mp|^2 - 1$ and formula (2.8). By virtue of Lemma 2.1 the Wronskian $W(\lambda) =< \hat{f}^-, \hat{f}^+ >$ is a holomorphic function with respect to λ in the domain $\mathbb{C}\backslash(\sigma(L_+)\bigcup\sigma(L_-)\bigcup\{\lambda_1, \dots, \lambda_p\}\bigcup\gamma_\pm\bigcup\delta_\pm)$. It takes complex conjugate values on the upper and lower sides of the spectrum bands. It is real on the gaps of spectrum and is properly defined everywhere except of the points $\gamma_\pm$ and $\delta_\pm$. In points where $W(\lambda) = 0$ the scattering coefficients T_+ and T_- have simple poles. Besides, by (2.2)–(2.12) one can be singularities of the order $O((\lambda - \delta_\pm)^{1/2})$ in the points $\delta_\pm$ such as $s_1^\pm(\delta_\pm) = 0$. In the points $\gamma_\pm$ the function $W(\lambda)$ has singularities of order $O((\lambda - \gamma_\pm)^{-1/2})$ and from (1.17) it follows that both T_+ and T_- have the same singularities in these points. Now we discuss the behavior of the transformation coefficients at the points of the auxiliary spectrum $\delta_\pm$. Consider, for example the point $\delta_+ \in [\nu^+, \mu]$. There are two possibilities. If δ_+ is the pole of the Weyl function $m_+(\lambda)$, then the Weyl solution $\hat{\psi}_n^+(\lambda)$ and corresponding Jost solution $\hat{f}_n^+(\lambda)$ has a singularity of the order $O((\lambda - \delta_+)^{-1/2})$. If the Weyl function has no pole at this point, then $\hat{f}_n^+(\lambda)$ is continuous in this point and has a behavior as $O((\lambda - \delta_+)^{1/2})$. From(2.2)–(2.4), (2.6), (2.7), (2.9), (2.1) and

(1.16) we conclude, that the function $\Psi_\pm T_\pm \dot\theta_\pm$ have the properties, described in **C** . Due condition (1.10) the discrete spectrum of operator L coincides with the set of zeros of function $W(\lambda)$. These zeros are simple, it can be proved by the approach of [7]. Property **C** is proved. Property **D** is a simple corollary of the formula $\frac{d}{d\lambda} < \hat f^-, \hat f^+ >= \sum_{n=-\infty}^{+\infty} \hat f_n^- \hat f_n^+$ that we obtain proceeding as in [6]. In the points of the discrete spectrum we have $\hat f_n^\pm(\lambda_k) = c_k^\pm \hat f_n^\mp(\lambda_k)$, where $c_k^+ c_k^- = 1$ and therefore, $\sum_{n=-\infty}^{+\infty} \hat f_n^- \hat f_n^+(\lambda_k) = c_k^\mp (\alpha_k^\pm)^{-2}$, from which **D** follows.

Now we study the behavior of the transformation coefficients as $\lambda \to \infty$. Namely,

$$< f^+, f^- >= a_{-1}\left(\sum_{m=-1}^{+\infty} K_+(-1,m)\psi_m^+ \sum_{s=-\infty}^{0} K_-(0,s)\psi_s^- - \sum_{m=0}^{+\infty} K_+(0,m)\psi_m^+ \sum_{s=-\infty}^{-1} K_-(-1,s)\psi_s^-\right)$$

$$= a_{-1}K_+(-1,-1)K_-(0,0)\hat\psi_{-1}^+\hat\psi_0^- + O(1). \tag{2.10}$$

Here we use formulas (2.2)–(2.4) and an asymptotic behavior $s_1^\pm(\dot u^\pm)^{-1}(a_{-1}^\pm)^{-1} = 1 + O(\lambda^{-1})$ as $\lambda \to \infty$. By formula (2.2) and (2.3) $\tilde\psi_{-1}^\pm = 1$, $\tilde\psi_0^\pm = m_\pm$. Using the identity $s_2^\pm c_1^\pm - s_1^\pm c_2^\pm = 1$, we see that

$$m_+(\lambda) = a_{-1}^+(\lambda)^{-1} + O(\lambda^{-2}),\ m_-(\lambda) = (\lambda - b_{-1}^-)(a_{-1}^-)^{-1} + O(\lambda^{-1}). \tag{2.11}$$

Since ([3])

$$K_+(n,n) = \prod_{s=n}^{\infty}\frac{a_s^+}{a_s}, \quad K_-(n,n) = \prod_{m=-\infty}^{n-1}\frac{a_m^-}{a_m}, \tag{2.12}$$

then (2.10)–(2.11) imply

$$< f^+, f^- >= \prod_{m=-\infty}^{-2}\frac{a_m^-}{a_m}\prod_{s=-1}^{\infty}\frac{a_s^+}{a_s}\lambda + O(1) = K_+(-1,-1)K_-(-1,-1)\lambda + O(1).$$

But $(i\dot\theta_\pm)^{-1} = 2(u_\pm^2 - 1)^{1/2}(\dot u_+)^{-1} = \lambda + O(1)$. From (1.17) and above considerations property **E** follows.

P r o o f of Lemma 1.1. Consider the "+" half axis case. Let C_ϵ be closed contour inside the domain $\mathbb{P}_+$ at a short distance to its boundary. The image of this contour under the map $\lambda(w_+)$ is two contours around the sets $[\mu_+, \nu]$ and $[\mu_-, \nu_-]$. Consider the function $T^+ f_n^- \psi_m^+ w_+^{-1}$, where $m \geq n$. According to

Lemma 2.1, formula (2.5) and property **E** this function is bounded as $w \to 0$ when $m > n$ and has as simple pole as $m = n$. Besides, according to property **C** this function is meromorphic in the domain enclosed by the contour C_ϵ. Since the Lyapunov function has the asymptotic behavior $u_\pm(\lambda) = (2\mathcal{K}_\pm^2)^{-1}\lambda^2 + O(\lambda) = 1/2(w_\pm^2 + w_\pm^{-2})$, therefore, $\lim_{w_+\to 0}(w_+/w_-)^{n+1} = (\mathcal{K}_+/\mathcal{K}_-)^{n+1}$. We see that

$$Res_{w_+=0}T^+ f_n^- \psi_m^+ w_+^{-1} = \delta_{nm} \prod_{s=-\infty}^{n-1} \frac{a_s^-}{a_s} . \frac{\mathcal{K}_-^{n+1} a_{-1}^+ \dots a_{n-1}^+}{\mathcal{K}_+^{n+1} a_{-1}^- \dots a_{n-1}^-} \prod_{s=-\infty}^{-2} \frac{a_s}{a_s^-} \prod_{s=-1}^{+\infty} \frac{a_s}{a_s^+}$$

$$= \lim_{w_+\to 0} (\frac{w_+}{w_-})^{n+1}\delta_{nm} = \prod_{s=-1}^{n-1} \frac{a_s^+}{a_s} \prod_{s=-1}^{+\infty} \frac{a_s}{a_s^+}\delta_{nm} = \frac{\delta_{nm}}{K_+(n,n)}.$$

The function $T^+ f_n^- \psi_m^+ = \dot\theta_+ f_n^- \psi_m^+ < f^-, f^+ >^{-1}$, considered as a function of λ has no singularities at the points $\gamma_-, \delta_+, \delta_-$ due to normalizing conditions (2.2)–(2.4) and due to property of functions $f_n^\pm$ to inherit properties of functions $\psi_n^\pm$. Therefore, $f_n^- \psi_m^+ < f^-, f^+ >^{-1}$ has no pole at δ_+ as well as in δ_- and γ_-. Let $w'_{+,k}$ and $w''_{+,k}$ be the images of the of the eigenvalues λ_k, $k = 1, \dots, s$. Let ω'_+ and ω''_+ be the images of points γ_+, i.e., the vertexes of cuts $\beta'(+)$ and $\beta''(+)$. The function $T^+ f_n^- \psi_m^+$ has singularities in all this points as well as in the points $\omega_{+,k} = \omega_+(\lambda_k) \in (-1,1)$, $k = s+1, \dots, p$. Since $\omega_+(\lambda) = \exp(i\theta_+(\lambda))$, then $\dot\omega_+ = i\omega_+\dot\theta_+$. When the part $C_\epsilon(1)$ of the contour C_ϵ that goes around the cuts $\beta'(+)$ and $\beta''(-)$ shrinks to these cuts as $\epsilon \to 0$, then the values of the function $T^+ f_n^- \psi_m^+$ outside of small neighbor of singularities tends to real values, that are equal in the symmetric points of cuts. Therefore, by (1.17) as $\epsilon \to 0$

$$\frac{1}{2\pi i} \int\limits_{C_\epsilon(1)} T^+ f_n^- \psi_m^+ \omega_+^{-1} d\omega_+ \to \sum_{k=1}^{s} Res_{\lambda_k} i T^+ f_n^- \psi_m^+ \dot\theta_+ + Res_{\gamma_+} i T^+ f_n^- \psi_m^+ \dot\theta_+$$

$$= -\sum_{k=1}^{s} Res_{\lambda_k} f_n^- \psi_m^+ W^{-1} - Res_{\gamma_+} f_n^- \psi_m^+ W^{-1} = -\sum_{k=1}^{s} Res_{\lambda_k} f_n^- \psi_m^+ W^{-1}, \tag{2.13}$$

because the function $f_n^- \psi_m^+ W^{-1}$ has no singularity at the points γ_+. By **D** and (1.8)

$$Res_{\lambda_k} f_n^- \psi_m^+ W^{-1} = f_n^-(\lambda_k)\psi_m^+(\lambda_k)\dot W(\lambda_k)^{-1} = c_k^- f_n^+(\lambda_k)\psi_m^+(\lambda_k)(\alpha_k^+)^2 (c_k^-)^{-1}$$

$$= (\alpha_k^+)^2 \sum_{l=n}^{+\infty} K_+(n,l)\psi_n^+(\lambda_k)\psi_m^+(\lambda_k). \tag{2.14}$$

Let $C_\epsilon(2)$ be the part of the contour C_ϵ that shrinks to the unit circle $\mathbb{T}_+$ as $\epsilon \to 0$. The function $T^+ f_n^- \psi_m^+$ is bounded and continuous on the set $\mathbb{T}_+ \setminus \{w_+ : w_+^4 = 1\}$. By use of the representation [3] $K_+(n,m) = 1/(2\pi i)\int_{\mathbb{T}_+}(w_+)^{-1}\overline{f_n^+}\psi_m^+ dw_+$ we obtain from (1.11) that as $\epsilon \to 0$

$$\frac{1}{2\pi i}\int\limits_{C_\epsilon(2)} T_+ f_n^- \psi_m^+ \frac{dw_+}{w_+} \to_{\epsilon\to 0} \frac{1}{2\pi i}\int\limits_{\mathbb{T}_+}(R_+ f_n^+ + \overline{f_n^+})\psi_m^+ \frac{dw_+}{w_+}$$

$$= \sum_{l=n}^{+\infty} K_+(n,l)\frac{1}{2\pi i}\int\limits_{\mathbb{T}_+} R_+ \psi_l^+ \psi_m^+ \frac{dw_+}{w_+} + K_+(n,m).$$

Now consider the part $C_\epsilon(3)$, corresponding to the contour going around the interval Δ_3^+. Since $\operatorname{Im}\psi_m^+ w_+^{-1} = 0$ and $(T_+ f_n^-)(w_+ + i0) = \overline{(T_+ f_n^-)(w_+ - i0)}$ as $w_+ \in (-1,1)$, then

$$\frac{1}{2\pi i}\oint_{C_\epsilon(3)} T_+ f_n^- \psi_m^+ \frac{dw_+}{w_+} = \frac{1}{\pi}\int_{\Delta_3^+} \operatorname{Im}[T_+ f_n^-](w+i0)\,\psi_m^+(w)\frac{dw}{w}. \qquad (2.15)$$

Using (1.11) and (1.17), we see that on the set Δ_3^+ a formula holds $T_+(\overline{T_+})^{-1} = \overline{W}/W = -R_-$, and, therefore

$$2i\operatorname{Im}[T_+ f_n^-]_{+0} = T_+ f_n^- - \overline{T_+}\overline{f_n^-} = \overline{T_-}(T_+/\overline{T_+} f_n^- - \overline{f_n^-})$$
$$= -\overline{T_+}(R_- f_n^- + \overline{f_n^-}) = -\overline{T_+}T_- f_n^+.$$

From (1.17) it follows also that

$$\overline{T_+}T_- = (\dot\theta_+\dot\theta_-|W|^2)^{-1} = -|T_+|^2\dot\theta_+(\dot\theta_-)^{-1}. \qquad (2.16)$$

The limit from the upper half disk to the points of Δ_3^+ with respect to the variable w_+ corresponds to the limit from the lower half plane to the points of band $[\mu^-, \nu^-]$. On this set we have $i\dot\theta_+ = \dot u_+(u_+^2 - 1)^{-1/2} > 0$ and $\dot\theta_- > 0$, therefore the function $h_+(w_+)$, defined by formula (1.21) is positive. By (2.16) and (2.15) we have

$$\frac{1}{2\pi i}\oint\limits_{C_\epsilon(3)} T_+ f_n^- \psi_m^+ \frac{dw_+}{w_+} = \frac{1}{\pi}\sum_{n=l}^{+\infty} K_+(n,l)\int\limits_{\Delta_3^+} h_+(w)\psi_l^+(w)\psi_m^+(w)dw. \qquad (2.17)$$

From the other hand, the Cauchy theorem implies

$$\frac{1}{2\pi i}\oint\limits_{C_\epsilon} T_+ f_n^- \psi_m^+ \frac{dw_+}{w_+} = -\sum_{k=s+1}^{p} Res_{\lambda_k}\hat f_n^- \hat\psi_m^+ W^{-1} - Res_{w=0} f_n^- \psi_m^+ T_+ \,.$$

Combining together formulas (2.11), (2.13), (2.14) and (2.17), we obtain the Marchenko equation (1.19)–(1.21) for the "+" half axis. The same is valid for "−" half axis. Lemma 1.1 is proved.

Put $\widetilde{F_\pm}(n,m) = F_\pm(n,m) - \widetilde{R_\pm}(n,m)$. From the properties of the Weyl solutions $\psi_n^\pm(\lambda)$ on their resolvent sets it follows immediately that the properties (1.23) are valid for the function $P(n,m) = \widetilde{F_\pm}(n,m)$. Since $F_\pm(n,m)$ can be considered as a solution of the Marchenko equation (1.19), with the kernel $K_\pm(n,m)$, satisfying (1.9) and (1.4), then the estimates (1.23) are valid also for $P(n,m) = F_\pm(n,m)$ (see [3]). Therefore, conditions (1.23) are fulfilled for the function $\widetilde{R_\pm}(n,m)$. The direct scattering problem is solved.

3. The inverse scattering problem

We proceed according to the scheme of solution, described in the introduction. In the framework of the Marchenko approach we have to prove following statements, given in Lemmas 3.1–3.3.

Lemma 3.1. *Let the functions $F_\pm(m,l)$ be defined by formula (1.20)–(1.21), where the data (1.24) satisfy conditions* **A**–**D**. *Then the equation $y(m) + \sum_{l=n\pm1}^{\pm\infty} \times F_\pm(m,l)y(l) = 0$ has no trivial solutions for any $n \in \mathbb{Z}$ fixed in the class $\ell^1(\mathbb{Z}(n,\pm\infty))$.*

The result of this Lemma is completely analogous to the result of Lemma 3.5.3 in [8], and we omit the prove. The compactness of corresponding operator is established in Theorem 10.10 of [5]. We conclude that the Marchenko equation (1.19)–(1.21) has a unique solution $K_|pm(n,m)$, and we can reconstruct $\tilde{a}_n^\pm, \tilde{b}_n^\pm$ by formulas (1.25)–(1.26).

Lemma 3.2. *Let $F_\pm(l,m)$ be as in Lemma 3.1. Put*

$$y_n^+(m) = a_{n-1}^+\kappa_+(n-1,m) + \frac{(\tilde{a}_n^+)^2}{a_n^+} + a_{m-1}^+\kappa_+(n,m-1)$$

$$-a_m^+\kappa_+(n,m+1) + (\tilde{b}_n^+ - b_m^+)\kappa_+(n,m), \tag{3.1}$$

$$y_n^-(m) = \frac{(\tilde{a}_{n-1}^-)^2}{a_{n-1}^-}\kappa_-(n-1,m) + a_n^-\kappa_-(n+1,m) + (\tilde{b}_n^- - b_m^-)\kappa_-(n,m)$$

$$-a_{m-1}^-\kappa_+(n,m-1) - a_m^-\kappa_-(n,m+1), \tag{3.2}$$

where $\kappa_\pm(n,m) = K_\pm(n,m)(K_\pm(n,n))^{-1}$ and $\tilde{a}_n^\pm, \tilde{b}_n^\pm$ are defined by (1.25)–(1.26). Then

$$y_n^\pm(m) + \sum_{l=n\pm1}^{\pm\infty} y_n^\pm(l)F_\pm(l,m) = 0.$$

This lemma can be proved by direct computation with the use of equality $a^{\pm}_{n-1}F_{\pm}(n-1,m)+b^{\pm}_{n}F_{\pm}(n,m)+a^{\pm}_{n}F_{\pm}(n+1,m)=a^{\pm}_{m-1}F_{\pm}(n,m-1)+b^{\pm}_{m}F_{\pm}(n,m)+a^{\pm}_{m}F_{\pm}(n,m+1)$. As a result of these two lemmas we have that $y^{\pm}_{n}(m)=0$ as $\pm m\geq n$. By formulas (3.1) and (3.2) the equality holds $\tilde{L}_{\pm}K_{\pm}=K_{\pm}L_{\pm}$, where $\tilde{L}_{\pm}$ is the operator with the coefficients (1.25)–(1.26). As it is mentioned at the end of Section 2, the condition **F** imply the same property **F** for the kernel $F_{\pm}(n,m)$ of the Marchenko equation. From this equation we can see, that property **F** is valid for $K_{\pm}(n,m)=P(n,m)$. By use of (1.25), (1.26) and (1.23) with $P(n,m)=K(n,m)$ we have (1.27) for the coefficients of operators $\tilde{L}_{\pm}$.
The proof of the following lemma is analogous to the proof of Lemma 3.5.3 [8].

Lemma 3.3. *Let conditions* **A-F** *be satisfied. Then for all* $\lambda\in\mathbb{C}$ *the functions* $\hat{f}^{\pm}_{n}(\lambda)=\sum_{m=n}^{\pm\infty}K_{\pm}(n,m)\hat{\psi}^{\pm}_{m}(\lambda)$ *satisfies the equations* $(\tilde{L}_{\pm}\hat{f})^{\pm}_{n}=\lambda\hat{f}^{\pm}_{n}$.

Lemma 3.4. *Let* $f^{\pm}_{n}(w_{\pm})=\hat{f}^{\pm}_{n}(w_{\pm}(\lambda))$ *be as in Lemma 3.3. Then they satisfy the scattering relations (1.11) with the scattering data (1.24).*

P r o o f. Let $\hat{\psi}^{\pm}_{n}(\lambda)$ be the solutions of equations (1.5), normalized by formulas (2.2)–(2.4) and let $\hat{\phi}^{\pm}_{n}(\lambda)$ be solutions of equation (1.5) such that $\hat{\phi}^{\pm}_{n}(\lambda)=\overline{\hat{\psi}^{\pm}_{n}(\lambda)}$ as $\lambda\in\sigma(L_{\pm})$. Put $\phi^{\pm}_{n}(w_{\pm})=\hat{\phi}^{\pm}_{n}(\lambda(w_{\pm}))$. By (1.6) we have $\phi^{\pm}_{n}\psi^{\pm}_{n}+\phi^{\pm}_{n+1}\psi^{\pm}_{n+1}=2$, $\forall n\in\mathbb{Z}$.

Let $\tilde{R}_{\pm}(n,m)$ be defined as in (1.22). According to condition **F** the functions $\tilde{R}_{\pm}(n,\cdot)$ and

$$\Phi_{\pm}(n,\cdot)=\sum_{l=n}^{\pm\infty}K_{\pm}(n,l)\tilde{R}_{\pm}(l,\cdot)$$

belong to the space $\ell^{2}(\mathbb{Z})$ for any fixed n. Moreover, $\sum_{m\in\mathbb{Z}}\Phi_{\pm}(n,m)\phi^{\pm}_{n}=R_{\pm}(w_{\pm})f^{\pm}_{n}(w_{\pm})$, $w_{\pm}\in\mathbb{T}_{\pm}$. On the other hand, by equation (1.19) for $\pm m>\pm n$ and $w_{\pm}\in\mathbb{T}_{\pm}$ we have

$$\sum_{m\in\mathbb{Z}}\Phi_{\pm}(n,m)\phi^{\pm}_{m}(w_{\pm})$$

$$=\sum_{m=n\mp1}^{\mp\infty}\Phi_{\pm}(n,m)\phi^{\pm}_{m}(w_{\pm})+(K_{\pm}(n,n))^{-1}\phi^{\pm}_{n}(w_{\pm})-\overline{f^{\pm}_{n}}(w_{\pm})$$

$$-\sum_{j=1}^{p}(\alpha^{\pm}_{j})^{2}\hat{f}^{\pm}_{n}(\lambda_{j})\sum_{m=n}^{\pm\infty}\phi^{\pm}_{m}(w_{\pm})\hat{\psi}^{\pm}_{m}(\lambda_{j})-\frac{1}{\pi}\int\limits_{\Delta^{\pm}_{3}}h_{\pm}(\zeta)f^{\pm}_{n}(\zeta)\sum_{m=n}^{\pm\infty}\phi^{\pm}_{m}(w_{\pm})\psi^{\pm}_{m}(\zeta)d\zeta. \quad (3.3)$$

Thus we obtain the equalities as $w_\pm \in \mathbb{T}_\pm$

$$R_\pm f_n^\pm + \overline{f_n^\mp} = T_\pm g_n^\mp, \tag{3.4}$$

where

$$g_n^\mp(w) = \frac{\phi_n^\pm(w)}{T_\pm(w)} \Bigg(\frac{1}{K_\pm(n,n)} + \sum_{m=n\mp1}^{\mp\infty} \Phi_\pm(n,m) \frac{\phi_m^\pm(w)}{\phi_n^\pm(w)}$$

$$- \sum_{j=1}^{p} (\alpha_j^\pm)^2 \hat{f}_n^\pm(\lambda_j) (\phi_n^\pm(w))^{-1} I_n^\pm(w_j, w)$$

$$+ \frac{1}{\pi} \int\limits_{\Delta_3^\pm} h_\pm(\zeta) f_n^\pm(\zeta) (\phi_n^\pm(w))^{-1} I_n^\pm(\zeta, w) d\zeta \Bigg), \tag{3.5}$$

$$I_n^\pm(v,w) = \sum_{m=n}^{\pm\infty} \psi_m^\pm(v)\phi_m^\pm(w), \quad v \in \Delta_3^\pm \bigcup \bigcup_j w_\pm(\lambda_j), w \in \mathbb{T}_\pm.$$

For the Floquet solutions the representation holds $\psi_m^\pm(w) = w^{\pm m} p_m^\pm(w)$, $\phi_m^\pm(w) = w^{\mp m} q_m^\pm(w)$, where $p_m^\pm(w) = p_{m+2}^\pm(w)$ and $q_m^\pm(w) = q_{m+2}^\pm(w)$ are periodic functions. Therefore

$$\sum_{m=n}^{\pm\infty} \psi_m^\pm(v)\phi_m^\pm(w) = S_n^\pm(v,w) \sum_{k=0}^{\pm\infty} v^{\pm 2k} w^{\mp 2k} = S_n^\pm(v,w) \frac{w^2}{w^2 - v^2}, \tag{3.6}$$

where $S_n^\pm(v,w) = \psi_n^\pm(v)\phi_n^\pm(w) + \psi_{n\pm1}^\pm(v)\phi_{n\pm1}^\pm(w)$. Remind, that according to the normalizing conditions $S_n^\pm(v,w) \to 2$ as $w \to v$. We see, the function $\phi_n^\pm(w_\pm))^{-1} I_n^\pm(v, w_\pm)$ as a function of parameter $w_\pm$ can be continued into the domain $\mathbb{P}_\pm \bigcup \partial\mathbb{W}_\pm$ with the singularities in the images of points $\lambda_1, \dots, \lambda_p$ with respect to mapping $w_\pm(\lambda)$. By property **C** the function $g^\mp(w_\pm)$ also can be continued in this domain. Therefore, being considered as a function of λ with notation $\hat{g}^\mp(\lambda)$ it can be continued in the domain $\mathbb{C} \setminus \sigma(L_+) \bigcup \sigma(L_-) \bigcup \{\lambda_1, \dots, \lambda_p\} \bigcup \delta_- \bigcup \gamma_-$. Determine the character of singularities of this function in the points $\lambda_1, \dots, \lambda_p$. Since $S_n(w_\pm(\lambda_j), w_\pm(\lambda_j)) = 2$ then by (3.5) and (3.6)

$$\lim_{\lambda\to\lambda_j} \hat{g}_n^\pm(\lambda) = 2\hat{f}_n^\pm(\lambda_j)(\alpha_j^\pm)^2 w_\pm(\lambda_j) \lim_{\lambda\to\lambda_j} (2T^\pm(w_\pm(\lambda))(w_\pm(\lambda) - w_\pm(\lambda_j)))^{-1}$$

$$= \hat{f}_n^\pm(\lambda_j)(\alpha_j^\pm)^2 w_\pm(\lambda_j) \lim_{\lambda\to\lambda_j} \frac{i\dot{\theta}_\pm(\lambda)\dot{W}(\lambda)}{i\dot{\theta}_\pm(\lambda) w_\pm(\lambda)} = \hat{f}_n^\pm(\lambda_j)(\alpha_j^\pm)^2 \dot{W}(\lambda_j). \tag{3.7}$$

Therefore, the functions $\hat{g}_n^{\pm}$ have removable singularities at the points λ_j. Besides, it takes real values as $\lambda \in \mathbb{R}\backslash(\sigma(L_+)\bigcup\sigma(L_-)\bigcup\{\lambda_1,\dots,\lambda_p\}\bigcup\delta_-\bigcup\gamma_-)$. From **A**, (3.4), and (1.12) it follows that $g_n^{\mp}(w_\pm)-g_n^{\mp}(w_\pm^{-1})=0$ as $w_\pm\in\Delta_1^{\pm}$ and, therefore, the function $\hat{g}_n^{\mp}$ has no a jump on the band $[\mu_\pm,\nu_\pm]$. Thus, the function $g_n^{\mp}$ is a single valued function of parameter $w_\mp$ as $w_\mp\in\mathbb{W}_\mp$. Property **F** implies that $\sup_{\pm n>0}\sum_{m=n\mp1}^{\mp\infty}|\Phi_\pm(n,m)|<\infty$ and, therefore $g^{\mp}\in\ell^2(\mathbb{Z}^{\mp})$ as $w_\pm\in\mathbb{P}^{\pm}\setminus\{0\}$. From equality (3.4) it follows that $(1-|R_\pm|^2)f_n^{\pm}=\overline{T_\pm g_n^{\mp}}-\overline{R_\pm}T_\pm g_n^{\mp}$, and by (1.14) as $w_\pm\in\Delta_2^{\pm}$ one has $|T_\pm|^2 f_n^{\pm}\dot\theta\pm(\dot\theta\mp)^{-1}=\overline{T_\pm g_n^{\mp}}-\overline{R_\pm}T_\pm g_n^{\mp}$. Therefore, due to (1.14) and (1.17)

$$T_\mp f_n^{\pm}=\overline{g_n^{\mp}}-T_\mp g_n^{\mp}\frac{\overline{R_\pm}\dot\theta_\mp}{\overline{T_\pm}\dot\theta_\pm}=\overline{g_n^{\mp}}+R_\mp g_n^{\mp},\quad w_\pm\in\Delta_2^{\pm}$$

and eliminating the reflection coefficients from (3.4), we obtain the equalities

$$T_\mp(f_n^{\pm}f_n^{\mp}-g_n^{\pm}g_n^{\mp})=(\overline{g_n^{\mp}}f_n^{\mp}-\overline{f_n^{\mp}}g_n^{\mp}),\qquad w_\pm\in\Delta_2^{\pm}.\tag{3.8}$$

The further considerations are identical for (+) and (–) equalities (3.8), and we give them for (+)-case. Consider the function $p_n(w_+)=T^+(f_n^+f_n^- - g_n^+g_n^-)$ which is equal to

$$p_n(w_+)=\overline{g_n^+}f_n^+-\overline{f_n^+}g_n^+\qquad\text{as } w_+\in\Delta_2^+.\tag{3.9}$$

With respect to variable λ the function p_n can be continued as meromorphic function $\tilde{p}_n(\lambda)$ on the upper sheet Γ of the hyperelliptic Riemman surface, associated with the function $((\lambda-\mu_+)(\lambda-\nu_+)(\lambda-\mu)(\lambda-\nu)(\lambda-\mu_-)(\lambda-\nu_-))^{1/2}$. To prove it, we use non-normalizing Weyl functions $\tilde\psi_n^{\pm}(\lambda)$, defined by formula (2.2) and $\tilde\phi_n^{\pm}(\lambda)=c_n+((s_2^{\pm}-c_1^{\pm})\pm2(u_\pm^2-1)^{1/2})(2s_1^{\pm})^{-1}s_n$ that are meromorphic on Γ. The same is valid for non-normalizing Jost solutions $\tilde f_n^{\pm}=\sum K_\pm(n,m)\tilde\psi_m^{\pm}$. From (3.5) and (3.7) we see, that $g_n^{\pm}(w)=\phi_n^{\pm}(w)(T_\pm(w))^{-1}S_\pm(n,\lambda)$, where $S_\pm(n,\lambda)$ is a meromorphic function on Γ with the poles in the points $\lambda_1,\dots,\lambda_p$. Put $\tilde T_\pm(\lambda)=\langle\tilde f^+,\tilde f^-\rangle^{-1}\langle\tilde f^{\pm},\bar{\tilde f}^{\pm}\rangle$. Formulas (2.4), (3.7) and (3.9) then imply that the function

$$\tilde p_n(\lambda)=s_1^{\pm}(\dot u_+)^{-1}\tilde T_+(\tilde\phi_n^+\tilde\phi_n^-(\tilde T_+\tilde T_-)^{-1}S_+(n)S_-(n)-\tilde f_n^+\tilde f_n^-)\tag{3.10}$$

is meromorphic on Γ with the pole at the point γ_+ and it has removable singularities at the points $\lambda_1,\dots,\lambda_p,\gamma_-,\delta_-,\delta_+$. Consider the behavior of this function as $\lambda\to\infty$, that is $w_\pm\to0$. By (1.8), (2.12) and (2.5) we see, that $f_n^+(0)f_n^-(0)=K_+(n,n)K_-(n,n)(a_{-1}^+\cdots a_{n-1}^+)(a_{-1}^-\cdots a_{n-1}^-)^{-1}$ and, by (1.18), (3.5)

$$g_n^+(0)g_n^-(0)=(T^+(0)T^-(0)K_+(n,n)K_-(n,n))^{-1}(a_{-1}^+\cdots a_{n-1}^+)^{-1}(a_{-1}^-\cdots a_{n-1}^-)$$

then $p_n(0) = 0$. Compute the jump of the function $\tilde{p}_n(\lambda)$ on the set $[\mu_-, \nu_-]$, i.e., the jump of the function $p_n(w)$ on the set Δ_3^+. Since the functions f_n^+ and g_n^+ have no jumps on this set, we consider the functions $T_+ f_n^-$ and $T_+ g_n^-$. According to **A**, (1.12), (3.4) and(1.17) we have as $w \in \Delta_3^+$

$$T_+ f_n^-(w + i0) - T_+ f_n^-(w - i0) = \frac{\dot{\theta}_-}{\dot{\theta}_+}(T_- f_n^- + \overline{T_- f_n^-})$$

$$= \frac{\dot{\theta}_-}{\dot{\theta}_+}(T_- f_n^- + |T_-|^2 g_n^+ - R_- \overline{T_-} f_n^-) = \frac{\dot{\theta}_-}{\dot{\theta}_+}|T_-|^2 g_n^+(w + i0).$$

From the other side, the only term for $T_+ g_n^-$, having the jump due to representation (3.5), is the last term. Compute the jump of function

$$h_n(w) = \frac{1}{\pi} \int\limits_{\Delta_3^\pm} h_\pm(\zeta) f_n^\pm(\zeta) I_n^\pm(\zeta, w) d\zeta).$$

From formula (3.6) using the Plemelj formula, we have $h_n(w+i0) - h_n(w-i0) = -\dot{\theta}_\pm |T_\pm|^2 (\dot{\theta}_\mp)^{-1} f_n^\pm(w + i0)$. Therefore, $p_n(w + i0) - p_n(w - i0) = f_n^+ g_n^+ (T_+ \overline{T_-} + T_- \overline{T_+}) = 0$ as $w \in \Delta_3^+$. Since the function $i\dot{\theta}_+ < 0$ has no jump on this interval, we conclude from (1.17), that the function $Q_n = i\dot{\theta}_+ p_n = i\dot{\theta}_+ T_+ (f_n^+ f_n^- - g_n^+ g_n^-) = i\dot{\theta}_- T_- (f_n^+ f_n^- - g_n^+ g_n^-)$ considered as the function of λ has no jumps on the bands $[\mu_+, \nu_+]$ and $[\mu_-, \nu_-]$. Besides, from (1.15), (3.10) and (1.12) we see that this function is holomorphic on the upper sheet of the Riemann surface of function $((\lambda - \mu)(\lambda - \nu))^{1/2}$. Consider the Jukovski transformation $z(\lambda)$, $z : [\mu, \nu] \to \mathbb{T}$. The function Q as a function of z is holomorphic in $\mathbb{D}$ and continuous up to the boundary. Due to (3.9) this function is odd as $z \in \mathbb{T} : Q_n(z^{-1}) = -Q_n(z)$. These properties allow us to continue the function Q_n in the domain $\mathbb{C} \setminus \mathbb{D}$. It is holomorphic in $\mathbb{C}$ and $Q_n(z) \to 0$ as $z \to \infty$. Thus by the Liouville theorem we have equalities $\hat{g}_n^+ \hat{g}_n^- = \hat{f}_n^+ \hat{f}_n^-$, $\lambda \in \mathbb{C}$ and $\overline{\hat{g}_n^+} \hat{f}_n^+ = \overline{\hat{f}_n^+} \hat{g}_n^+$, $\lambda \in \sigma(L_+)$. Applying now the arguments of [8, p. 279], we conclude that $\hat{g}_n^\pm = \hat{f}_n^\pm$ and, therefore, scattering relations (1.11) are fulfilled. Theorem 1.1 is proved.

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