Matematicheskaya fizika, analiz, geometriya
2003, v. 10, No. 4, p. 469–480

On the regularity of the Bäcklund transformation for pseudospherical surfaces

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Received May 17, 2002

We give estimates concerning the regularity of the Bäcklund image of regular regions of $L^2 \subset E^3$ and $L^2 \subset S^3$.

1. Introduction

A large class of integrable systems is of geometric origin. In fact one can see a growing overlapping of the classical differential geometry of immersions with the modern theory of integrable equations.

In this paper we consider the Bäcklund transformation for isometric immersions of L^2 into E^3 and S^3. We derive corresponding differential equations (reconstructing an old result of Bianchi) and show that the angles between normal planes at a point and at its Bäcklund image are constant. We propose to study

Mathematics Subject Classification 2000: 53A07 (primary), 37K25, 37K35 (secondary).
[1]Author supported partially by the State Fund of Fundamental Research of Ukraine (project No. 01.07/00132).
[2]The research supported partially by the KBN (grant 2 P03B 126 22).
Key words and phrases: Lobachevsky spaces, Bäcklund transformation, regularity, estimates, geodesic disc.

the regularity of the image of a region of L^n in the Bäcklund transformation. We estimate the range of the regular image in terms of the geometry (e.g., curvatures) of the considered region.

2. The Bäcklund transformation for pseudospherical surfaces

The Bianchi transformation for isometric immersions of L^n into E^{2n-1} has been constructed by Aminov [1]. Terng and Tenenblat proved an existence of the Bäcklund transformation for immersions of L^n into E^{2n-1} [14], and Tenenblat obtained the Bäcklund transformation for immersions of L^n into space forms [12]. In this section we present a short and straightforward derivation of the system of differential equations describing the Bäcklund transformation in the case of immersions of L^2 into S^3. In the next section we are going to study some interesting properties of this system. The system has been first written by Bianchi (compare [6, 7, p. 378, formula(A)], where these equations are given without proof).

Theorem 1. *Let F^2 be a regular surface with Gaussian curvature $K = -1$ in a sphere S^3 of radius 1. F^2 is described by the radius vector $r = r(u, v)$. In the curvature coordinates u, v the metric of F^2 can be written as*

$$ds^2 = \cos^2 \omega du^2 + \sin^2 \omega dv^2 \ . \tag{1}$$

Define $\tau := \cos\varphi\tau_1 + \sin\varphi\tau_2$, where τ_1 and τ_2 are unit vectors along principal directions. Let φ satisfy the following system of equations:

$$\begin{aligned} \frac{\partial\varphi}{\partial u} + \frac{\partial\omega}{\partial v} &= \alpha\cos\omega\sin\varphi + \beta\sin\omega\cos\varphi \ , \\ \frac{\partial\varphi}{\partial v} + \frac{\partial\omega}{\partial u} &= -\alpha\sin\omega\cos\varphi - \beta\cos\omega\sin\varphi \ , \end{aligned} \tag{2}$$

where α and β are constants such that $\alpha^2 - \beta^2 = 1$. Then the transformation $\psi : r \to \tilde{r}$, where

$$\tilde{r} = r\sin\sigma + \tau\cos\sigma \ , \qquad \sigma = \arctan\alpha \ , \tag{3}$$

transforms F^2 into a surface $\tilde{F}^2$ with $\tilde{K} = -1$.

Therefore, ψ is the Bäcklund transformation for pseudospherical surfaces in a sphere. The point of F^2 given by the radius vector r is mapped into the point $\tilde{r}$. The length of the great circle arc γ between r and $\tilde{r}$ is constant, and τ is tangent to γ.

Remark 1. *We note that if ω belongs to the regularity class C^{k-1} then φ belongs to the class C^{k-1} as well.*

P r o o f. Let $b = -\tau_1 \sin\varphi + \tau_2 \cos\varphi$ be unit tangent vector orthogonal to τ. Taking into account the Gauss decomposition for second derivatives of r, we obtain

$$\tilde{r}_{,u} = r_{,u} \sin\sigma + \cos\sigma(Ab + A^1 n_1 + A^2 n_2) ,$$

$$\tilde{r}_{,v} = r_{,v} \sin\sigma + \cos\sigma(Bb + B^1 n_1 + B^2 n_2) ,$$

where

$$\begin{aligned} A &= \tfrac{\partial\varphi}{\partial u} + \tfrac{\partial\omega}{\partial v} , \qquad & B &= \tfrac{\partial\varphi}{\partial v} + \tfrac{\partial\omega}{\partial u} , \\ A^\mu &= \tfrac{\cos\varphi L^\mu_{11}}{\cos\omega} , \qquad & B^\mu &= \tfrac{\sin\varphi L^\mu_{22}}{\sin\omega} . \end{aligned} \tag{4}$$

Here n_μ are normal vectors to F^2 and L^μ_{ii} are coefficients of its second quadratic forms. From general theory of surfaces:

$$L^1_{11} = -L^1_{22} = \sin\omega\cos\omega\,; \quad L^2_{11} = \cos^2\omega\,, \quad L^2_{22} = \sin^2\omega\,.$$

Let $c = (-r\cos\sigma + \tau\sin\sigma)$ be the unit vector tangent to the arc γ in the point $\tilde{r}$. We assume (3) and (4). To construct the Bäcklund transformation we impose two additional conditions:

1. Vector c is tangent to $\tilde{F}^2$.

2. The integrability conditions for the system (2) are given by $K = -1$.

Consider the first condition. We easily see that $\tilde{r}$ and unit vector $\xi := \lambda[\tilde{r}_{,u}\,\tilde{r}_{,v}\,\tilde{r}]$ (where $\lambda := 1/|[\tilde{r}_{,u}\,\tilde{r}_{,v}\,\tilde{r}]|$) are normal vectors to $\tilde{F}^2$. The vector c is by definition orthogonal to $\tilde{r}$. Hence the first conditions means that $(\xi c) = 0$. Substituting c and $\tilde{r}$, we have

$$0 = (\xi c) = \lambda(\tilde{r}_{,u}\,\tilde{r}_{,v}\,\tilde{r}\,\tau) = (\tau_1\tau_2 n_1 n_2)\cos\sigma\Lambda ,$$

where

$$\Lambda = \sin\sigma(\sin\varphi B^1\sqrt{g_{11}} + \cos\varphi A^1\sqrt{g_{22}}) + \cos\sigma(A^1 B - B^1 A) .$$

Therefore $\Lambda = 0$. Taking into account (4) we substitute $A^1 = \sin\omega\cos\varphi$, $B^1 = -\cos\omega\sin\varphi$, $\sqrt{g_{11}} = \cos\omega$, $\sqrt{g_{22}} = \sin\omega$. Then the equation $\Lambda = 0$ assumes the form

$$\cos\sigma(A\cos\omega\sin\varphi + B\sin\omega\cos\varphi) + \sin\sigma(\cos^2\varphi\sin^2\omega - \sin^2\varphi\cos^2\omega) = 0.$$

Finally, using (2), we substitute $A = \alpha\cos\omega\sin\varphi + \beta\sin\omega\cos\varphi$ and $B = -\alpha\sin\omega\cos\varphi - \beta\cos\omega\sin\varphi$:

$$(\alpha\cos\sigma - \sin\sigma)(\cos^2\omega - \cos^2\varphi) = 0 .$$

Hence: $\alpha = \tan\sigma$, β arbitrary (note that the case $\omega = \pm\varphi$ is not possible because then from (2) we have $\varphi = 0$ which is excluded by the assumption of regularity).

Let us proceed to the second condition. The compatibility conditions for equations (2) are given by

$$\omega_{,vv} - \omega_{,uu} = (\beta^2 - \alpha^2)\sin\omega\cos\omega \ .$$

If (according to our asumption) $\alpha^2 - \beta^2 = 1$, then this condition is equivalent to the sine-Gordon equation, i.e., $K = -1$. Note that it implies $|\alpha| > 1$ or $\tan\sigma > 1$.

Let us compute the metric of $\tilde{F}^2$. The tangent vectors are given by:

$$\tilde{r}_{,u} = \cos\varphi\,\{\tau_1(\cos\omega\cos\varphi\sin\sigma - \beta\cos\sigma\sin\omega\sin\varphi)$$

$$+\tau_2\cos\sigma(\alpha\cos\omega\sin\varphi + \beta\sin\omega\cos\varphi)\} + \cos\sigma A^\nu n_\nu \ ,$$

$$\tilde{r}_{,v} = \sin\varphi\,\{\tau_1\cos\sigma(\alpha\sin\omega\cos\varphi + \beta\cos\omega\sin\varphi)$$

$$+\tau_2(\sin\omega\sin\varphi\sin\sigma - \beta\cos\sigma\cos\varphi\cos\omega)\} + \cos\sigma B^\nu n_\nu \ .$$

Now, we will find $\tilde{E} = \tilde{r}_{,u}^2$:

$$\tilde{E} = \cos^2\varphi(\cos^2\omega\sin^2\sigma + \beta^2\cos^2\sigma\sin^2\omega) + \cos^2\sigma((A^1)^2 + (A^2)^2) \ ,$$

and, because $(A^1)^2 + (A^2)^2 = \cos^2\varphi(\sin^2\omega + cos^2\omega)$, we have $\tilde{E} = \cos^2\varphi$. Similarly, $\tilde{F} = 0$ and $\tilde{G} = \sin^2\varphi$. Finally,

$$d\tilde{s}^2 = \cos^2\varphi du^2 + \sin^2\varphi dv^2 \ .$$

To find the Gaussian curvature of this metric it is necessary to consider (2). The compatibility conditions (now for the existence of a solution ω) read

$$\varphi_{,uu} - \varphi_{,vv} = (\alpha^2 - \beta^2)\cos\varphi\sin\varphi \ ,$$

which means, because of $\alpha^2 - \beta^2 = 1$, that the Gaussian curvature of $\tilde{F}^2$ equals $\tilde{K} = -1$. The proof is completed.

Remark 2. *Because the regularity classes of φ and ω are the same, we construct by the above procedure an infinite sequence of metrics of the class C^{k-1} (provided that we start from the immersion F^2 of the class C^k).*

Remark 3. *The above theorem is valid for pseudospherical immersions in E^3 as well. In this case*

$$\alpha = \frac{1}{\cos\sigma} \ , \qquad \beta = \frac{\sin\sigma}{\cos\sigma} \ , \tag{5}$$

where $\cos\sigma$ is the length of the segment joining the points corresponding under the Bäcklund transformation. The differential equations defining the Bäcklund transformation in the Euclidean case are the same as the equations in S^3-case.

3. Normal planes

It is well known that in the case of local isometric immersions of L^n into E^{2n-1} the angles $\theta_1, \dots, \theta_{n-1}$ between the normal space N_X at a point $X \in L^n \subset E^{2n-1}$ and the normal space at its Bäcklund transform $\tilde{N}_{\tilde{X}}$ are all equal: $\theta_1 = \theta_2 = \dots = \theta_{n-1}$ [14]. Now, we are going to consider this problem in the case of local isometric immersions of L^2 into $S^3 \subset E^4$.

Lemma 1 (Compare [5, Sect. 6]). *Let $T, E \subset E^4$ be two planes, and suppose that we choose orthonormal bases $e_1, e_2 \subset T$ and $f_1, f_2 \subset E$ in such a way that e_1, e_2 are projected onto E along f_1, f_2, respectively. Then the angles θ_j between e_j and f_j are exactly (principal) angles between T and E.*

Let N_X be the normal plane at the point $X \in E^4$ described by the radius vector r (we assume $|r| = 1$) and $\tilde{N}_{\tilde{X}}$ be the normal plane at its Bäcklund image $\tilde{X} \in E^4$ described by $\tilde{r}$ (and $|\tilde{r}| = 1$). Consider the orthogonal basis $r, n_1 \in N_X$, where $n_1 := \mu[r_{,u}\, r_{,v}\, r]$ (μ is the normalizing factor, i.e., $|n_1| = 1$). Similarly, in the space $\tilde{N}_{\tilde{X}}$ we have orthogonal basis $\tilde{r}, \xi$, where $\xi := \nu[\tilde{r}_{,u}\, \tilde{r}_{,v}\, \tilde{r}]$ (where ν normalizes ξ, i.e., $|\xi| = 1$, we assume $\nu > 0$). Therefore $r \perp n_1$ and $\tilde{r} \perp \xi$. Moreover, because of $\tilde{r} = r \sin\sigma + \tau \cos\sigma$, we have also $\tilde{r} \perp n_1$. Thus the angles between r and $\tilde{r}$ and between n_1 and ξ are exactly the (principal) angles between N_X and $\tilde{N}_{\tilde{X}}$. In the sequel we denote them by θ_1 and θ_2.

The first angle can be computed immediately:

$$\cos\theta_1 = \frac{(r\,\tilde{r})}{|r||\tilde{r}|} = \sin\sigma\,.$$

Thus $\theta_1 = \frac{\pi}{2} - \sigma$. Similarly

$$\cos\theta_2 = (\xi\, n_1) = \nu(\tilde{r}_{,u}\, \tilde{r}_{,v}\, \tilde{r}\, n_1)\,.$$

A straightforward (although cumbersome) calculation yields

$$(\tilde{r}_{,u}\, \tilde{r}_{,v}\, \tilde{r} n_1) = \beta \cos\sigma \sin\varphi \cos\varphi\,,$$

moreover,

$$\nu = \frac{1}{|[\tilde{r}_{,u}\, \tilde{r}_{,v}\, \tilde{r}]|} = \frac{1}{\sin\varphi\cos\varphi}\,.$$

Hence:

Corollary 1. *The principal angles between N_X and $\tilde{N}_{\tilde{X}}$ are constant (do not depend on X) but $\theta_1 \neq \theta_2$:*

$$\cos\theta_1 = \sin\sigma = \alpha\cos\sigma\ ,$$

$$\cos\theta_2 = \tfrac{\beta}{\alpha}\sin\sigma = \beta\cos\sigma.$$

4. The regularity of the Bäcklund transformation image

Given a pseudosherical immersion F^2 in curvature coordinates with the metric (1) let us consider (2) as a system of equations for the unknown φ. Let $T(t)$ be a geodesic disc on F^2 with the radius t and φ_0 be an initial value of φ taken in the center O of $T(t)$. We assume $0 < \varphi_0 < \pi/2$. The surface $\tilde{F}^2$ is regular at the point corresponding to O. Let us formulate the following problem:

What is an estimate for the radius t_r such that the Bäcklund transform of $T(t_r)$ is a regular surface ?

We are going to express this estimate in terms of geometry of F^2 and the initial value of φ_0.

Theorem 2. *Let a pseudospherical immersion F^2 with principal curvatures κ_1, κ_2 belongs to the class of regularity C^k $(k \geq 2)$ and let there exist positive constants κ_0 and N such that*

$$|\kappa_i| \leq \kappa_0 \,, \qquad \left| \frac{\partial \log \sqrt{1+\kappa_i^2}}{\partial s} \right| \leq N \,, \quad i = 1, 2,$$

for any curve Γ starting from a fixed point $O \in F^2$ (where s is the arc length parameter along Γ). Let us denote $M_1(\varphi_0) := \min(\varphi_0, \frac{\pi}{2} - \varphi_0)$ and define

$$t_r = \frac{M_1(\varphi_0)}{\alpha + \beta\sqrt{1+\kappa_0^2} + 2N} \ .$$

Then the image of the geodesic disc $T(t_r)$ (with the center O) under the Bäcklund transformation is a regular surface of the class C^{k-1}. If F^2 is analytic, then its image is analytic as well.

In the remark to his well-known article "On the surfaces of constant Gauss curvature" D. Hilbert wrote that by his proposition G. Lütkemeyer proved the fact of existence of nonanalitic surfaces of constant negative curvature.

If $F^2 \in C^k$, then $\omega \in C^{k-1}$ and the solution φ of the system (2) is also from the class C^{k-1}. Note that the moduli of $\varphi_{,u}$ and $\varphi_{,v}$ given by (2) have uniform estimates from above in the disc $T(t)$. These estimates do not depend on φ. Therefore the solution φ of (2) exists in the whole disc $T(t)$ and the vector function $\tilde{r}$ given by (3) belongs to the class C^{k-1}. So, the surface $\tilde{F}^2$ will be regular if the coefficients of its metric $d\tilde{s}^2 = \cos^2 \varphi du^2 + \sin^2 \varphi dv^2$ do not vanish. In the sequel we estimate the region where this assertion is true.

If the principal curvatures, as in our case, are bounded from above (i.e., $|\kappa_i| \leq \kappa_0$), then the radius of the geodesic circle is bounded from above as well. It follows from the following theorem

Theorem 3 (Efimov [10]). *Let the Gaussian curvature K satisfies (at any point of the geodesic circle of the radius t) $K \le -1$ and $|k_i| \le k_0$. Then*

$$t \le \pi\sqrt{1+\kappa_0^2}\,. \tag{6}$$

Our estimate obtained in the context of the Bäcklund transform, bounds the radius of the geodesic circle from below. Therefore both these results are, in a sense, complementary.

The proof of Theorem 2 will be divided into a series of lemmas.

Lemma 2. *Let $\Gamma \subset F^2$ be a geodesic curve with the beginning at the center of $T(t)$ and $\tilde{\Gamma} \subset \tilde{F}^2$ be its Bäcklund image. Let $\theta(t)$ be the angle between Γ and the first principal direction, and $\tilde{t}$ be the arc length parameter on $\tilde{\Gamma}$. Then the following estimate holds*

$$|\varphi - \varphi_0 - \theta + \theta_0| \le \alpha t + \beta \tilde{t}\ .$$

Remark 4. *The angle $\varphi - \theta$ has a clear geometric interpretation as the angle between the vector τ pointing to the Bäcklund image and the considered geodesic.*

To prove Lemma 2 we add the first equation of (2) multiplied by du/ds and the second equation multiplied by dv/ds. Thus

$$\begin{aligned} \tfrac{d\varphi}{ds} &= -Q + \alpha\left(\cos\omega\sin\varphi\tfrac{du}{ds} - \sin\omega\cos\varphi\tfrac{dv}{ds}\right) \\ &+\beta\left(\sin\omega\cos\varphi\tfrac{du}{ds} - \cos\omega\sin\varphi\tfrac{dv}{ds}\right)\ , \end{aligned} \tag{7}$$

where $Q = \frac{\partial\omega}{\partial v}\frac{du}{ds} + \frac{\partial\omega}{\partial u}\frac{dv}{ds}$. The system of equations for geodesic curves of the metric $ds^2 = \cos^2\omega du^2 + \sin^2\omega dv^2$ has the following form:

$$\begin{aligned} &\tfrac{d^2u}{ds^2} - \tan\omega\left(\tfrac{d\omega}{ds}\tfrac{du}{ds} + \tfrac{dv}{ds}Q\right) = 0\ , \\ &\tfrac{d^2v}{ds^2} + \cot\omega\left(\tfrac{du}{ds}Q + \tfrac{dv}{ds}\tfrac{d\omega}{ds}\right) = 0\ . \end{aligned} \tag{8}$$

The system (8) can be rewritten as

$$\begin{aligned} &\tfrac{d}{ds}\left(\cos\omega\tfrac{du}{ds}\right) = \sin\omega\tfrac{dv}{ds}Q\ , \\ &\tfrac{d}{ds}\left(\sin\omega\tfrac{dv}{ds}\right) = -\cos\omega\tfrac{du}{ds}Q\ . \end{aligned} \tag{9}$$

We denote

$$A = \cos\omega\frac{du}{ds}\ , \qquad B = \sin\omega\frac{dv}{ds}\ .$$

Because of $A^2 + B^2 = 1$ there exists θ such that

$$A = \cos\theta\ , \qquad B = \sin\theta\ , \tag{10}$$

and the system (9) transforms into the following single equation:

$$\frac{d\theta}{ds} = -Q\ . \tag{11}$$

By considering the infinithesimal rectangular triangle with the hypotenuse along Γ and other sides along principal directions we can see that θ is the angle between Γ and u-curve (i.e., $v = \text{const}$).

Let $\tilde{\Gamma} \subset \tilde{F}^2$ be the image of Γ under the Bäcklund transformation and $\tilde{\theta}$ be the angle between $\tilde{\Gamma}$ and u-curve on $\tilde{F}^2$. The metric of $\tilde{F}^2$ has the form $d\tilde{s}^2 = \cos^2\varphi du^2 + \sin^2\varphi dv^2$. Hence

$$\cos\tilde{\theta} = \cos\varphi \frac{du}{d\tilde{s}}\ , \quad \sin\tilde{\theta} = \sin\varphi\frac{dv}{d\tilde{s}}\ ,$$

and the equation (7) assumes the form

$$\frac{d(\varphi - \theta)}{ds} = \alpha\sin(\varphi - \theta) + \beta\sin(\omega - \tilde{\theta})\frac{d\tilde{s}}{ds}\ . \tag{12}$$

Integrating this equation along Γ starting from O and taking obvious estimates, we obtain Lemma 2.

Lemma 3. *If the moduli of the principal curvatures κ_i of F^2 are limited from above, i.e., $|\kappa_i| \leq \kappa_0 = \text{const}$, then*

$$\tilde{t} \leq t\sqrt{1 + \kappa_0^2}\ . \tag{13}$$

To prove Lemma 3 we take into account that

$$\kappa_1 = \tan\omega\ , \qquad \kappa_2 = \cot\omega\ ,$$

which implies

$$\frac{1}{\cos^2\omega} \leq 1 + \kappa_0^2\ , \qquad \frac{1}{\sin^2\omega} \leq 1 + \kappa_0^2\ .$$

If $L = \min_{F^2}(\cos^2\omega, \sin^2\omega)$, then $1/L \leq 1 + \kappa_0^2$. Finally, from

$$\frac{d\tilde{s}^2}{ds^2} = \frac{\cos^2\varphi du^2 + \sin^2\varphi dv^2}{\cos^2\omega du^2 + \sin^2\omega dv^2} \leq \frac{du^2 + dv^2}{L(du^2 + dv^2)} = \frac{1}{L}$$

we obtain (13).

Lemma 4. *Let*

$$N = \max_{T(t),\nu} \left| \frac{\partial \log \sqrt{1+\kappa_0^2}}{\partial s} \right| ,$$

where $\partial/\partial s$ is the derivative with respect the arc length for arbitrary direction ν. Then

$$|\theta - \theta_0| \le 2Nt.$$

Let $\partial/\partial s_1$ and $\partial/\partial s_2$ are derivatives along the lines of curvature with respect to the length of arcs. It is easy to find

$$\max_{T(t)} \left(\left| \frac{\partial \log \sin \omega}{\partial s_1} \right|, \left| \frac{\partial \log \cos \omega}{\partial s_2} \right| \right) \le N \ .$$

Rewriting the equation (11), we have

$$\frac{d\theta}{ds} = \frac{\partial \omega}{\partial s_1} \frac{\cos \omega}{\sin \omega} \sin \omega \frac{dv}{ds} - \frac{\partial \omega}{\partial s_2} \frac{\sin \omega}{\cos \omega} \cos \omega \frac{du}{ds} .$$

Consequently,

$$|\theta - \theta_0| \le \left| \int_0^t \frac{d\theta}{ds} ds \right| \le 2Nt \ .$$

Let us return to the proof of Theorem 2. By Lemmas 2–4 we have

$$|\varphi - \varphi_0| \le |\varphi - \varphi_0 - \theta + \theta_0| + |\theta - \theta_0| \le (\alpha + \beta\sqrt{1+\kappa_0^2} + 2N)t. \tag{14}$$

Therefore by the assumption of the theorem we have

$$|\varphi - \varphi_0| < \min(\varphi_0, \frac{\pi}{2} - \varphi_0) \ .$$

Hence, the function φ satisfies $0 < \varphi < \pi/2$. The coefficients of the metric $d\tilde{s}^2 = \cos^2 \varphi du^2 + \sin^2 \varphi dv^2$ are different from zero and, therefore, $\tilde{F}^2$ is a regular surface. In consequence of $\varphi \in C^{k-1}$ we have $\tilde{F}^2 \in C^{k-1}$. The proof is completed.

The next problem is to estimate the radius of a geodesic disc on $\tilde{F}^2$ which does not contain singular points.

Theorem 4. *Let*

$$M_1(\varphi_0) = \min(\varphi_0, \frac{\pi}{2} - \varphi_0), \quad M_2(\varphi_0) = \min\left(\sin \frac{\varphi_0}{2}, \cos(\frac{\pi}{4} + \frac{\varphi_0}{2})\right) .$$

Then the radius $\tilde{\rho}$ of the geodesic disc on $\tilde{F}^2$ which does not contain singular points satisfies the inequality

$$\tilde{\rho} \ge \frac{M_1(\varphi_0) M_2(\varphi_0)}{2(\alpha + \beta\sqrt{1+\kappa_0^2} + 2N)} \ .$$

Indeed, let us consider the Bäcklund transform $\tilde{T}(t_r/2)$ of the geodesic disc of radius $t_r/2$ contained in the disc $T(t_r)$. Let $\tilde{\gamma}$ be the shortest curve joining $\tilde{O}$ and the boundary of $\tilde{T}(t_r/2)$. We denote its length by $\tilde{\rho}$. We are going to find an estimate for $\tilde{\rho}$. Let γ (of the length ρ) be the co-image of $\tilde{\gamma}$. Note that in general γ is not geodesic on F^2 but anyway $t_r \le 2\rho$. The arc length elements of γ and $\tilde{\gamma}$ satisfy

$$\frac{ds^2}{d\tilde{s}^2} = \frac{\cos^2\omega du^2 + \sin^2\omega dv^2}{\cos^2\varphi du^2 + \sin^2\varphi dv^2} \le \frac{1}{\Lambda^2} ,$$

where $\Lambda^2 := \min(\sin^2\varphi, \cos^2\varphi)$. Therefore $\Lambda ds \le d\tilde{s}$ and integrating this inequality along γ and $\tilde{\gamma}$, we obtain

$$\Lambda\rho \le \tilde{\rho} .$$

Let us estimate Λ from below. Applying the estimate (14) to the disc $T(t_r/2)$, we get

$$|\varphi - \varphi_0| \le (\alpha + \beta\sqrt{1+\kappa_0^2} + 2N)\frac{t_r}{2} .$$

We can fix

$$\frac{t_r}{2} = \frac{M_1(\varphi_0)}{2(\alpha + \beta\sqrt{1+\kappa_0^2} + 2N)} , \tag{15}$$

which means that

$$|\varphi - \varphi_0| \le \frac{\varphi_0}{2} , \quad |\varphi - \varphi_0| \le \frac{\pi}{4} - \frac{\varphi_0}{2} .$$

Therefore

$$\varphi \ge \varphi_0 - |\varphi - \varphi_0| \ge \frac{\varphi_0}{2} .$$

Obviously,

$$\frac{\pi}{2} - \varphi = \frac{\pi}{2} - \varphi_0 + \varphi_0 - \varphi \ge \frac{\pi}{2} - \varphi_0 - |\varphi - \varphi_0| \ge \frac{\pi}{4} - \frac{\varphi_0}{2} ,$$

i.e., $\varphi \le \frac{\pi}{4} + \frac{\varphi_0}{2}$. Therefore $\Lambda \ge M_2(\varphi_0)$. Hence

$$\frac{1}{2} t_r M_2(\varphi_0) \le \Lambda\rho \le \tilde{\rho}.$$

The proof is finished.

Let us consider a special case of the Bäcklund transformation (2), namely $\beta = 0$. In this case θ_2, defined in the previous chapter, equals $\pi/2$ and the equation (12) for the derivative along a geodesic line assumes the form:

$$\frac{d(\varphi - \theta)}{ds} = \sin(\varphi - \theta) .$$

Solving this equation, we obtain either

$$\varphi - \theta = 2 \arctan e^{s-s_0} \ ,$$

or $\varphi \equiv \theta$ (which can be treated as a limiting case of the general solution for $s_0 \to \infty$). Hence we have the following result.

Corollary 2. *In the case $\beta = 0$ the integral curves of the vector field τ (vectors pointing to the Bäcklund images) are either geodesics (the case $\varphi \equiv \theta$) or the total change (when s is changing from $-\infty$ to $+\infty$) of the angle between τ and a given geodesic equals π.*

In the conclusion let us remark that it would be interesting to estimate the range of the regular Bäcklund image after a sequence of transformations.

Acknowledgements. We are grateful to Antoni Sym for interest in our work and for the reference to the Bianchi papers.

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