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On a criterion of belonging to the Hardy class $H_p(\mathbb{C}_+)$ up to exponential factor

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A criterion of belonging to the Hardy class $H_p(\mathbb{C}_+)$ up to factor e^{ikz} is obtained. It deals with functions f analytic in $\mathbb{C}_+$, having Blaschke zero-sets, and satisfying the condition $|f(z)| \le \exp\{|\mathrm{Im} z|^{-1} \exp(o(|z|))\}$, $z \to \infty$, $z \in \mathbb{C}_+$.

1. Introduction

According to the classical definition, a function f analytic in the upper half-plane $\mathbb{C}_+$ belongs to the Hardy class $H_p(\mathbb{C}_+)$, $0 < p \le \infty$, if

$$\sup_{0<y<\infty} \|f(\cdot + iy)\|_p < \infty, \tag{1}$$

where

$$\|h(\cdot)\|_p = \left(\int_{-\infty}^{\infty} |h(x)|^p dx \right)^{\min(1,1/p)}, \quad 0 < p < \infty,$$

$$\|h(\cdot)\|_\infty = \text{ess sup}_{x\in\mathbb{R}} |h(x)|.$$

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This class is very important in analysis and its properties has been studied in detail (see, [2, 5, 6]).

We are going to consider a bit wider class $\overline{H}_p(\mathbb{C}_+)$ consisting of functions f belonging to $H_p(\mathbb{C}_+)$ up to an exponential factor. More precisely, we define

$$\overline{H}_p(\mathbb{C}_+) = \{f : \ f(z)e^{ikz} \in H_p(\mathbb{C}_+) \text{ for some } k \in \mathbb{R}\}.$$

The following properties of a function $f \in \overline{H}_p(\mathbb{C}_+)$ are easy corollaries of well-known properties of functions of $H_p(\mathbb{C}_+)$.

(A) *Zeros $\{z_k\}$ of f satisfy the Blaschke condition*

$$\sum_k \frac{\mathrm{Im} z_k}{1 + |z_k|^2} < \infty.$$

(B) *For some $H > 0$,*

$$\sup_{0<y<H} \int_{-\infty}^{\infty} \frac{\log^- |f(x+iy)|}{1+x^2} dx < \infty.$$

(C) *The estimate holds*

$$|f(x+iy)| \le C_f y^{-1/p} e^{k_f y}, \quad z = x + iy \in \mathbb{C}_+, \tag{2}$$

where C_f and k_f are constants.

Assume that a function f is analytic in $\mathbb{C}_+$ and satisfies (A), (B) and (C). Then it is easy to check that f will belong to $\overline{H}_p(\mathbb{C}_+)$ if it satisfies only a "small part" of (1), namely

(D) *For some $H > 0$,*

$$\sup_{0<y<H} \|f(\cdot + iy)\|_p < \infty.$$

Observe that simple examples show that there exist functions satisfying (A), (B) and (D) which do not belong to $\overline{H}_p(\mathbb{C}_+)$ (see, e.g., Example 4 below). Thus, condition (C) cannot be dropped. On the other hand, it turns out that it can be substantially weakened. Roughly speaking, right hand side of (2) can be replaced with a function growing as $\exp[y^{-1}\exp(o(|z|))]$, $z \to \infty$, and this bound is in some sense sharp.

The main result of the paper is the following.

Theorem. *Let f be a function analytic in $\mathbb{C}_+$ and satisfying (A), (B), (D) and moreover,* *(E) There exists a sequence* $\{r_k\}$, $r_k \to \infty$, *such that*

$$\int_0^\pi \log^+ |f(re^{i\varphi})| \sin\varphi d\varphi \le \exp\{o(r)\}, \quad r = r_k \to \infty.$$

Then $f \in \overline{H}_p(\mathbb{C}_+)$.

Let us consider examples showing that conditions (A), (B), (D), (E) are independent, and moreover, (D) and (E) cannot be substantially weakened.

E x a m p l e 1. Let

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k/\rho)}, \quad \rho > 1,$$

be Mittag–Leffler's entire function. It is known (see, e.g., [1, p. 275]) that the following asymptotic formula holds as $z \to \infty$:

$$E_\rho(z) = \begin{cases} -\frac{1}{z\Gamma(1-1/\rho)} + O(\frac{1}{|z|^2}), & \frac{\pi}{2\rho} \le \arg z \le 2\pi - \frac{\pi}{2\rho}, \\ \rho e^{z^\rho} + O(\frac{1}{|z|}), & -\frac{\pi}{2\rho} \le \arg z \le \frac{\pi}{2\rho}. \end{cases}$$

This implies that the function

$$f(z) = (z+i)^{-2/p} E_\rho(-iz)$$

satisfies (B), (D), (E). Nevertheless, it does not belong to $\overline{H}_p(\mathbb{C}_+)$ because

$$f(iy) = \rho(y+1)^{-2/p} e^{y^\rho}(1+o(1)) \; y \to \infty,$$

and this contradicts to (C). Here (A) is violated.

E x a m p l e 2. The function

$$f(z) = e^{-z^2}$$

satisfies (A), (D), (E). But it does not belong to $\overline{H}_p(\mathbb{C}_+)$ (e.g., since (C) is violated). Here (B) is violated.

E x a m p l e 3. The function

$$f(z) = (z+i)^{-2/p} e^{-iz^3}$$

satisfies (A), (B), (E), but f does not belong to $\overline{H}_p(\mathbb{C}_+)$. Here (D) is violated. This example also shows that our Theorem ceases to be true if (D) is replaced with $\|f(\cdot + i0)\|_p < \infty$.

E x a m p l e 4. The function

$$f(z) = (z+i)^{-2/p} \exp\exp(-ciz), \quad c > 0,$$

satisfies (A), (B), (D), but evidently it does not belong to $\overline{H}_p(\mathbb{C}_+)$. Here (E) is violated. This example also shows that our Theorem ceases to be true if "o" is replaced with "O" in (E).

2. Proof of the theorem

I. Firstly, we prove the theorem under additional condition that $f(z)$ is analytic in the closed half-plane $\overline{\mathbb{C}}_+$ and does not vanish on $\mathbb{R}$.

Using (A), we can form the Blaschke product $B(z)$ with the same zero-set as $f(z)$. Set

$$g(z) = \frac{f(z)}{B(z)}, \quad u(z) = \log|g(z)|. \tag{3}$$

The function $g(z)$ is analytic and non-vanishing in the closed half-plane $\overline{\mathbb{C}}_+$ and therefore $u(z)$ is harmonic in $\overline{\mathbb{C}}_+$. We will apply to $u(z)$ the following result of [3, 4].

Theorem A. *Let $u(z)$ be a function harmonic in $\mathbb{C}_+$ and satisfy the following conditions:*

(α) *There is a sequence $\{r_k\}$, $r_k \to \infty$, such that*

$$\int_0^\pi u^+(re^{i\varphi}) \sin\varphi d\varphi \le \exp(o(r)), \ r = r_k \to \infty.$$

(β) *There is $H > 0$ such that*

$$\sup_{0<y<H} \int_{-\infty}^{\infty} \frac{|u(x+iy)|}{1+x^2} dx < \infty.$$

Then the following representation holds

$$u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{(x-t)^2 + y^2} + ky, \quad z = x+iy \in \mathbb{C}_+, \tag{4}$$

where $k \in \mathbb{R}$ and ν is a real-valued Borel measure on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+t^2} < \infty.$$

Let us check that conditions (α), (β) are satisfied for the function $u(z)$ defined by (3).

We have

$$\int_0^\pi u^+(re^{i\varphi})\sin\varphi d\varphi \le \int_0^\pi \log^+|f(re^{i\varphi})|\sin\varphi d\varphi + \int_0^\pi \log^-|B(re^{i\varphi})|\sin\varphi d\varphi.$$

It is well-known (see, e.g.,[4]) that the second integral in the right hand side is $O(r)$, as $r \to \infty$. Therefore the condition (α) follows from (E).

Further,

$$\int_{-\infty}^{\infty} \frac{|u(x+iy)|}{1+x^2}dx \le \int_{-\infty}^{\infty} \frac{|\log|f(x+iy)||}{1+x^2}dx + \int_{-\infty}^{\infty} \frac{|\log|B(x+iy)||}{1+x^2}dx.$$

It is well-known (see, e.g., [4]) that the second integral in the right hand side is bounded on any finite interval of values of $y > 0$. For the first integral, we have

$$\int_{-\infty}^{\infty} \frac{|\log|f(x+iy)||}{1+x^2}dx = \int_{-\infty}^{\infty} \frac{\log^+|f(x+iy)|}{1+x^2}dx + \int_{-\infty}^{\infty} \frac{\log^-|f(x+iy)|}{1+x^2}dx$$

$$\le \frac{1}{p}\int_{-\infty}^{\infty} |f(x+iy)|^p dx + \int_{-\infty}^{\infty} \frac{\log^-|f(x+iy)|}{1+x^2}dx.$$

Therefore the condition (β) follows from (D) and (B).

So, Theorem A is applicable to function $u(z)$ defined by (3), and hence the representation (4) holds. Since $u(z)$ is harmonic in the closed half-plane $\overline{\mathbb{C}}_+$, we have

$$d\nu(t) = u(t)dt = \log|g(t)|dt.$$

Since $|B(t)| = 1$ for $t \in \mathbb{R}$, we have

$$|g(t)| = |f(t)|, \quad d\nu(t) = \log|f(t)|dt.$$

Hence the representation (4) can be rewritten in the form

$$\log|g(z)| = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(t)|}{(t-x)^2+y^2}dt + ky, \quad z = x+iy \in \mathbb{C}_+.$$

Taking into account that $|f(z)| \le |g(z)|$, $z \in \mathbb{C}_+$, we obtain

$$\log|f(z)| \le \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(t)|}{(t-x)^2+y^2}dt + ky.$$

Hence,

$$|f(z)e^{ikz}|^p \le \exp\left\{\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log(|f(t)|^p)}{(x-t)^2+y^2}dt\right\}$$
$$\le \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{|f(t)|^p}{(x-t)^2+y^2}dt, \quad z=x+iy\in\mathbb{C}_+ .$$

Using the well-known properties of the Poisson integral, we conclude that $f(z)e^{ikz}\in H_p(\mathbb{C}_+)$.

II. Now, we consider the general case.

Let $s\in(0,H)$ be such that $f(t+is)\ne 0$ for $t\in\mathbb{R}$. Set

$$f_s(z)=f(z+is).$$

This function is analytic in the closed half-plane $\overline{\mathbb{C}}_+$ and does not vanish on $\mathbb{R}$. It suffices to show that $f_s(z)\in\overline{H}_p(\mathbb{C}_+)$ because this means that

$$\|f(\cdot+iy)e^{ik(\cdot+iy)}\|_p \tag{5}$$

is bounded for $s\le y<\infty$ (for some $k\in\mathbb{R}$). Boundedness of (5) for $0<y<s$ follows from (D) because $s<H$.

In order to apply the result of Part I of the proof, we should check that conditions (A), (B), (D), (E) are satisfied for $f_s(z)$. It is trivial that (B) and (D) are satisfied with $H-s$ instead of H because $0<s<H$.

To check (A), note that, if z_k is a zero of $f(z)$, then z_k-is is a zero of $f_s(z)$. Therefore,

$$\sum_{\mathrm{Im} z_k>s}\frac{\mathrm{Im}(z_k-is)}{1+|z_k-is|^2}\le\left(\max_{\mathrm{Im} z_k>s}\frac{1+|z_k|^2}{1+|z_k-is|^2}\right)\sum_{\mathrm{Im} z_k>s}\frac{\mathrm{Im} z_k}{1+|z_k|^2}<\infty.$$

It remains to check (E).

Let

$$Q_{R,s}=\{z:|z+is|<R\}\cap\mathbb{C}_+,\ R>s.$$

Since the function $\log^+|f_s(z)|$ is subharmonic in the closure of $Q_{R,s}$, we have

$$\log|f_s(z)|\le\frac{1}{2\pi}\int_{\partial Q_{R,s}}\log^+|f_s(\zeta)|\frac{\partial G_{Q_{R,s}}}{\partial n}(\zeta,z)|d\zeta|,\ \ z\in Q_{R,s},$$

where $G_{Q_{R,s}}(\zeta,z)$ is the Green function of $Q_{R,s}$ and $\partial/\partial n$ is derivative in the direction of the inner normal.

Let

$$K_{R,s} = \{z : |z + is| < R,\ \mathrm{Im} z > -s\}.$$

According to the principle of extension of domains, we have

$$\frac{\partial G_{Q_{R,s}}}{\partial n}(\zeta, z) \leq \frac{\partial G_{K_{R,s}}}{\partial n}(\zeta, z),\ \text{for}\ \zeta \in \partial Q_{R,s} \cap \partial K_{R,s},\ z \in Q_{R,s}.$$

Using the well-known explicit expression for the Green function of a half-disc, we get ($z_s = z + is,\ \varphi_s = \arg z_s$)

$$\int\limits_{\partial Q_{R,s} \cap \partial K_{R,s}} \log^+ |f_s(\zeta)| \frac{\partial G_{Q_{R,s}}}{\partial n}(\zeta, z) |d\zeta|$$

$$\leq \int\limits_{\partial Q_{R,s} \cap \partial K_{R,s}} \log^+ |f_s(\zeta)| \frac{\partial G_{K_{R,s}}}{\partial n}(\zeta, z) |d\zeta|$$

$$= \int\limits_{\arcsin(s/R)}^{\pi - \arcsin(s/R)} \log^+ |f(Re^{i\theta})| \frac{4R|z_s|(R^2 - |z_s|^2) \sin\theta \sin\varphi_s d\theta}{|Re^{i\theta} - z_s|^2 |Re^{-i\theta} - z_s|^2}$$

$$\leq \frac{4(R + |z| + s)^3}{(R - |z| - s)^3} \int\limits_0^{\pi} \log^+ |f(Re^{i\theta})| \sin\theta d\theta. \tag{6}$$

Further, since $Q_{R,s} \subset \mathbb{C}_+$, we have

$$\frac{\partial G_{Q_{R,s}}}{\partial n}(\zeta, z) \leq \frac{\partial G_{\mathbb{C}_+}}{\partial n}(\zeta, z),\ \text{for}\ \zeta \in \partial Q_{R,s} \backslash \mathbb{C}_+,\ z \in Q_{R,s}.$$

Hence,

$$\int\limits_{\partial Q_{R,s} \backslash \mathbb{C}_+} \log^+ |f_s(\zeta)| \frac{\partial G_{Q_{R,s}}}{\partial n}(\zeta, z) |d\zeta|$$

$$\leq \int\limits_{\partial Q_{R,s} \backslash \mathbb{C}_+} \log^+ |f_s(\zeta)| \frac{\partial G_{\mathbb{C}_+}}{\partial n}(\zeta, z) |d\zeta|$$

$$= \int\limits_{-\sqrt{R^2 - s^s}}^{\sqrt{R^2 + s^2}} \log^+ |f(t + is)| \frac{2y}{(x - t)^2 + y^2} dt$$

$$\leq \int\limits_{-\infty}^{\infty} \log^+ |f(t + is)| \frac{2y dt}{(x - t)^2 + y^2},\quad (z = x + iy). \tag{7}$$

Joining (6) and (7), we obtain

$$\log^+|f_s(re^{i\varphi})| \le \frac{2(R+r+s)^3}{\pi(R-r-s)^3}\int_0^\pi \log^+|f(Re^{i\theta})|\sin\theta d\theta$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\log^+|f(t+is)|\frac{r\sin\varphi dt}{r^2+t^2-2rt\cos\varphi},\ re^{i\varphi}\in Q_{R,s}. \quad (8)$$

For $0<r<R-s$, we have $re^{i\varphi}\in Q_{R,s}$ for $0<\varphi<\pi$. Let us multiply both sides of (8) by $\sin\varphi$ and integrate with respect to φ from 0 to π. Using the relation

$$\int_0^\pi \frac{r\sin^2\varphi d\varphi}{r^2+t^2-2rt\cos\varphi} = \frac{\pi r}{2}\min\left(\frac{1}{r^2},\frac{1}{t^2}\right) \le \frac{\pi r}{1+t^2},\ \text{for } r\ge 1,$$

we obtain

$$\int_0^\pi \log^+|f_s(re^{i\varphi})|\sin\varphi d\varphi \le \frac{8(R+r+s)^3}{\pi(R-r-s)^3}\int_0^\pi \log^+|f(Re^{i\theta})|\sin\theta d\theta$$
$$+\ r\int_{-\infty}^{\infty}\log^+|f(t+is)|\frac{dt}{1+t^2}. \quad (9)$$

Let $\{r_k\}$ be the sequence staying in (E). Put $R=r_k$, $r=r_k/2-s$ in (9). Then we see that condition (E) is satisfied for $f_s(z)$ with $\{r_k/2-s\}$ instead of $\{r_k\}$. ■

References

[1] *L. Bieberbach*, Lehrbuch der Funktionentheorie II. Chelsea Publ. Co, New York (1945).

[2] *P. Duren*, Theory of H^p spaces. Academic Press, New York,(1970).

[3] *S. Gergün and I.V. Ostrovskii*, On the Poisson representation of a function harmonic in the upper half-plane. — *Comput. Meth. Funct. Theory* (2002), v. 2, p. 191–213.

[4] *S. Gergün, I.V. Ostrovskii, and A. Ulanovskii*, On the Titchmarsh convolution theorem. — *Ark. Mat.* (2002), v. 40, p. 55–71.

[5] *K. Hoffman*, Banach spaces of analytic functions. Dover Publ., Inc., New York (1962).

[6] *P. Koosis*, Introduction to H_p spaces. Cambridge University Press, Cambridge (1998).