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Spaces of holomorphic almost periodic functions on a strip

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The notions of almost periodicity in the sense of Weyl and Besicovitch of the order $p \geq 1$ are extended to holomorphic functions on a strip. We prove that the spaces of holomorphic almost periodic functions in the sense of Weyl for various orders p are the same. These spaces are considerably wider than the space of holomorphic uniformly almost periodic functions and considerably narrower than the spaces of holomorphic almost periodic functions in the sense of Besicovitch. Besides we construct examples showing that the spaces of holomorphic almost periodic functions in the sense of Besicovitch for various orders p are all different.

A continuous function f on a strip $\Pi_{[a,b]} = \{z = x + iy \in \mathbf{C} : a \leq y \leq b\}$ is called almost periodic if for any $\varepsilon > 0$ the set of ε-almost periods

$$\left\{\tau \in \mathbf{R} : d^U_{[a,b]}(f(z), f(z+\tau)) < \varepsilon\right\}$$

is relatively dense on $\mathbf{R}$, i.e., its intersection with any segment of the length $L = L(\varepsilon)$ is nonempty. Here

$$d^U_{[a,b]}(g,h) = \sup_{z \in \Pi_{[a,b]}} |g(z) - h(z)| \tag{1}$$

is the standard uniform metric on $\Pi_{[a,b]}$.

We denote by $U_{[a,b]}$ the space of almost periodic functions on $\Pi_{[a,b]}$. By the Approximation Theorem, $U_{[a,b]}$ is equal to the closure of the set of all finite exponential sums

$$\sum_{n=1}^{N} c_n(y) e^{i\lambda_n x}, \quad \lambda_n \in \mathbf{R}, \quad c_n(y) \in C_{[a,b]} \tag{2}$$

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with respect to the metric $d^U_{[a,b]}$.

Note that any continuous periodic function on a strip $\Pi_{[a,b]}$ with real periods is almost periodic since any period is an ε-almost period for every $\varepsilon > 0$, and the set of periods forms a dual-sided arithmetical progression.

We can replace the uniform metric $d^U_{[a,b]}$ by either the Stepanov metric

$$d^{S^p}_{[a,b]}(g,h) = \sup_{z\in\Pi_{[a,b]}} \left(\int_0^1 |g(z+t) - h(z+t)|^p dt \right)^{\frac{1}{p}}, \ p \geq 1, \tag{3}$$

or the Weyl metric

$$d^{W^p}_{[a,b]}(g,h) = \overline{\lim_{T\to\infty}} \sup_{z\in\Pi_{[a,b]}} \left(\frac{1}{2T} \int_{-T}^{T} |g(z+t) - h(z+t)|^p dt \right)^{\frac{1}{p}}, \ p \geq 1, \tag{4}$$

or the Besicovitch metric

$$d^{B^p}_{[a,b]}(g,h) = \overline{\lim_{T\to\infty}} \sup_{a\leq y\leq b} \left(\frac{1}{2T} \int_{-T}^{T} |g(t+iy) - h(t+iy)|^p dt \right)^{\frac{1}{p}}, \ p \geq 1. \tag{5}$$

We suppose that the functions $g(z)$ and $h(z)$ are measurable and $|g(x+iy)|^p$ and $|h(x+iy)|^p$ are locally integrable with respect to the variable x for fixed y.

The closure of the set of sums (2) in metrics (3)–(5)will be called the space of almost periodic functions in the sense of Stepanov of order p, the space of almost periodic functions in the sense of Weyl of order p, the space of almost periodic functions in the sense of Besicovitch of order p, and , respectively, will be denote by

$$S^p_{[a,b]}, W^p_{[a,b]} \text{ and } B^p_{[a,b]}. \tag{6}$$

In the case $a = b = 0$, i.e., for functions defined on the real axis, these spaces are well-known (see [2, 3, 6, p. 189–247].) Note that we can also define the spaces (6) by using the concept of ε-almost period. But for the space of $B_{[a,b]}$ this definition is much more complicated (see [2, p. 91–104]).

It is clear that every sum (2) is uniformly bounded on $\Pi_{[a,b]}$, and therefore an almost periodic function $f(z)$ from the space $U_{[a,b]}$ (or from spaces (6)) is bounded in the corresponding metric, i.e., $d^U_{[a,b]}(f,0) < \infty$ (or $d^{S^p}_{[a,b]}(f,0) < \infty$, $d^{W^p}_{[a,b]}(f,0) < \infty$, $d^{B^p}_{[a,b]}(f,0) < \infty$). Observe that in the definitions of spaces (6) we can replace sums (2) by functions from $U_{[a,b]}$.

It is easy to see that the elements of metric spaces (6) are equivalent classes of functions with the corresponding zero distances. For example, the equivalent class for the space $S^p_{[a,b]}$ consists of the functions which coincide a.e. on every horizontal straight line. Note that every two functions with the difference $e^{-(x+iy)^2}$ always belong to the same class in the spaces $W^p_{[a,b]}$ and $B^p_{[a,b]}$, since $d^{W^p}_{[a,b]}(e^{-(x+iy)^2},0)=0$ for all $p<\infty$ and $a\le b$. Nevertheless we will use the notation $f\in W^p_{[a,b]}$ (or $f\in B^p_{[a,b]}$); this means that the equivalent class of the function f belongs to the corresponding space.

The Hölder inequality implies that for all $p<p'$

$$d^{S^p}_{[a,b]}(g,h)\le d^{S^{p'}}_{[a,b]}(g,h),\ \ d^{W^p}_{[a,b]}(g,h)\le d^{W^{p'}}_{[a,b]}(g,h),\ \ d^{B^p}_{[a,b]}(g,h)\le d^{B^{p'}}_{[a,b]}(g,h).$$

Besides for all $p\ge 1$

$$d^{B^p}_{[a,b]}(g,h)\le d^{W^p}_{[a,b]}(g,h)\le d^{S^p}_{[a,b]}(g,h)\le d^{U}_{[a,b]}(g,h),$$

the inequality between $d^{W^p}_{[a,b]}(g,h)$ and $d^{S^p}_{[a,b]}(g,h)$ follows from the relation

$$\begin{aligned}\frac{1}{2L}\int_{-L}^{L}|f(x+t)-g(x+t)|^p dt &\le \frac{1}{2L}\sum_{j=-[L]-1}^{[L]}\int_{j}^{j+1}|f(x+t)-g(x+t)|^p dt\\ &\le \frac{[L]+1}{L}d^{S^p}_{[a,b]}(f,g).\end{aligned}$$

Hence we have

$$S^{p'}_{[a,b]}\subset S^p_{[a,b]},W^{p'}_{[a,b]}\subset W^p_{[a,b]},B^{p'}_{[a,b]}\subset B^p_{[a,b]} \tag{7}$$

and

$$U_{[a,b]}\subset S^p_{[a,b]}\subset W^p_{[a,b]}\subset B^p_{[a,b]}. \tag{8}$$

Here all the inclusions mean that each equivalent class in a "narrower" space is contained in some equivalent class in a "wider" space. All the inclusions are strict even in the case $a=b=0$: there exists the equivalent class in a "wider" space containing no class from a "narrower" space (see [4]).

Note that there exists the mean value

$$(M_t f)(y)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t,y)dy \tag{9}$$

uniformly in $y\in[a,b]$ for the before mentioned functions. Indeed, the mean value of finite exponential sums (2) equals the coefficient at e^{i0t}; in the general case, the existence of limits (9) follows easily from the definition of almost periodicity. By the same way, we can prove the relation

$$\lim_{T\to\infty}\left|\sup_{\substack{x\in\mathbf{R}^m,\\ a\le y\le b}}\frac{1}{2T}\int_{-T}^{T} f(x+t,y)dt-(M_t f)(y)\right|=0 \tag{10}$$

for the space W^1 and for all "narrower" spaces.

Let $\Pi_{(a,b)} = \{z = x + iy \in \mathbf{C} : -\infty \le a < y < b \le \infty\}$ be an open strip, may be infinite width. A function $f(z)$ is uniformly almost periodic on $\Pi_{(a,b)}$ (or almost periodic in the sense of Stepanov, Weyl, and Besicovitch), if the restriction $f(z)$ to every strip $\Pi_{[\alpha,\beta]}$ for $a < \alpha < \beta < b$ is almost periodic in metrics (1), (3)–(5). We denote by $HU_{(a,b)}$, $HS^p_{(a,b)}$, $HW^p_{(a,b)}$, $HB^p_{(a,b)}$ the corresponding spaces of holomorphic almost periodic functions on $\Pi_{(a,b)}$. The inclusions similar to (7) and (8) hold for these spaces. The spaces $HB^p_{(a,b)}$ were studied earlier in [7]. These spaces were also defined in [1] as sets of holomorphic functions on a strip with the following property: the restrictions to each straight line in the strip are almost periodic functions in the sense of Besicovitch of order p. However as it follows from [1, Theorem 3.4], for functions growing as $O(e^{e^{|z|^N}})$ on a strip these definitions coincide.

The following Linfoot's theorem is well-known:

Theorem L (see [2, p. 146]). *Spaces $HU_{(a,b)}$ and $HS^p_{(a,b)}$ coincide for all $p \ge 1$.*

Here we obtain the similar result:

Theorem 1. *The spaces $HW^p_{(a,b)}$ coincide for all $p \ge 1$.*

The proof of this theorem is based on the following proposition.

Proposition 1. *Each function $f \in HW^1_{(a,b)}$ is uniformly bounded on every substrip $\Pi_{[\alpha,\beta]}$, $a < \alpha < \beta < b$.*

Next, we prove that inclusions $HU_{(a,b)} \subset HW^1_{(a,b)}$, $HW^1_{(a,b)} \subset HB^p_{(a,b)}$, $HB^{p'}_{(a,b)} \subset HB^p_{(a,b)}$, $p' > p \ge 1$, are strict in the same sense as inclusions (7) and (8).

Theorem 2. *There exists a function $f \in HW^1_{(-\infty,\infty)}$ such that every function g equivalent to f in any space $W^1_{[-H,H]}$, does not belong to $U_{[-H,H]}$ for all $H > 0$.*

Theorem 3. *There exists a function $f \in \bigcap_{p\ge 1} HB^p_{(-\infty,\infty)}$ such that every function g equivalent to f in any space $B^p_{[-H,H]}$, $p \ge 1$, does not belong to $W^1_{[-H,H]}$ for all $H > 0$.*

Theorem 4. *For all $p' > p \ge 1$ there exists a function $f \in HB^p_{(-\infty,\infty)}$ such that every function g equivalent to f in any space $B^p_{[-H,H]}$, does not belong to $B^{p'}_{[-H,H]}$ for all $H > 0$.*

The proof of Proposition 1. Suppose $a < \alpha' < \alpha < \beta < \beta' < b$. Since $d^{W^1}_{[\alpha',\beta']}(f,0) < \infty$, we have

$$\frac{1}{2T_0}\int_{-T_0}^{T_0} |f(z+u)|du \le C$$

for some $C < \infty$, $T_0 < \infty$ and all $z \in \Pi_{[\alpha',\beta']}$.

Fix $r < \min\{T_0,\ \alpha-\alpha',\ \beta'-\beta\}$.

Since the function $f(z)$ is holomorphic, we have for all $z_0 \in \Pi_{[\alpha,\beta]}$

$$|f(z_0)| \le \frac{1}{\pi r^2}\int_{|z-z_0|<r} |f(z)|dxdy$$

and

$$\begin{aligned} |f(z_0)| &\le \frac{1}{\pi r^2}\int_{-T_0}^{T_0}\int_{|v|<r} |f(z_0+u+iv)|dudv \\ &\le \frac{1}{\pi r^2}\sup_{z\in\Pi_{[\alpha,\beta]}}\int_{-T_0}^{T_0} |f(z+u)|du \le \frac{2T_0C}{\pi r^2}. \end{aligned}$$

Proposition 1 is proved.

Consider the sums

$$K^{(q)}(t) = \sum_n k_n^q e^{-i\lambda_n t},$$

which are called Bochner–Fejer kernels (see, for example, [6, p. 66–71]). Here λ_n runs over the linear envelope of the countable set $\Lambda \subset \mathbf{R}$ over the field $\mathbf{Q}$. The following properties of Bochner–Fejer kernels are fulfilled:

1) $0 \le k_n^q \le 1$;
2) there is only a finite number of nonzero coefficients k_n^q for every fixed q;
3) $k_n^q \to 1$ as $q \to \infty$ for every fixed n;
4) $K^{(q)}(-t) = K^{(q)}(t)$;
5) $K^{(q)}(t) \ge 0$ for all $t \in \mathbf{R}$;
6) $M_t\left\{K^{(q)}(t)\right\} = 1$.

Clearly, $M_t\left\{e^{-i\lambda t}e^{i\mu t}\right\} = 0$ for $\lambda \ne \mu$. Hence for any finite sum $Q(x,y)$ of type (2) with all exponents $\lambda_n \in \Lambda$, we have

$$(Q * K^{(q)})(x,y) = M_t\left\{Q(x+t,y)K^{(q)}(t)\right\} = \sum_{n=1}^{N} c_n(y)k_n^q e^{i\lambda_n x}.$$

It follows from condition 3) that this sum converges to $Q(x,y)$ as $q\to\infty$ uniformly on the strip $\Pi_{[a,b]}$.

Take a function $f\in W^1_{[a,b]}$. Fix $\varepsilon>0$ and take any sum $Q(x,y)$ of type (2) such that $d^{W^1}_{[a,b]}(f,Q)<\varepsilon$. Then choose q such that

$$\sup_{(x,y)\in\Pi_{[a,b]}}\left|Q(x,y)-(Q*K^{(q)})(x,y)\right|<\varepsilon.$$

Put $(f*K^{(q)})(x,y)=M_t\left\{f(x+t,y)K^{(q)}(t)\right\}$. It follows from (10) that the mean value does not change under shift on $x\in\mathbf{R}$. Since

$$M_t\left\{f(x+t,y)e^{-i\lambda t}\right\}=M_t\left\{f(x+t,y)e^{-i\lambda(x+t)}\right\}e^{i\lambda x},$$

we see that $\left\{f*K^{(q)}\right\}(x,y)$ is a finite sum of type (2). Next, we have

$$|(f*K^{(q)})(x,y)|\le\varliminf_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^{T}|f(x+t,y)|K^{(q)}(t)dt$$

for all $x\in\mathbf{R}$, $y\in K$. By Fatou's lemma, we obtain

$$\frac{1}{2X}\int\limits_{-X}^{X}|(f*K^{(q)})(x+\tau,y)|dx\le\varliminf_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^{T}\frac{1}{2X}\int\limits_{-X}^{X}|f(x+t+\tau,y)|K^{(q)}(t)dtdx$$

$$\le\varlimsup_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^{T}K^{(q)}(t)\left\{\sup_{t\in\mathbf{R},\,y\in[a,b]}\frac{1}{2X}\int\limits_{-X}^{X}|f(x+t,y)|dx\right\}dt$$

for every $\tau\in\mathbf{R}$ and $X<\infty$. The expression in the curly brackets does not exceed $d^{W^1}_{[a,b]}(f,0)+\varepsilon$ for large X. Consequently, passing to the limit as $X\to\infty$, and then $\varepsilon\to 0$, gives the inequality

$$d^{W^1}_{[a,b]}(f*K^{(q)},0)\le d^{W^1}_{[a,b]}(f,0)\lim_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^{T}K^{(q)}(t)dt=d^{W^1}_{[a,b]}(f,0).$$

Replace f by $f-Q$ to get

$$d^{W^1}_{[a,b]}(f*K^{(q)},f)\le d^{W^1}_{[a,b]}((f-Q)*K^{(q)},0)+d^{W^1}_{[a,b]}(Q*K^{(q)},Q)+d^{W^1}_{[a,b]}(f-Q,0)\le 3\varepsilon,$$

thus the exponential sums $f*K^{(q)}$ approximate the function f with respect to the metric $d^{W^1}_{[a,b]}$ as well.

T h e p r o o f o f T h e o r e m 1. Let $f \in HW^1_{(a,b)}$, $[\alpha,\beta] \subset (a,b)$. According with Proposition 1 we have $\sup\limits_{z\in\Pi_{[\alpha,\beta]}} |f(z)| = C < \infty$, therefore for any $K^{(q)}$

$$\left|\left(f * K^{(q)}\right)(z)\right| = \left|M_t\left\{f(z+t)K^{(q)}(t)\right\}\right| \le CM_t\left\{K^{(q)}(t)\right\} = C.$$

Hence for any $p > 1$

$$\left(d^{W^p}_{[\alpha,\beta]}(f * K^{(q)}, f)\right)^p = \overline{\lim_{T\to\infty}} \sup_{z\in\Pi_{[\alpha,\beta]}} \frac{1}{2T}\int\limits_{-T}^{T} |f(z+t) - (f*K^{(q)})(z+t)|^p dt$$

$$\le \overline{\lim_{T\to\infty}} \sup_{z\in\Pi_{[\alpha,\beta]}} \frac{1}{2T}\int\limits_{-T}^{T} |f(z+t) - (f*K^{(q)})(z+t)|$$

$$\times |f(z+t) - (f*K^{(q)})(z+t)|^{p-1}dt \le (2C)^{p-1}\, d^{W^1}_{[\alpha,\beta]}(f*K^{(q)}, f).$$

Thus the exponential sums $f*K^{(q)}$ approximate f in the metric $d^{W^p}_{[\alpha,\beta]}$, as claimed.

T h e p r o o f o f T h e o r e m 2. Consider the function $f(z) = \sum\limits_{n\in I} e^{-4(z-n)^2}$, where $I = \left\{n = 3^{l-1}(3k+1),\ k\in\mathbf{Z},\ l\in\mathbf{N}\right\}$. For any $z = x + iy \in \mathbf{C}$ we have $|f(z)| \le e^{4y^2}\sum\limits_{n\in I} e^{-4(x-n)^2} \le e^{4y^2}\sum\limits_{n\in\mathbf{Z}} e^{-4(x-n)^2}$. The series $\sum\limits_{n=-\infty}^{\infty} e^{-4(x-n)^2}$ converges for every $x\in\mathbf{R}$ and is a periodic function with the period 1, therefore $f(z)$ is bounded in every strip $\{z = x+iy : x\in\mathbf{R},\ |y| < H\}$ and $f(z)$ is an entire function. In particular, $f(z)$ is uniformly continuous on each strip.

Let us check that $f(z) \in HW^1_{(-\infty,\infty)}$. Put

$$\varphi_l(z) = \sum_{\substack{n=3^{l-1}(3k+1),\\ k\in\mathbf{Z}}} e^{-4(z-n)^2}.$$

The function $f_m(z) = \sum\limits_{1\le l\le m}\varphi_l(z)$ is the sum of the functions with periods 3^l, $l \le m$, therefore $f_m(z)$ is a periodic function with the real period 3^m, and belongs to the space $HU_{(-\infty,\infty)}$. It is sufficient to prove that

$$d^{W^p}_{[-H,H]}(f, f_m) \to 0 \tag{11}$$

as $m \to \infty$ for each $H < \infty$. We have

$$|f(z) - f_m(z)| = \sum_{l=m+1}^{\infty} \sum_{n\in\mathbf{Z}} \left| e^{-4(z-3^l n - 3^{l-1})^2} \right| \le \sum_{n=-\infty}^{\infty} e^{-4(x-3^m n)^2} e^{4H^2}$$

and $d^{W^1}_{[-H,H]}(f, f_m) \le e^{4H^2} \overline{\lim_{T\to\infty}} \sup_{x\in\mathbf{R}} \frac{1}{2T} \sum_{n=-\infty}^{\infty} \int_{-T}^{T} e^{-4(x+t-3^m n)^2} dt.$

Put

$$E_1 = \left\{ n \in \mathbf{Z} : n \le 3^{-m}x - 3^{-m}T - \frac{1}{2} \right\},\ n_1 = \sup E_1,$$

$$E_2 = \left\{ n \in \mathbf{Z} : 3^{-m}x - 3^{-m}T - \frac{1}{2} < n < 3^{-m}x + 3^{-m}T + \frac{1}{2} \right\},$$

$$E_3 = \left\{ n \in \mathbf{Z} : n \ge 3^{-m}x + 3^{-m}T + \frac{1}{2} \right\},\ n_2 = \inf E_3$$

for any fixed $x \in \mathbf{R}$, $T > 0$. Denote by $\operatorname{card} E$ the number of elements of the set E. Note that $\operatorname{card} E_2 \le 2 \cdot 3^{-m}T + 2$.

For $n \in E_1$ and $t \in [-T, T]$

$$(x + t - 3^m n) \ge 3^m \left(n_1 + \frac{1}{2} - n \right)$$

and

$$e^{-4(x+t-3^m n)^2} \le e^{-3^{2m}(2(n_1-n)+1)^2}.$$

Similarly,

$$(3^m n - x - t) \ge 3^m \left(n - n_2 + \frac{1}{2} \right)$$

and

$$e^{-4(x+t-3^m n)^2} \le e^{-3^{2m}(2(n-n_2)+1)^2}$$

for $n \in E_3$, $t \in [-T, T]$. Consequently,

$$\sum_{n\in E_1} \frac{1}{2T} \int_{-T}^{T} e^{-4(x+t-3^m n)^2} dt \le \sum_{n=1}^{\infty} e^{-(3^m n)^2} \le \int_0^{\infty} e^{-(3^m u)^2} du = \frac{\sqrt{\pi}}{2 \cdot 3^m},$$

and the same estimate holds for the sum

$$\sum_{n\in E_3} \frac{1}{2T} \int_{-T}^{T} e^{-4(x+t-3^m n)^2} dt.$$

Next, we have

$$\int_{-T}^{T} e^{-4(x+t-3^m n)^2}dt \le \int_{-\infty}^{\infty} e^{-4u^2}du = \frac{\sqrt{\pi}}{2}$$

and

$$\sum_{n\in E_2} \frac{1}{2T}\int_{-T}^{T} e^{-4(x+t-3^m n)^2}dt \le \frac{1}{2T}\cdot\frac{\sqrt{\pi}}{2}\operatorname{card} E_2 \le \frac{1}{2T}\cdot\frac{\sqrt{\pi}}{2}\left(3^{-m}2T+2\right)$$

for $n \in E_2$. Thus $d^{W^1}_{[-H,H]}(f, f_m) \le \frac{3\sqrt{\pi}}{2}\cdot 3^{-m}e^{4H^2}$, and we obtain (11).

Choose $H < \infty$. Let us check that $d^{W^1}_{[-H,H]}(f,g) = 0$ for no function $g \in U_{(-H,H)}$. We will prove the stronger result: there are no almost periodic functions $g(x)$ in the sense of Stepanov on $\mathbf{R}$ with the property $d^{W^1}_{\{0\}}(f,g) = 0$.

We need some auxiliary lemmas

Lemma 1.

$$\sup_{x\in\mathbf{Z}\setminus I} f(x) \le \frac{\sqrt{\pi}}{2} < 1 \le \inf_{x\in I} f(x). \tag{12}$$

The proof of Lemma 1. We have

$$f(x) = \sum_{l\in\mathbf{N}} \varphi_l(z) \le \sum_{n\in\mathbf{Z}\setminus\{0\}} e^{-4n^2} \le \int_{-\infty}^{\infty} e^{-4t^2}dt = \frac{\sqrt{\pi}}{2}$$

for any $x \in \mathbf{Z}\setminus I$, and $f(x) \ge e^{-4(x-n_0)^2} = 1$ for $x = n_0 \in I$.

Lemma 2. *For all $q \in \mathbf{Z}\setminus\{0\}$ there exists a two-sided arithmetical progression $I(q) \subset I$ such that $(I(q)+q)\cap I = \emptyset$.*

The proof of Lemma 2. Every positive integer q has the form $q = 3^{r-1}(3m+1)$ or $q = 3^{r-1}(3m-1)$, $r \in \mathbf{N}$, $m \in \mathbf{Z}$. In the first case take $n_j = 3^{r-1}(3j+1)$, $j \in \mathbf{Z}$, because

$$3^{r-1}(3m+1) + 3^{r-1}(3j+1) = 3^{r-1}\left(3(m+j+1)-1\right) \notin I.$$

In the second case take $n_j = 3^{r-1}(3(3j-m-1)+1)$, $j \in \mathbf{Z}$, because $n_j + q = 3^r(3j-1) \notin I$ for any $j \in \mathbf{Z}$.

Lemma 3. *There exists $\gamma > 0$ such that for each $\tau \in \mathbf{R}$, $|\tau| \geq 1$ there is a relatively dense set $I(\tau) \subset \mathbf{R}$, with the property*

$$\inf_{x \in I(\tau)} |f(x+\tau) - f(x)| > \gamma. \tag{13}$$

T h e p r o o f o f L e m m a 3. Since $f(x)$ is uniformly continuous on $\mathbf{R}$, there exists $N < \infty$ such that

$$|f(x) - f(t)| < \frac{1}{2}\left(1 - \frac{\sqrt{\pi}}{2}\right) \tag{14}$$

whenever $|x - t| \leq \frac{1}{N}$.

Let τ be an arbitrary real number, $|\tau| \geq 1$. We show that inequality (13) takes place with $\gamma = \frac{1}{2N}\left(1 - \frac{\sqrt{\pi}}{2}\right)$ for all points from some relatively dense set in $\mathbf{R}$.

Since the fractional parts of numbers $0, \tau,\ 2\tau, \dots,\ N\tau$ belong to the half-open interval $[0,1)$, there are two numbers $k\tau$ and $k'\tau$, $0 \leq k < k' \leq N$, such that the distance between their fractional parts is at most $\frac{1}{N}$, i.e., for some $q \in \mathbf{Z} \setminus \{0\}$ the inequality

$$\left| k\tau - k'\tau - q \right| \leq \frac{1}{N}$$

holds. For $M = k' - k$ we obtain

$$|M\tau - q| \leq \frac{1}{N}. \tag{15}$$

Let L be the difference of the arithmetic progression $I(q)$ from Lemma 2. Fix a real number $a \in \mathbf{R}$, and $n \in I(q) \cap [a, a+L)$. Taking into account Lemma 1 and 2, we see that

$$|f(n) - f(n+q)| \geq 1 - \frac{\sqrt{\pi}}{2}. \tag{16}$$

On the other hand,

$$|f(n+q) - f(n)| \leq |f(n+q) - f(n+M\tau)| + \sum_{k=0}^{M-1} |f(n+k\tau) - f(n+(k+1)\tau)|. \tag{17}$$

By (14) and (15), we have $|f(n+q) - f(n+M\tau)| \leq \frac{1}{2}\left(1 - \frac{\sqrt{\pi}}{2}\right)$. Hence inequalities (16) and (17) imply that

$$|f(n+k''\tau) - f(n+(k''+1)\tau)| \geq \frac{1}{2N}\left(1 - \frac{\sqrt{\pi}}{2}\right)$$

for some k'', $0 \le k'' \le M \le N$.

Thus there exists the point $x = n + k''\tau$, $x \in [a, a + L + N\tau]$ such that $|f(x+\tau) - f(x)| > \gamma$. The lemma is proved.

We continue the proof of Theorem 2. Let γ be the constant from Lemma 3. Take $\delta > 0$ such that the inequality

$$|f(x) - f(t)| \le \frac{\gamma}{5} \tag{18}$$

holds whenever $x,\ t \in \mathbf{R}$, $|x - t| < \delta$. Let us check that an arbitrary function $g(x)$ from the equivalent class of the function $f(x)$ in the space $W^1_{\{0\}}$ satisfies the inequality

$$d^{S^1}_{\{0\}}\left(g(t+\tau), g(t)\right) = \sup_{x\in\mathbf{R}} \int\limits_x^{x+1} |g(t+\tau) - g(t)| dt \ge \frac{\gamma\delta}{10} \tag{19}$$

for arbitrary $\tau \in \mathbf{R}$, $|\tau| \ge 1$. Then the set of ε-almost periods for the function g in the Stepanov metric for $\varepsilon < \frac{\gamma\delta}{10}$ is contained in the segment $[-1, 1]$, and so $g(x) \notin S^1_{\{0\}}$.

In order to prove (19) for fixed $\tau \in \mathbf{R}$ put

$$F_1 = \left\{x \in \mathbf{R} : |g(x) - f(x)| \ge \frac{\gamma}{5}\right\},$$

$$F_2 = \left\{x \in \mathbf{R} : |g(x+\tau) - f(x+\tau)| \ge \frac{\gamma}{5}\right\}.$$

Take $L < \infty$ such that the set $I(\tau)$ from Lemma 3 has nonempty intersection with every interval of the length L. Since

$$\begin{aligned}\frac{1}{2nL} mes\,(F_1 \cap [-nL,\ nL]) \ &\le\ \frac{5}{2nL\gamma} \int\limits_{F_1\cap[-nL,\ nL]} |f(x) - g(x)|\, dx \\ &\le\ \frac{5}{2nL\gamma} \int\limits_{-nL}^{nL} |f(x) - g(x)|\, dx,\end{aligned}$$

and f, g belong to the same equivalent class in the space $W^1_{\{0\}}$, we get

$$\overline{\lim_{n\to\infty}} \frac{1}{2nL} mes\,(F_1 \cap [-nL,\ nL]) \le d^{W^1}_{\{0\}}\{f, g\} = 0.$$

The same equality holds for the set F_2. Hence

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n-1} \frac{mes\,\{(F_1 \cup F_2) \cap [kL, (k+1)L]\}}{L} \\ &= \lim_{n\to\infty} \frac{1}{2nL} mes\,\{(F_1 \cup F_2) \cap [-nL, nL]\} = 0.\end{aligned}$$

Therefore for n sufficiently large there exists an interval $[k_0L,(k_0+1)L]$ such that

$$mes\,\{(F_1\cup F_2)\cap[k_0L,(k_0+1)L]\}<\frac{\delta}{2}. \qquad (20)$$

Take a real number $x\in[k_0L,(k_0+1)L]\cap I(\tau)$, where $I(\tau)$ is defined in Lemma 3. By (18) we get the inequality

$$|f(t+\tau)-f(t)|\ge|f(x+\tau)-f(x)|-|f(x+\tau)-f(t+\tau)|-|f(x)-f(t)|\ge\frac{3\gamma}{5}$$

for each point $t\in(x-\delta,x+\delta)$.

Note that the length of the interval $(x-\delta,x+\delta)\cap(k_0L,(k_0+1)L)$ is at least δ. It follows from (20) that the measure of the set $(x-\delta,x+\delta)\setminus[F_1\cup F_2]$ is at least $\frac{\delta}{2}$. Next, for $t\in(x-\delta,x+\delta)\setminus[F_1\cup F_2]$ we have

$$|g(t+\tau)-g(t)|\ge|f(t+\tau)-f(t)|-|f(t+\tau)-g(t+\tau)|-|f(t)-g(t)|\ge\frac{\gamma}{5}.$$

Thus,

$$\int\limits_{x-\delta}^{x+\delta}|g(t+\tau)-g(t)|\,dt\ge\int\limits_{[x-\delta,x+\delta]\setminus(F_1\cup F_2)}|g(t+\tau)-g(t)|\,dt\ge\frac{\gamma\delta}{10}.$$

The last inequality implies (19). The theorem is proved.

For the proof of other theorems we need following lemmas:

Lemma 4. *Any collection of functions* $æ_n(x)=e^{-4(x-3n)^2}$ *satisfy the inequality*

$$\left(\sum_{k=1}^{\infty}æ_k(x)\right)^p\le 2^{p-1}\left(\sum_{k=1}^{\infty}æ_k(x)\right).$$

T h e p r o o f o f L e m m a 4. Fix $x\in\mathbf{R}$ and put $n_0=\left[\frac{x}{3}+\frac{1}{2}\right]$. Since $3n_0-\frac{3}{2}\le x\le 3n_0-\frac{1}{2}$, we obtain

$$\sum_{n=-\infty}^{n_0-1}e^{-4(x-3n)^2}\le\sum_{n=-\infty}^{n_0-1}e^{-(6(n_0-1-n)+3)^2}\le\sum_{n=1}^{\infty}e^{-(3n)^2}\le\frac{\sqrt{\pi}}{6}$$

and

$$\sum_{n=n_0+1}^{\infty}e^{-4(x-3n)^2}\le\sum_{n=n_0+1}^{\infty}e^{-(6(n-n_0-1)+3)^2}\le\sum_{n=1}^{\infty}e^{-(3n)^2}\le\frac{\sqrt{\pi}}{6},$$

hence $\sum\limits_{n\in\mathbf{Z}\setminus\{n_0\}}^{\infty} e^{-4(x-3n)^2} \le \frac{\sqrt{\pi}}{3} < 1$. The statement of the lemma follows from the inequality

$$(a+b)^p \le 2^{p-1}(a+b)$$

with $a = æ_{n_0}(x)$ and $b = \sum\limits_{n\neq n_0} æ_n(x)$.

T h e p r o o f o f T h e o r e m 3. For $z = x+iy \in \mathbf{C}$, $l \in \mathbf{N}$ put

$$\varphi_l(z) = \sum_{\substack{n=3^{l-1}(3k+1),\\ k\in\mathbf{Z}}} e^{-4(z-n)^2},$$

$$f(z) = \sum_{l=1}^{\infty} l\varphi_l(z). \tag{21}$$

First of all, for $|y| \le H$ and any $x \in \mathbf{R}$ we have

$$|\varphi_l(z)| \le e^{4H^2} \sum_{k\in\mathbf{Z}} e^{-4(x-3^{l-1}(3k+1))^2} \le e^{4H^2} \sum_{n\in\mathbf{Z}} e^{-4(x-n)^2}. \tag{22}$$

Hence each term of sum (21) is uniformly bounded on every horizontal strip. If $|x| \le 3^{l_0-2}$ for some $l_0 \in \mathbf{N}$, then for all $k \in \mathbf{Z}$, $l \ge l_0$ we have

$$\left|x - 3^{l-1}(3k+1)\right| \ge 3^{l-1}|3k+1| - 3^{l-2} \ge 3^{l-2}(4|k|+1)$$

and for $z = x+iy$, $|x| \le 3^{l_0-2}$, $|y| \le H$

$$\begin{aligned} |\varphi_l(z)| &\le e^{4H^2} \sum_{k\in\mathbf{Z}} e^{-4(3^{l-2}(4|k|+1))^2} \le e^{4H^2} 2\sum_{n=1}^{\infty} e^{-4(3^{l-2}n)^2} \\ &\le e^{4H^2} 2\int_0^{\infty} e^{-4(3^{l-2}u)^2} du = \frac{e^{4H^2}\sqrt{\pi}}{2\cdot 3^{l-2}}, \end{aligned}$$

so all terms of series (21) with indices $l \ge l_0$ are majorized by the terms of the convergent series

$$\sum_{l=1}^{\infty} \frac{9\sqrt{\pi}e^{4H^2}}{2} \cdot \frac{l}{3^l}.$$

Thus series (21) uniformly converge on every compact set in $\mathbf{C}$, and $f(z)$ is an entire function.

Next, $\varphi_l(z)$ is an entire function with the period 3^l, and the sum $\sum\limits_{l=1}^{m} l\varphi_l(z)$ is a periodic function with the period 3^m, therefore this sum belongs to the space $HU_{(-\infty,\infty)}$. Hence if

$$\overline{\lim_{T\to\infty}}\left(\frac{1}{2T}\sup_{|y|\le H}\int_{-T}^{T}\left|\sum_{l=m+1}^{\infty} l\varphi_l(z)dx\right|^p\right)^{\frac{1}{p}} = d^{B^p}_{[-H,H]}\left(\sum_{l=1}^{m} l\varphi_l(z), f(z)\right) \to 0 \quad (23)$$

as $m\to\infty$ for all $H>0$, then $f(z)\in HB^p_{(-\infty,\infty)}$, $p\ge 1$.

Fix $T>0$ and consider the integral

$$\int_{-T}^{T}\varphi_l(x)dx = \sum_{k\in\mathbf{Z}}\int_{-T}^{T} e^{-4(x-3^{l-1}(3k+1))^2}dx.$$

Put

$$E_1=\left\{n\in\mathbf{Z}: n\le -3^{1-l}T-\frac{1}{2}\right\},\ n_1=\sup E_1,$$

$$E_2=\left\{n\in\mathbf{Z}: -3^{1-l}T-\frac{1}{2}<n<3^{1-l}T+\frac{1}{2}\right\}\setminus\{0\},$$

$$E_3=\left\{n\in\mathbf{Z}: n\ge 3^{1-l}T+\frac{1}{2}\right\},\ n_2=\inf E_3.$$

Note that

$$\sum_{k\in\mathbf{Z}}\int_{-T}^{T} e^{-4(x-3^{l-1}(3k+1))^2}dx \le \sum_{n\in E_1}\int_{-T}^{T} e^{-4(x-3^{l-1}n)^2}dx$$

$$+\sum_{n\in E_2}\int_{-T}^{T} e^{-4(x-3^{l-1}n)^2}dx+\sum_{n\in E_3}\int_{-T}^{T} e^{-4(x-3^{l-1}n)^2}dx.$$

Note also that the number $card\,E_2$ is at most $2T\cdot 3^{1-l}$.

For $n\in E_1$ and $x\in[-T,T]$ we have

$$\left(x-3^{l-1}n\right)\ge 3^{l-1}\left(n_1+\frac{1}{2}-n\right)$$

and

$$e^{-4(x-3^{l-1}n)^2}\le e^{-3^{2(l-1)}(2(n_1-n)+1)^2},$$

by the same way, for $n \in E_3$, $x \in [-T, T]$, we have

$$\left(3^{l-1}n - x\right) \geq 3^{l-1}\left(n - n_2 + \frac{1}{2}\right)$$

and

$$e^{-4(x-3^{l-1}n)^2} \leq e^{-3^{2(l-1)}(2(n-n_2)+1)^2}.$$

Therefore

$$\sum_{n\in E_1} \frac{1}{2T}\int_{-T}^{T} e^{-4(x-3^{l-1}n)^2}dx \leq \sum_{n=1}^{\infty} e^{-3^{2(l-1)}n^2} \leq \int_{0}^{\infty} e^{-(3^{l-1}u)^2}du = \frac{\sqrt{\pi}}{2\cdot 3^{l-1}}.$$

The same estimate is true for the sum over $n \in E_3$. Next, for $n \in E_2$ we get

$$\int_{-T}^{T} e^{-4(x-3^{l-1}n)^2}dx \leq \int_{-\infty}^{\infty} e^{-4u^2}du = \frac{\sqrt{\pi}}{2}$$

and

$$\sum_{n\in E_2} \frac{1}{2T}\int_{-T}^{T} e^{-4(x-3^{l-1}n)^2}dx \leq \frac{\sqrt{\pi}}{4T}\operatorname{card} E_2 \leq \frac{\sqrt{\pi}}{4T}3^{1-l}2T.$$

Thus $\frac{1}{2T}\int_{-T}^{T}\varphi_l(x)dx = \sum_{k\in\mathbf{Z}}\frac{1}{2T}\int_{-T}^{T} e^{-4(x-3^{l-1}(3k+1))^2}dx \leq \frac{3\sqrt{\pi}\cdot 3^{1-l}}{2}$.

Applying Lemma 4 to the functions $e^{-4(x-3^{l-1}(3k+1))^2}$, $k \in \mathbf{Z}$, we obtain

$$\frac{1}{2T}\int_{-T}^{T}\varphi_l^p(x)dx \leq 2^{p-1}\frac{1}{2T}\int_{-T}^{T}\varphi_l(x)dx \leq 2^{p-2}9\sqrt{\pi}3^{-l}. \tag{24}$$

The Hölder inequality implies

$$\left|\sum_{l=m+1}^{\infty} l\varphi_l(x)\right| \leq \left(\sum_{l=m+1}^{\infty} l^{2p}\varphi_l^p(x)\right)^{\frac{1}{p}}\left(\sum_{l=m+1}^{\infty}\frac{1}{l^q}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore for sufficiently large m we have

$$\frac{1}{2T}\int_{-T}^{T}\left(\sum_{l=m+1}^{\infty} l\varphi_l(x)\right)^p dx \leq \sum_{l=m+1}^{\infty} l^{2p}\frac{1}{2T}\int_{-T}^{T}\varphi_l^p(x)dx \leq 2^{p-2}9\sqrt{\pi}\sum_{l=m+1}^{\infty}\frac{l^{2p}}{3^l}.$$

If we combine the letter with the inequality

$$\sup_{|y|\le H}\left|\sum_{l=m+1}^{\infty} l\varphi_l(z)\right| \le e^{4H^2}\sum_{l=m+1}^{\infty} l\varphi_l(x),$$

we get (23).

Let us check that an arbitrary function $g(x)$ from the equivalent class of $f(x)$ in the space $B^p_{\{0\}}$ does not belong to the space $W^p_{\{0\}}$. Then it follows that g does not belong to the space $W^p_{[-H,H]}$ for all $H>0$.

Assume the contrary. Then

$$\overline{\lim_{x\to\infty}}\frac{1}{2T}\sup_{x\in\mathbf{R}}\int_{-T}^{T}|g(x+t)|^p dt = \left(d^{W^p}_{\{0\}}(0,g)\right)^p < \infty,$$

and for some $T_0<\infty$, $c=c(T_0)$ and all $x\in\mathbf{R}$

$$\int_{-T_0}^{T_0}|g(x+t)|^p dt \le c. \tag{25}$$

Take an integer l such that

$$l > \max\left\{\left(\frac{2c}{\int\limits_{-T_0}^{T_0} e^{-4pt^2}dt}\right)^{\frac{1}{p}}, \frac{\log 2T_0}{\log 3}\right\}. \tag{26}$$

For each fixed $x_n = 3^l n + 3^{l-1}$, $n\in\mathbf{Z}$, we have

$$\int_{-T_0}^{T_0}|f(x_n+t)|^p dt \ge l^p\int_{-T_0}^{T_0}\left(e^{-4(x_n+t-3^{l-1}(3n+1))^2}\right)^p dt \ge l^p\int_{-T_0}^{T_0} e^{-4pt^2}dt \ge 2c. \tag{27}$$

Take $T_n = x_n + T_0$, $n\in\mathbf{N}$. It follows from (26) that $2T_0 < 3^l$, hence the intervals $[x_k - T_0,\ x_k + T_0]$ are mutually disjoint. By the Minkowski inequality for $E = \bigcup\limits_{k=-n}^{n}[x_k - T_0, x_k + T_0]$, (25) and (27), we get

$$\left(\int_{-T_n}^{T_n}|f(t)-g(t)|^p\,dt\right)^{\frac{1}{p}} \ge \left(\int_E |f(t)-g(t)|^p\,dt\right)^{\frac{1}{p}}$$

$$\geq \left(\int_E |f(t)|^p \, dt \right)^{\frac{1}{p}} - \left(\int_E |g(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\geq \left(\sum_{k=-n}^{n} \int_{-T_0}^{T_0} |f(x_k+t)|^p \, dt \right)^{\frac{1}{p}} - \left(\sum_{k=-n}^{n} \int_{-T_0}^{T_0} |g(x_k+t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\geq ((2n+1)\,2c)^{\frac{1}{p}} - ((2n+1)\,c)^{\frac{1}{p}} = (2n+1)^{\frac{1}{p}} c^{\frac{1}{p}} (2^{\frac{1}{p}} - 1),$$

therefore

$$\int_{-T_n}^{T_n} |f(t) - g(t)|^p \, dt \geq (2n+1)c(2^{\frac{1}{p}} - 1)^p. \tag{28}$$

Since $T_n = T_0 + 3^{l-1} + n3^l$, we see that $\frac{2n+1}{2T_n} \to 3^{-l}$ as $n \to \infty$. Hence inequality (28) contradicts to the equality

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |f(t) - g(t)|^p \, dt = \left(d_{\{0\}}^{B^p}(f,g) \right)^p = 0,$$

which is true for each function g from the equivalent class of f. The theorem is proved.

T h e p r o o f o f T h e o r e m 4. Fix $p_0 \in (p, p')$ and take

$$f(z) = \sum_{l=1}^{\infty} 3^{\frac{l}{p_0}} \varphi_l(z), \tag{29}$$

here the functions $\varphi_l(z)$ are the same as in the proof of Theorem 3. As in that proof we see that the terms of series (29) are majorized by the terms of series

$$\sum_{l=1}^{\infty} \frac{9\sqrt{\pi} e^{4H^2} 3^{l\left(\frac{1}{p_0}-1\right)}}{2}$$

on compact sets $\left\{|x| \leq 3^{l_0-2},\ |y| \leq H\right\}$. Therefore $f(z)$ is an entire function on $\mathbf{C}$.

As in the proof of Theorem 3, we have for any $H < \infty$

$$d_{[-H,H]}^{B^p} \left(\sum_{l=1}^{m} 3^{\frac{l}{p_0}} \varphi_l(z), f(z) \right) \to 0 \ as\ m \to \infty. \tag{30}$$

Indeed, by the Hölder equality

$$\left|\sum_{l=m+1}^{\infty} 3^{\frac{l}{p_0}}\varphi_l(x)\right| \le \left(\sum_{l=m+1}^{\infty} 3^{\frac{lp}{p_0}} l^p \varphi_l^p(x)\right)^{\frac{1}{p}} \left(\sum_{l=m+1}^{\infty} \frac{1}{l^q}\right)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1.$$

Hence for m sufficiently large, for all $T>0$ inequality (24) implies

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\left(\sum_{l=m+1}^{\infty} 3^{\frac{l}{p_0}}\varphi_l(x)\right)^p dx &\le \sum_{l=m+1}^{\infty} 3^{\frac{lp}{p_0}} l^p \frac{1}{2T}\int_{-T}^{T}\varphi_l^p(x)dx \\ &\le 2^{p-1}9\sqrt{\pi}\sum_{l=m+1}^{\infty}\frac{l^p}{3^{l\left(1-\frac{p}{p_0}\right)}}.\end{aligned}$$

The convergence of the last series yields (30).

Let us show that an arbitrary function g from the equivalent class of $f(x)$ in the space $B^p_{\{0\}}$ does not belong to $B^{p'}_{\{0\}}$.

Assume the contrary. Then for sufficiently large T we have

$$\frac{1}{2T}\int_{-T}^{T}|g(t)|^{p'}dt \le C_1 < \infty.$$

Let x_n be the same as in the proof of Theorem 3, and $T_n = x_n + \frac{1}{2}$.

Applying the Hölder inequality for the set

$$E = \bigcup_{k=-n}^{n}\left[x_k - \frac{1}{2},\ x_k + \frac{1}{2}\right],$$

gives for sufficiently large n

$$\begin{aligned}\int_E |g(t)|^p dt &\le \left(\int_E |g(t)|^{p'}dt\right)^{\frac{p}{p'}}\left(\int_E dt\right)^{1-\frac{p}{p'}} \\ &\le (2T_n)^{\frac{p}{p'}}\left(\frac{1}{2T_n}\int_{-T_n}^{T_n}|g(t)|^{p'}dt\right)^{\frac{p}{p'}}\left(\int_E dt\right)^{1-\frac{p}{p'}} \\ &\le C_1^{\frac{p}{p'}}(2T_n)^{\frac{p}{p'}}(2n+1)^{1-\frac{p}{p'}}.\end{aligned}$$

As in the previous theorem, we have for all $l \in \mathbf{N}$

$$\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} |f(x_n+t)|^p dt \geq 3^{\frac{pl}{p_0}} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k \in \mathbf{Z}} e^{-4(x_n+t-3^{l-1}(3k+1))^2} \right)^p dt$$

$$\geq 3^{\frac{lp}{p_0}} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} e^{-4pt^2} dt = 3^{\frac{lp}{p_0}} C_2.$$

Now the Minkowski inequality yields

$$\left(\int\limits_{-T_n}^{T_n} |f(t)-g(t)|^p \, dt \right)^{\frac{1}{p}} \geq \left(\int\limits_{E} |f(t)-g(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\geq \left(\int\limits_{E} |f(t)|^p \, dt \right)^{\frac{1}{p}} - \left(\int\limits_{E} |g(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\geq \left(\sum_{k=-n}^{n} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} |f(x_k+t)|^p \, dt \right)^{\frac{1}{p}} - \left(\int\limits_{E} |g(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\geq \left(3^{\frac{lp}{p_0}} C_2 (2n+1) \right)^{\frac{1}{p}} - \left(C_1^{\frac{p}{p'}} \left(\frac{2T_n}{2n+1} \right)^{\frac{p}{p'}} (2n+1) \right)^{\frac{1}{p}},$$

therefore

$$\left(\frac{1}{2T_n} \int\limits_{-T_n}^{T_n} |f(t)-g(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\geq \left(\frac{3^{\frac{lp}{p_0}} C_2 (2n+1)}{2T_n} \right)^{\frac{1}{p}} - \left(C_1^{\frac{p}{p'}} \left(\frac{2T_n}{2n+1} \right)^{\frac{p}{p'}} \frac{2n+1}{2T_n} \right)^{\frac{1}{p}}.$$

Since $T_n = \frac{1}{2} + 3^{l-1} + n3^l$, we obtain $\frac{2T_n}{2n+1} \to 3^l$ as $n \to \infty$, hence

$$\overline{\lim_{T \to \infty}} \left(\frac{1}{2T} \int\limits_{-T}^{T} |f(t)-g(t)|^p dt \right)^{\frac{1}{p}} \geq C_2^{\frac{1}{p}} \cdot 3^{l\left(\frac{1}{p_0} - \frac{1}{p}\right)} - C_1^{\frac{1}{p'}} \cdot 3^{l\left(\frac{1}{p'} - \frac{1}{p}\right)}.$$

For $p' > p_0$, and l sufficiently large the last inequality contradicts to the equality

$$\overline{\lim_{T\to\infty}} \left(\frac{1}{2T} \int_{-T}^{T} |f(t) - g(t)|^p \, dt \right)^{\frac{1}{p}} = d_{\{0\}}^{B^p}(f,g) = 0,$$

which is true for all $g(x)$ from the equivalent class of f in the space $B^p_{\{0\}}$. The theorem is proved.

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