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Weak cluster points of a sequence and coverings by cylinders

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Let H be a Hilbert space. Using Ball's solution of the "complex plank problem" we prove that the following properties of a sequence $a_n > 0$ are equivalent:

1. There is a sequence $x_n \in H$ with $\|x_n\| = a_n$, having 0 as a weak cluster point;
2. $\sum_1^\infty a_n^{-2} = \infty$.

Using this result we show that a natural idea of generalization of Ball's "complex plank" result to cylinders with k-dimensional base fails already for $k = 3$. We discuss also generalizations of "weak cluster points" result to other Banach spaces and relations with cotype.

1. Introduction

Let H be an infinite-dimensional Hilbert space. It is well known that the weak topology of H has bad sequential properties: a sequence $h_n \in H$ having a weak cluster point x, can be free from weakly convergent to x subsequences. Moreover, for a sequence $h_n \in H$ with a weak cluster point it is possible to have $\|h_n\| \to \infty$ [5]. In this paper we study the following question: if a sequence has a weak cluster point, how quick can tend to infinity the norms of the sequence elements? The main tool for estimation of this speed from above will be Ball's "complex plank" theorem — a recent and really beautiful statement from Hilbert space geometry.

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Recall, that by a plank of width w in H one means a set of the form

$$P = \{h \in H : |\langle h - h_0\,,\, e\rangle| \leq \frac{w}{2}\},$$

where $\|e\| = 1$. According to T. Bang's theorem [4], if a sequence P_n of planks of widths w_n covers a ball of diameter w, then $\sum w_n \geq w$.

Bang's theorem is true both in real and complex spaces, but in complex spaces it can be dramatically improved. The following theorem is proved by K. Ball [3].

Theorem 1.1. *Let*

$$P_n = \{h \in H : |\langle h, e_n\rangle| \leq \frac{w_n}{2}\},$$

where $\|e_n\| = 1$, be planks of widths w_n in a complex Hilbert space, and let $\bigcup P_n$ cover a ball of diameter w centered in origin. Then $\sum w_n^2 \geq w^2$.

We are going not only to apply the Ball's theorem to our problem, but also to apply our result on weak topology to give some limitations for possible generalizations of the Theorem 1.1. After that we discuss generalizations of our weak topology result to other Banach spaces. Allover the paper the letter X will be used for infinite dimensional Banach space, S_X and B_X — for its unit sphere and unit ball respectively.

2. Weighted weak convergence

Let $P = \{p_{n,m}\}$ be an infinite matrix of nonnegative numbers with finite rows, satisfying conditions

1. $\sum_{m=1}^{\infty} p_{n,m} = 1$,
2. $\lim_{n\to\infty} p_{n,m} = 0$.

We say that a sequence $\{a_n\}$ of reals is P-convergent to 0 ($P \lim a_n = 0$) if

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} p_{n,m} a_m = 0.$$

Some evident well-known properties of P-convergence are collected in the proposition below.

Proposition 2.1. *1. If $\lim a_n = 0$, then $P \lim a_n = 0$.*

2. If $P \lim a_n = 0$ and $P \lim b_n = 0$, then $P \lim (a_n + b_n) = 0$.

3. If $a_n \geq 0$ and $P \lim a_n = 0$, then 0 is a cluster point of $\{a_n\}$.

Let X be a Banach space, $1 \leq p < \infty$. A sequence $x_n \in X$ is said to be weakly (P,p)-convergent to 0 ($w(P,p) \lim x_n = 0$), if for every $f \in X^*$ the sequence $|f(x_n)|^p$ P-converges to 0.

Proposition 2.2. *If $w(P,p) \lim x_n = 0$, then 0 is a weak cluster point of $\{x_n\}$.*

P r o o f. We must prove that for arbitrary finite set $\{f_k\}_{k=1}^m \subset X^*$ and arbitrary $\varepsilon > 0$ there is an arbitrarily big $n \in \mathbb{N}$ with $\sum_{k=1}^m |f_k(x_n)|^p \leq \varepsilon$. By definition of $w(P,p)$-convergence and item (2) of Proposition 2.1

$$P \lim_n \sum_{k=1}^m |f_k(x_n)|^p = 0.$$

The rest we deduce from item (3) of Proposition 2.1. ■

3. The main result

Theorem 3.1. *The following properties of a sequence $a_n > 0$ are equivalent: (a) $\sum_1^\infty a_n^{-2} = \infty$. (b) There is a sequence x_n in a Hilbert space H with $\|x_n\| = a_n$, and there is a sequence $P = \{p_{n,m}\}$ of weights such that $w(P,2) \lim x_n = 0$. (c) There is a sequence x_n in a Hilbert space H with $\|x_n\| = a_n$, having 0 as a weak cluster point.*

P r o o f. (a) $\Longrightarrow$ (b). Assume $\sum_1^\infty a_n^{-2} = \infty$ and consider the sequence $x_n = a_n e_n$, where e_n is an orthonormal sequence in H. Introduce the following sequence $P = \{p_{n,m}\}$ of weights:

$$p_{n,m} = \frac{a_m^{-2}}{\sum_{j=1}^n a_j^{-2}}$$

when $m \leq n$ and $p_{n,m} = 0$ for $m > n$. Then for every $f \in H$ we have

$$\sum_{m=1}^\infty p_{n,m} |\langle x_m, f\rangle|^2 = \frac{\sum_{m=1}^n |\langle e_m, f\rangle|^2}{\sum_{m=1}^n a_m^{-2}} \leq \frac{\|f\|^2}{\sum_{m=1}^n a_m^{-2}} \to 0, \text{ as } n \to \infty$$

which means $w(P,2)$-convergence of x_n to 0. (b) $\Longrightarrow$ (c). This is given by the Proposition 2.2. (c) $\Longrightarrow$ (a). Let

$$\sum_1^\infty a_n^{-2} = R^2 < \infty \tag{1}$$

and let $x_n \in H$ be vectors with $\|x_n\| = a_n$. We may assume that H is a complex Hilbert space, since otherwise we may embed H into its complexification. Define planks

$$P_n = \{h \in H : |\langle h\,,\, x_n\rangle| \le \frac{1}{2}\}.$$

The width of P_n equals a_n^{-1}. Using (1) and the Theorem 1.1 we deduce that the planks P_n cannot cover the whole space H (they even cannot cover a ball of radius $R + \varepsilon$). So there is an element $h \in H$ for which all the inequalities

$$|\langle h\,,\, x_n\rangle| > \frac{1}{2}$$

hold true at the same time. This h separates our sequence x_n from 0. ■

4. A comment to the Ball's Theorem 1.1

Why estimates for coverings by real and complex planks differ so strongly? A possible explanation looks as follows: a complex plank of width r looks not like a slice between two hyperplanes, but like an orthogonal cylinder having as a base a circle of the radius $r/2$. If one tries to cover a circle of radius R by circles of radiuses r_n, then calculating corresponding areas one can easily see, that $\sum r_n^2 \ge R^2$. This explanation leads to the following hypothesis: let $k \in \mathbb{N}$ be a fixed number, and let C_n be orthogonal cylinders in a real Hilbert space H, having as their bases k-dimensional balls of radiuses r_n respectively and $\bigcup_{n\in\mathbb{N}} C_n = H$. Then $\sum_{n=1}^{\infty} r_n^k = \infty$. With the help of the Theorem 3.1 we can show that this natural hypothesis fails already for $k = 3$. In fact, assume that the abovementioned hypothesis is true for $k = 3$. Consider a sequence $x_n \in H$, $\|x_n\| = a_n$, having 0 as a weak cluster point, but with $\sum_1^{\infty} a_n^{-3} < \infty$ (such a sequence exists due to the Theorem 3.1). Consider an auxiliary Hilbert space $H_1 = H \oplus H \oplus H$ — the orthogonal direct sum of 3 copies of the original space H. Introduce cylinders C_n in H_1 as follows:

$$C_n = \{h = (h_1, h_2, h_3) \in H_1 : \sum_{j=1}^{3} |\langle h_j\,,\, x_n\rangle|^2 \le 1\}.$$

C_n are orthogonal cylinders in H_1, having as their bases 3-dimensional balls of radiuses a_n^{-1}. According to our hypothesis, these cylinders do not cover the whole space H_1, i.e., there is an $g = (g_1, g_2, g_3) \in H_1$ which does not belong to any of these cylinders. But this means, that the weak neighborhood of 0

$$W = \{x \in H : \sum_{j=1}^{3} |\langle g_j\,,\, x\rangle|^2 \le 1\}$$

separates all the x_n from 0. Contradiction.

5. Generalization to Banach spaces: the role of finite representability and cotype

Let X be a Banach space, $2 \le p \le \infty$. The space l_p is finitely representable in X if for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$ there are elements $e_1, e_2, \dots, e_n \in S_X$ such that

$$(1-\varepsilon)\left(\sum_{j=1}^{n}|b_j|^p\right)^{\frac{1}{p}} \le \|\sum_{j=1}^{n} b_j e_j\| \le (1+\varepsilon)\left(\sum_{j=1}^{n}|b_j|^p\right)^{\frac{1}{p}}$$

for all selections of coefficients b_j.

Theorem 5.1. *Let X be a Banach space, $2 \le p \le \infty$ and let $1 \le p' \le 2$ be dual to p exponent. Let moreover l_p be finitely representable in X and $a_n > 0$ satisfy condition $\sum_1^\infty a_n^{-p'} = \infty$. Then there is a sequence $x_n \in X$ with $\|x_n\| = a_n$, having 0 as a weak cluster point. Moreover there is a matrix $P = \{p_{n,m}\}$ of weights such that $w(P,p')\lim x_n = 0$.*

P r o o f. Fix an $\varepsilon < 1/8$. First choose $0 = n_1 < n_2 < n_3 < \dots$ to satisfy condition

$$\lim_{k\to\infty} \sum_{j=n_k+1}^{n_{k+1}} a_j^{-p'} = \infty.$$

Define $P = \{p_{k,m}\}$ as follows:

$$p_{k,m} = \frac{a_m^{-p'}}{\sum_{j=n_k+1}^{n_{k+1}} a_j^{-p'}}$$

when $n_k < m \le n_{k+1}$ and $p_{k,m} = 0$ otherwise. Using step-by-step finite representability of l_p in X select sequence $e_n \in S_X$, for which

$$(1-\varepsilon)\left(\sum_{j=n_k+1}^{n_{k+1}}|b_j|^p\right)^{\frac{1}{p}} \le \|\sum_{j=n_k+1}^{n_{k+1}} b_j e_j\| \le (1+\varepsilon)\left(\sum_{j=n_k+1}^{n_{k+1}}|b_j|^p\right)^{\frac{1}{p}}$$

for all k and b_j. Now define $x_n = a_n e_n$. Then for every $f \in X^*$ we have

$$\sum_{m=1}^{\infty} p_{k,m}\,|f(x_m)|^{p'} = \frac{\sum_{j=n_k+1}^{n_{k+1}} |f(e_m)|^{p'}}{\sum_{j=n_k+1}^{n_{k+1}} a_j^{-p'}} \le \frac{2\|f\|^{p'}}{\sum_{j=n_k+1}^{n_{k+1}} a_j^{-p'}} \to 0,\ as\ n\to\infty,$$

which means $w(P,p')$-convergence of x_n to 0. By the Proposition 2.2 this means that 0 is a weak cluster point of $\{x_n\}$. ■

Together with Dvoretzky's theorem (l_2 is finitely representable in every infinite-dimensional Banach space) this gives us the following:

Corollary 5.2. *For every infinite-dimensional Banach space X and every selection of $a_n > 0$ satisfying condition $\sum_1^\infty a_n^{-2} = \infty$ there is a sequence $x_n \in X$ with $\|x_n\| = a_n$, having 0 as a weak cluster point.*

Recall, that a Banach space X has M-cotype $p < \infty$ if there is a constant $C > 0$ such that for every finite collection of vectors $\{x_k\}_{k=1}^n \subset X$ there are coefficients $\gamma_k = \pm 1$ for which

$$\|\sum_{k=1}^n \gamma_k x_x\| \geq C \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}.$$

Due to Maurey–Pisier theorem a space X has an M-cotype if and only if l_∞ is not finitely representable in X. A small survey of facts concerning M-cotype can be found in [6].

The Theorem 5.1 together with another Ball's result [2] (generalization of the Bang's real plank theorem to arbitrary Banach spaces) gives us an analog of Theorem 3.1 for spaces without cotype (in particular for c_0). In this case the characterization does not involve square exponentials.

Corollary 5.3. *Let X be a Banach spaces in which l_∞ is finitely representable. Then the following properties for a sequence $a_n > 0$ are equivalent:*

1. *There is a sequence $x_n \in X$ with $\|x_n\| = a_n$, and there is a sequence $P = \{p_{n,m}\}$ of weights such that $w(P,1)\lim x_n = 0$.*
2. *There is a sequence $x_n \in X$ with $\|x_n\| = a_n$, having 0 as a weak cluster point;*
3. $\sum_1^\infty a_n^{-1} = \infty$.

The next theorem gives us a relationship between weak weighted limit and cotype.

Theorem 5.4. *Let X be a Banach space of M-cotype $2 \leq p \leq \infty$ and let $1 \leq p' \leq 2$ be dual to p exponential. Let $x_n \in X$ be a sequence such that $w(P,p')\lim x_n = 0$ for some matrix $P = \{p_{n,m}\}$ of weights. Then $\sum_1^\infty \|x_n\|^{-p'} = \infty$.*

P r o o f. Since $\lim_{n\to\infty} p_{n,m} = 0$, using small perturbation argument we may assume that there are $m_1(n) < m_2(n)$, $m_1(n) \to \infty$, such that $p_{n,m} = 0$ for m outside the interval $(m_1(n), m_2(n))$. By the closed graph theorem, applied to the operator $T : X^* \to (\sum_{n=1}^{\infty} l_1)_{\infty}$,

$$Tx^* = \big((p_{1,1}|x^*(x_1)|, p_{1,2}|x^*(x_2)| \ldots);\ (p_{2,1}x^*(x_1), p_{2,2}x^*(x_2) \ldots);\ \ldots\big),$$

there is a constant $C > 0$ such that

$$\sum_{m=1}^{\infty} p_{n,m}|x^*(x_m)| \le C\|x^*\|$$

for all $n \in \mathbb{N}$ and all $x^* \in X^*$. So

$$C \ge \sup_{x^*\in S_{X^*}} \sum_{m=1}^{\infty} p_{n,m}|x^*(x_m)| = \sup_{x^*\in S_{X^*}} \sum_{m=m_1(n)}^{m_2(n)} p_{n,m}|x^*(x_m)|$$

$$= \sup_{x^*\in S_{X^*}} \sup_{\gamma_m=\pm 1} \sum_{m=m_1(n)}^{m_2(n)} p_{n,m}\gamma_m x^*(x_m) = \sup_{\gamma_m=\pm 1} \|\sum_{m=m_1(n)}^{m_2(n)} p_{n,m}x_m\|$$

$$\ge C_1 \left(\sum_{m=m_1(n)}^{m_2(n)} (p_{n,m}\|x_m\|)^p \right)^{1/p},$$

where C_1 is the constant from the definition of M-cotype. Applying Hölder inequality to $\sum_{m=m_1(n)}^{m_2(n)} p_{n,m} = 1$, we deduce

$$1 \le \left(\sum_{m=m_1(n)}^{m_2(n)} (p_{n,m}\|x_m\|)^p \right)^{1/p} \left(\sum_{m=m_1(n)}^{m_2(n)} \|x_m\|^{-p'} \right)^{1/p'}$$

$$\le \frac{C}{C_1} \left(\sum_{m=m_1(n)}^{m_2(n)} \|x_m\|^{-p'} \right)^{1/p'}$$

which means that $\sum_1^{\infty} \|x_n\|^{-p'} = \infty$. ■

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