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A remark to the construction of canonical products of minimal growth

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For a sequence (a_n) of complex numbers, $|a_n| \nearrow +\infty$, we construct an entire function f of minimal growth such that $f(a_n) = 0$. Similar result is obtained for analytic functions in the unit disk.

1. Let $A = (a_n)$ be a sequence of complex numbers, $0 < |a_n| \leq |a_{n+1}| \nearrow +\infty$ $(n \to \infty)$, and let $n(r, A) = \sum\limits_{|a_n| \leq r} 1$ be the counting function of A. By $G(A)$ we denote a class of entire functions whose zero set coincides with A. By L we denote the class of all subsets of $[1, +\infty)$, which have a finite logarithmic measure. A.A. Gol'dberg [1] proved that if $\varepsilon > 0$ is an arbitrary number and

$$\varliminf_{r \to +\infty} \frac{\ln n(r, A)}{\ln r} > 0, \tag{1}$$

then there exists a function $f \in G(A)$ such that

$$\ln \ln M_f(r) = O((\ln n(r, A))^{2+\varepsilon}), \quad r \to +\infty \quad (r \notin E \in L), \tag{2}$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$. W. Bergweiler [2] showed in particular that the number $2 + \varepsilon$ cannot be replaced by 2.

The following problem is natural: find a counterpart of Gol'dberg's result when condition (1) does not hold. The following theorem is true.

Theorem 1. *For any sequence A and any $\varepsilon > 0$ there exists a function $f \in G(A)$ such that*

$$\ln \ln M_f(r) = O\left(\int\limits_1^{r(1+\varepsilon)} \frac{\ln n(t, A)}{t} dt \right), \quad r \to +\infty. \tag{3}$$

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We remark that $1+\varepsilon$ cannot be replaced by 1. Indeed, otherwise, we have $\ln \ln M_f(r) = O(\ln n(r,A) \ln r)$, $r \to +\infty$, and we obtain contradiction to the mentioned result of W. Bergweiler.

2. We need a lemma from the Dirichlet series theory.

Let $0 \le \lambda_n \nearrow +\infty \, (n \to \infty)$, $b_n \ge 0 \, (n \ge 1)$ and let

$$F(\sigma) = \sum_{n=1}^{\infty} b_n \exp\{\sigma \lambda_n\} \tag{4}$$

be a Dirichlet series absolutely convergent for all $\sigma \in \mathbb{R}$.

Let $\mu(\sigma, F) = \max\{b_n \exp\{\sigma\lambda_n\} : n \ge 1\}$ be the maximal term and $\nu(\sigma, F) = \max\{n : b_n \exp\{\sigma\lambda_n\} = \mu(\sigma, F)\}$ be the central index of series (4). Functions $\nu(\sigma, F)$ and $\lambda_{\nu(\sigma,F)}$ are piecewise constant and right continuous.

Lemma 1 [3, p. 16–17]. *For $-\infty < \sigma_0 \le \sigma < +\infty$, the following formula holds*

$$\ln \mu(\sigma, F) = \ln \mu(\sigma_0, F) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t,F)} dt. \tag{5}$$

P r o o f o f T h e o r e m 1. Let $\varepsilon > 0$ be an arbitrary number, $\eta = \ln(1+\varepsilon)$, $m = [1/\eta] + 1$ and $p_n = m([\ln n] + 1) - 1$. We consider canonical product

$$f(z) = \prod_{n=1}^{\infty} E(z/a_n, p_n), \quad |z| = r, \tag{6}$$

where $E(z,p)$ is the Weierstrass primary factor. It is well known that f is an entire function. O. Blumenthal [4, p. 131] proved that $\ln |E(z,p)| \le |z|^{p+1}$ for any $p \ge 0$ and $z \in \mathbb{C}$. Therefore, for $r = e^\sigma$

$$\ln M_f(e^\sigma) \le \sum_{n=1}^{\infty} \left(\frac{e^\sigma}{|a_n|}\right)^{p_n+1} = \sum_{n=1}^{\infty} \left(\frac{e^{\sigma+\eta}}{|a_n|}\right)^{m([\ln n]+1)} e^{-\eta m([\ln n]+1)}$$
$$\le \mu(\sigma+\eta) \sum_{n=1}^{\infty} e^{-\eta m \ln n} = K(\varepsilon)\mu(\sigma+\eta), \quad K(\varepsilon) = \text{const} > 0,$$

where $\mu(\sigma)$ is the maximal term of Dirichlet series (4) with $b_n = (1/|a_n|)^{m([\ln n]+1)}$ and $\lambda_n = m([\ln n]+1)$. Since $\mu(\sigma) \to +\infty$, $\sigma \to +\infty$, we have $e^\sigma/|a_{\nu(\sigma)}| > 1$, that is $\nu(\sigma) \le n(e^\sigma, A)$. Hence $\lambda_{\nu(\sigma)} = m([\ln \nu(\sigma)] + 1) \le m(\ln n(e^\sigma, A) + 1)$, $\sigma > \sigma_0$. Thus, in view of (5),

$$\ln \mu(\sigma) \le \ln \mu(\sigma_0) + m \int_{\sigma_0}^{\sigma} (\ln n(e^t, A) + 1) dt = \ln \mu(\sigma_0) + m \int_{\sigma_0}^{e^\sigma} \frac{\ln n(t,A) + 1}{t} dt$$

and

$$\ln \ln M_f(r) \le \ln K(\varepsilon) + \ln \mu(\ln r + \eta)$$
$$\le \ln K(\varepsilon) + \ln \mu(\sigma_0) + m \int_{\sigma_0}^{re^{\eta}} \frac{\ln n(t, A) + 1}{t} dt,$$

whence we obtain (3). Theorem 1 is proved.

3. Let $A = (a_n)$ be a sequence in $\mathbb{D} = \{z : |z| < 1\}$ such that $0 < |a_n| \nearrow 1\,(n \to \infty)$, and let $G(A)$ be the class of analytic functions in $\mathbb{D}$ whose zero set coincides with A. The following counterpart of Theorem 1 is true.

Theorem 2. *For any sequence* $A \subset \mathbb{D}$ *and any* $\varepsilon > 0$ *there exists a function* $f \in G(A)$ *such that*

$$\ln \ln M_f(r) = O\left(\int_0^{\sqrt{1-(1-r)/(1+\varepsilon)}} \frac{\ln n(t, A)}{1-t} dt \right), \quad r \to 1. \tag{7}$$

P r o o f. Let $\varepsilon > 0$ be an arbitrary number, $\eta = \ln(1+\varepsilon)$, $m = [1/\eta] + 1$ and $p_n = m([\ln n] + 1) - 1$. Consider the infinite product

$$f(z) = \prod_{n=1}^{\infty} E\left(\frac{1 - |a_n|^2}{1 - \bar{a}_n z}, p_n\right), \quad |z| = r \in [0, 1). \tag{8}$$

(Products of form (8) with $p_n \equiv$const are known as Naftalevich–Tsuji products [5–7].)

In view of Blumenthal inequality we have

$$\ln M_f(r) \le \sum_{n=1}^{\infty} \left(\frac{1 - |a_n|^2}{|1 - \bar{a}_n z|}\right)^{p_n+1} \le \sum_{n=1}^{\infty} \left(\frac{1 - |a_n|^2}{1 - r}\right)^{p_n+1}. \tag{9}$$

Put $\sigma = \ln \dfrac{1}{1-r}$, $b_n = (1 - |a_n|^2)^{m([\ln n]+1)}$ and $\lambda_n = m([\ln n] + 1)$. Then the last series in (9) becomes Dirichlet series of form (4). Therefore, as above, $(1 - |a_{\nu(\sigma)}|^2)e^{\sigma} > 1$, $\nu(\sigma) \le n(\sqrt{1 - e^{-\sigma}}, A)$, $\lambda_{\nu(\sigma)} \le m(\ln n(\sqrt{1 - e^{-\sigma}}, A) + 1)$

and for $\sigma > \sigma_0$ we obtain

$$\ln\ln M_f(r) \le \ln K(\varepsilon) + \ln\mu\left(\ln\frac{1}{1-r}+\eta\right)$$
$$\le \ln K(\varepsilon) + \ln\mu(\sigma_0) + m\int_{\sigma_0}^{\ln(1/(1-r))+\eta}(\ln n(\sqrt{1-e^{-t}},A)+1)dt$$
$$\le \ln K(\varepsilon) + \ln\mu(\sigma_0) + m\int_0^{\sqrt{1-(1-r)/(1+\varepsilon)}}\frac{\ln n(t,A)+1}{1-t}\frac{2tdt}{1+t},$$

whence (7) follows.

4. Let Ω be the class of positive unbounded functions Φ on $(-\infty,\infty)$ with positive continuous derivative $\Phi' \nearrow +\infty$. Let φ be the inverse function of Φ' and $\Psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(\sigma)$ be the function associated with Φ in the sense of Newton. Then [8] Ψ is continuous and increasing to $+\infty$ function on $(-\infty,+\infty)$ and φ is continuous and increasing to $+\infty$ function on $(0,+\infty)$.

Theorem 3. *If a sequence A satisfies*

$$\overline{\lim_{n\to\infty}}\frac{1}{\ln|a_n|}\Psi\left(\varphi\left(\frac{\ln n}{\ln|a_n|}\right)\right) \le \gamma < 1, \tag{10}$$

then there exists $f \in G(A)$ such that

$$\overline{\lim_{r\to\infty}}\frac{\ln\ln M_f(r)}{\Phi(\ln r)} \le \frac{1}{1-\gamma}. \tag{11}$$

We need some lemmas from the Dirichlet series theory.

Lemma 2 [9]. *Let (4) be a Dirichlet series such that*

$$\overline{\lim_{n\to\infty}}\frac{\ln n}{-\ln b_n} \le h < 1. \tag{12}$$

Then for any $\varepsilon \in (0, 1-h)$ there exists $A_0(\varepsilon) > 0$ such that

$$F(\sigma) \le A_0(\varepsilon)\mu(\sigma/(1-h-\varepsilon),F)^{1-h-\varepsilon},\ \sigma \ge 0. \tag{13}$$

Lemma 3 [8]. *Let $\Phi \in \Omega$. In order that $\ln\mu(\sigma,F) \le \Phi(\sigma)$ for $\sigma \ge \sigma_0$, it is necessary and sufficient that $\ln b_n \le -\lambda_n\Psi(\varphi(\lambda_n))$ for $n \ge n_0$.*

The following lemma is an immediate corollary.

Lemma 4. *Let* $\Phi \in \Omega$, $0 < A$, $B < +\infty$. *In order that* $\ln \mu(\sigma, F) \le A\Phi(B\sigma)$, $\sigma \ge \sigma_0$, *it is necessary and sufficient that* $\ln b_n \le -\frac{\lambda_n}{B}\Psi\left(\varphi\left(\frac{\lambda_n}{AB}\right)\right)$, $n \ge n_0$.

P r o o f o f T h e o r e m 3. Let $\varepsilon \in (0, (1-\gamma)/2)$, $p_n = \left[\frac{\Phi'(\Psi^{-1}((\gamma+\varepsilon)\ln|a_n|))}{1-\gamma-2\varepsilon}\right] - 1$ and let f be represented by (6). Then, as in the proof of Theorem 1, we have

$$\ln M_f(e^\sigma) \le F(\sigma) = \sum_{n=1}^{\infty}\left(\frac{e^\sigma}{|a_n|}\right)^{\lambda_n}, \quad \lambda_n = \left[\frac{\Phi'(\Psi^{-1}((\gamma+\varepsilon)\ln|a_n|))}{1-\gamma-2\varepsilon}\right].$$

Since $b_n = |a_n|^{-1/n}$ and, by (10), $\ln n \le \Phi'(\Psi^{-1}((\gamma+\varepsilon)\ln|a_n|))\ln|a_n|$ for $n \ge n_0(\varepsilon)$, we have

$$\overline{\lim_{n\to\infty}}\frac{\ln n}{-\ln b_n} = \overline{\lim_{n\to\infty}}\frac{\ln n}{\lambda_n \ln|a_n|} \le 1-\gamma-2\varepsilon.$$

Therefore, by Lemma 2 with $h = 1-\gamma-2\varepsilon$,

$$\ln\ln M_f(e^\sigma) \le (\gamma+\varepsilon)\ln\mu\left(\frac{\sigma}{\gamma+\varepsilon}\right) + \ln A_0(\varepsilon), \quad \sigma \ge \sigma_0. \tag{14}$$

We put $A = 1/((\gamma+\varepsilon)(1-\gamma-2\varepsilon))$ and $B = \gamma+\varepsilon$. Then

$$\begin{aligned}\ln b_n = -\lambda_n \ln|a_n| &\le -\frac{\lambda_n}{\gamma+\varepsilon}\Psi(\varphi((1-\gamma-2\varepsilon)\lambda_n)) \\ &= -\frac{\lambda_n}{B}\Psi\left(\varphi\left(\frac{\lambda_n}{AB}\right)\right), \quad n \ge n_0,\end{aligned}$$

and by Lemma 4

$$\ln\mu\left(\frac{\sigma}{\gamma+\varepsilon}\right) \le A\Phi\left(\frac{B\sigma}{\gamma+\varepsilon}\right) = \frac{1}{(\gamma+\varepsilon)(1-\gamma-2\varepsilon)}\Phi(\sigma), \quad \sigma \ge \sigma_0.$$

Therefore (14) implies

$$\ln\ln M_f(e^\sigma) \le \frac{1}{(1-\gamma-2\varepsilon)}\Phi(\sigma) + \ln A_0(\varepsilon), \quad \sigma \ge \sigma_0,$$

whence inequality (11) follows. Theorem 3 is proved.

5. The proof of the following result is based on the same ideas as proofs of Theorems 2 and 3.

Theorem 4. *If a sequence A satisfies condition*

$$\varlimsup_{n\to\infty} \frac{1}{-\ln(1-|a_n|)}\Psi\left(\varphi\left(\frac{\ln n}{-\ln(1-|a_n|)}\right)\right) \le \gamma < 1,$$

then there exists a function $f \in G(A)$ such that

$$\varlimsup_{n\to\infty} \frac{\ln\ln M_f(r)}{\Phi(-\ln(1-r))} \le \frac{1}{1-\gamma}.$$

R e m a r k. The author cannot show that estimate (11) is sharp.

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