

## Strong stabilizability for a class of linear time delay systems of neutral type

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We consider the strong stabilizability problem for delayed systems of neutral type. For simplicity the case of one delay in state is studied. We distinguish a special class of such systems for which we give a constructive solution, without using the derivative of the localized delayed state. Our results are based on an abstract theorem on the strong stabilizability of contractive systems in Hilbert space. An illustrative example is also given.

### 1. Introduction

The problems of stability and stabilizability are of great importance in the theory of delayed systems [2, 9, 6]. In this context note that the majority of works deals with the so-called exponential stability or stabilizability. In this case the conditions of stability (stabilizability) are well explored for both systems with ordinary delay and of neutral type [2, 10, 9, 6, 17, 15]. Note also that this type of stability is similar to the stability for finite-dimensional linear systems. However, for systems of neutral type appears an essentially different kind of stability — the so-called strong stability.

Consider a system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_{-1} \dot{x}(t-1), \quad (1)$$

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where  $x \in \mathbb{R}^n$ ,  $A_0, A_1$  and  $A_{-1}$  are  $n \times n$ -matrices. It is well known [6] that (1) is exponentially stable iff  $\operatorname{Re}(\sigma) \leq -\alpha < 0$ , where  $\sigma = \{\lambda : \det(\lambda I - A_{-1}\lambda e^{-\lambda} - A_0 - A_1 e^{-\lambda}) = 0\}$ , and  $\operatorname{Re}(\sigma)$  is the real part of all values in  $\sigma$ .

In particular, this condition holds [9] if  $\|A_{-1}\| < 1$  and  $\operatorname{Re}(\sigma) < 0$ .

On the other hand, the system (1) is unstable if matrix  $A_{-1}$  possesses at least one eigenvalue  $\mu$  such that  $|\mu| > 1$ . At the same time, it turns out [2, 3] that asymptotic stability of (1) is possible under the weaker condition  $\operatorname{Re}(\sigma) < 0$ , which, in particular, may appear when there are some eigenvalues  $\mu_j$ ,  $j = 1, \dots, k$  of matrix  $A_{-1}$  such that  $|\mu_j| = 1$ ,  $j = 1, \dots, k$ . As it is shown in [3], in this case solutions of the system (1) decay essentially slower than exponentials, namely as functions  $1/t^\beta$ ,  $\beta > 0$ .

An explanation of this effect can be found using the model of neutral type systems as abstract differential equations in Banach space and the results of the theory of strong asymptotic stability originated in [1, 13, 20] (see also the bibliography in [14]).

Note also that the results on the strong stability find a natural application in the control theory for analysis of the strong stabilizability of contractive semi-groups, for example [4, 16, 12, 11, 19].

The main goal of the present paper is to show an extension of the stabilizability theory to the case of control systems of neutral type. To justify this point we consider a special class of neutral type systems (including, however, all the one-dimensional systems) and give a constructive solution of the strong stabilizability problem for this class based on an abstract theorem from [11].

The paper is organized as follows. In Section 2 we formulate the problem of strong stabilizability and interpret it in the language of abstract control systems. In Section 3 an operator analysis of the obtained system is given. The main results are given in Sect. 4. We design explicitly a strong stabilizing control and give an illustrative example.

Finally notice that the analysis of the strong stabilizability problem under some more general assumption is to be given in one of our forthcoming works.

## **2. The model and the statement of stabilizability problem**

For simplicity we consider a control neutral type system with one delay in the state

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_{-1} \dot{x}(t-1) + B u(t), \quad (2)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^r$ ,  $A_j, j = 0, 1, -1$  are  $n \times n$ -matrices,  $B$  is a  $n \times r$ -matrix.

The stabilizability problem amounts on finding a linear feedback control law  $u = p(x(\cdot))$  such that the closed loop system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_{-1} \dot{x}(t-1) + B p(x(\cdot))$$

becomes an asymptotic stable one. In order to formulate the problem more precisely let us consider an abstract functional model of the system (2). Following Yamamoto and Ueshima [21] (see also [10] and a more general system in [18]), we put

$$x_t(\cdot) : \theta \mapsto x(t + \theta), \quad \theta \in [-1, 0]$$

and  $y(t) = x(t) - A_{-1}x(t - 1)$ . Let  $Z = \mathbb{C}^n \times L_2[(-1, 0), \mathbb{C}^n]$ . For  $z_i = \begin{pmatrix} q_i \\ \varphi_i(\cdot) \end{pmatrix} \in Z$ ,  $i = 1, 2$ , the scalar product in  $Z$  is given by

$$\langle z_1, z_2 \rangle_Z = \langle q_1, q_2 \rangle_{\mathbb{C}^n} + \int_{-1}^0 \langle \varphi_1(\theta), \varphi_2(\theta) \rangle_{\mathbb{C}^n} d\theta.$$

The corresponding norm is denoted by  $\|\cdot\|_Z$ . The indices will be omitted if it is not necessary.

Introduce an operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow Z$  defined by

$$\mathcal{A} \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 q + (A_1 + A_0 A_{-1})\varphi(-1) \\ \frac{\partial}{\partial \theta} \varphi(\cdot) \end{pmatrix},$$

where

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} : q = \varphi(0) - A_{-1}\varphi(-1), \quad \varphi(\cdot) \in W_2^{(1)}[(-1, 0), \mathbb{C}^n] \right\}.$$

With these notations the system (2) can be rewritten as

$$\frac{d}{dt} \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix} = \mathcal{A} \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix} + \mathcal{B}u(t), \quad (3)$$

where  $\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$  is a linear operator  $\mathcal{B} : \mathbb{C}^n \rightarrow Z$ .

It is known [21, 22] that  $\mathcal{A}$  generates a  $C_0$ -semigroup in  $Z$  and that its spectrum  $\sigma(\mathcal{A})$  is the set

$$\sigma(\mathcal{A}) = \sigma = \{ \lambda : \det(\lambda I - A_{-1}\lambda e^{-\lambda} - A_0 - A_1 e^{-\lambda}) = 0 \}.$$

and consists of eigenvalues only. Denote further by  $\Sigma$  the set of all nonzero eigenvalues of matrix  $A_{-1}$ . Then [2] for any  $\mu \in \Sigma$  the set  $\sigma$  includes a family of eigenvalues

$$\sum^\mu = \{ \lambda_k^\mu = \log |\mu| + i(\arg \mu + 2\pi k) + \bar{o}(1), \quad k \in \mathbb{Z} \}, \quad (4)$$

where  $\bar{o}$  is meant as  $k \rightarrow \pm\infty$ .

The substitution of a feedback control  $u = p(x(\cdot))$  into (2) amounts to transforming system (3) to the form

$$\frac{d}{dt} \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix} = \tilde{\mathcal{A}} \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix}, \quad (5)$$

where  $\tilde{\mathcal{A}}$  is a perturbation of the infinitesimal operator  $\mathcal{A}$  by an operator of the form  $\mathcal{B}\mathcal{P}$ , where  $\mathcal{P} : Z \rightarrow \mathbb{C}^n$ . There are three different kinds of such a perturbation.

1. *Perturbation with a bounded operator* (class 1).

This corresponds to the case where we admit feedback controls

$$u = \mathcal{P}(x(\cdot)) = P(x(t) - A_{-1}x(t-1)) + \int_{-1}^0 \hat{P}(\theta)x(t+\theta)d\theta,$$

where  $P$  is a real  $(r \times n)$ -matrix,  $\hat{P}(\theta), \theta \in [-1, 0]$  is a real square-integrable  $(r \times n)$ -matrix-function.

In this case  $\mathcal{B}\mathcal{P}$  is a bounded operator and so [7] the perturbed operator  $\tilde{\mathcal{A}}$  is infinitesimal and  $\mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A})$ . Note, however, that possibilities of stabilization in this class of controls are rather restricted.

2. *Perturbation with an operator bounded with respect to  $\mathcal{A}$*  (class 2).

This corresponds to the choice

$$u = \mathcal{P}(x(\cdot)) = \int_{-1}^0 \tilde{P}(\theta)\dot{x}(t+\theta)d\theta + \int_{-1}^0 \hat{P}(\theta)x(t+\theta)d\theta,$$

where  $\tilde{P}(\theta), \hat{P}(\theta), \theta \in [-1, 0]$  are real square-integrable  $(r \times n)$ -matrix-function. In this case one can easily check that the operator  $\mathcal{B}\mathcal{P}$  is a bounded with respect to  $\mathcal{A}$  [7], i.e., for some  $a, b > 0$

$$\left\| \mathcal{B}\mathcal{P} \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\| \leq a \left\| \mathcal{A} \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\| + b \left\| \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\|.$$

This implies  $\mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A})$ . At the same time, the infinitesimality of  $\tilde{\mathcal{A}}$  must be proved separately (see [7]). One can observe, however, an important particular case for our further purpose, for which this infinitesimality is obvious. Let

$$u = \mathcal{P}(x(\cdot)) = P_0x(t) + P_1x(t-1) + \int_{-1}^0 \hat{P}(\theta)x(t+\theta)d\theta,$$

$P_0, P_1$  are  $(r \times n)$ -matrices and  $\widehat{P}(\theta)$  is a square-integrable  $(r \times n)$ -matrix-function. Then the operator  $\widetilde{\mathcal{A}}$  can be represented as a perturbation of the infinitesimal operator  $\widetilde{\mathcal{A}}_1$ :

$$\widetilde{\mathcal{A}}_1 \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} (A_0 + BP_0)q + ((A_1 + BP_1) + (A_0 + BP_0)A_{-1})\varphi(-1) \\ \frac{\partial}{\partial \theta} \varphi(\cdot) \end{pmatrix}, \quad (6)$$

by a bounded operator. So it is also infinitesimal. Consider the possibilities of stabilization by feedback controls of class 2. It can be proved that the spectrum  $\sigma(\widetilde{\mathcal{A}}) = \widetilde{\sigma}$  of the perturbed operator  $\widetilde{\mathcal{A}}$  is given by

$$\widetilde{\sigma} = \left\{ \lambda : \det \left( \lambda I - A_{-1} \lambda e^{-\lambda} - A_0 - A_1 e^{-\lambda} - B \lambda \widetilde{\Pi} - B \widehat{\Pi} \right) = 0 \right\},$$

where

$$\widetilde{\Pi} = \int_{-1}^0 e^{\lambda \theta} \widetilde{P}(\theta) d\theta, \quad \widehat{\Pi} = \int_{-1}^0 e^{\lambda \theta} \widehat{P}(\theta) d\theta$$

and then it also includes the families  $\sum^\mu$  of the form (4) for any  $\mu \in \sum$ . This means that the exponential stability of the closed loop system (5) is possible only in the case when  $|\mu| < 1$  for all  $\mu \in \sum$ . On the other hand, it is clear that the system (5) is unstable if there exists at least one  $\mu \in \sum$  such that  $|\mu| > 1$ . It remains one more case to be considered. Let us make the following assumption: (a1)  $\sum \subset \{w : |w| \leq 1\}$  and there exists  $\mu \in \sum : |\mu| = 1$ .

In this case system (5) cannot be exponentially stable by a feedback of class 1 or 2 but probably be strongly stable. That leads us to the following statement:

**Problem of strong stabilizability (PSS).** *Let the matrix  $A_{-1}$  satisfies (a1). Find conditions on system (2) (or (3)) under which there exists a feedback control of class 2 such that the perturbed operator  $\widetilde{\mathcal{A}}$  in (5) is infinitesimal and all the solutions of this equation tend to 0 as  $t \rightarrow +\infty$  in the norm of  $Z$ .*

We consider PSS in the further sections. Now we mention one more way to formulate the stabilizability problem.

### 3. Perturbation with an operator unbounded with respect to $\mathcal{A}$ (class 3).

It is shown [17, 15] that the possibilities of stabilization of system (2) are essentially wider if we admit feedback controls of the form

$$u = \mathcal{P}(x(\cdot)) = P_{-1} \dot{x}(t-1) + \int_{-1}^0 \widetilde{P}(\theta) \dot{x}(t+\theta) d\theta + \int_{-1}^0 \widehat{P}(\theta) x(t+\theta) d\theta. \quad (7)$$

This kind of stabilization is out of our consideration. We only notice that the use of a control such as in (7) means, from the operator point of view, a perturbation

of  $\mathcal{A}$  by an operator  $\mathcal{BP}$  which is not bounded with respect to  $\mathcal{A}$ . In particular, that implies  $\mathcal{D}(\tilde{\mathcal{A}}) \neq \mathcal{D}(\mathcal{A})$ . So even if we prove infinitesimality of  $\tilde{\mathcal{A}}$ , the domains of solutions of the initial and closed loop systems are different.

### 3. Operator analysis of the model

We consider PSS and complete (a1) with the following assumptions characterizing the class of systems (2) we deal with:

(a2) All the eigenvalues  $\mu \in \Sigma$  such that  $|\mu| = 1$  are simple in the sense that there are no Jordan chains corresponding to such eigenvalues.

(a3) The finite-dimensional system

$$\dot{x}(t) = A_0x(t) + Bu(t), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^r \quad (8)$$

is controllable, i.e.,  $\text{rank}(B A_0 B \cdots A_0^{n-1} B) = n$ . In particular, this implies that (8) is stabilizable, i.e. there exists a linear feedback control  $u = P_0^0 x$  such that  $\text{Re}(\sigma(A + BP_0^0)) < 0$ .

(a4)  $\text{rank}(A_1 + A_0 A_{-1} B) = \text{rank} B$ .

Let us put into (2) a control  $u(t) = P_0 x(t) + P_1 x(t-1) + v(t)$ . That leads to replace (3) by the system

$$\frac{d}{dt} \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix} = \tilde{\mathcal{A}}_1 \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix} + \mathcal{B}v(t),$$

where  $\tilde{\mathcal{A}}_1$  is given by (6). Due to (a4) for any  $P_0 \in \mathbb{R}^{(r \times n)}$  there exists a matrix  $P_1 = P_1(P_0) \in \mathbb{R}^{(r \times n)}$  such that

$$A_1 + BP_1 + (A_0 + BP_0)A_{-1} = (A_1 + A_0 A_{-1}) + BP_0 A_{-1} + BP_1 = 0.$$

For this choice of  $P_1$  operator  $\tilde{\mathcal{A}}_1$  takes the form

$$\tilde{\mathcal{A}}_1 \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 + BP_0 & 0 \\ 0 & \frac{\partial}{\partial \theta} \end{pmatrix} \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}. \quad (9)$$

**Proposition 3.1.** *Let  $\tilde{\mathcal{A}}_1$  be given by (9). Then*

i)  $\sigma(\tilde{\mathcal{A}}_1) = \sigma(A_0 + BP_0) \cup \log \Sigma$ .

ii) *Under the assumption  $\sigma(A_0 + BP_0) \cap \log \Sigma = \emptyset$  the set of eigenvectors of  $\tilde{\mathcal{A}}_1$  is as follows:*

a) *to each eigenvector  $d \in \mathbb{C}^n$  of  $A_0 + BP_0$  with eigenvalue  $\lambda$  there corresponds an eigenvector*

$$\tilde{d} = \begin{pmatrix} d \\ (I - e^{-\lambda} A_{-1})^{-1} e^{\lambda \theta} \end{pmatrix}$$

*of  $\tilde{\mathcal{A}}_1$  with the same eigenvalue;*

b) to each eigenvector  $g \in \mathbb{C}^n$  of  $A_{-1}$  with eigenvalue  $\mu$  there corresponds a family  $\{\tilde{g}_k\}_{k \in \mathbb{Z}}$  of eigenvectors of  $\tilde{A}$ :

$$\tilde{g}_k = \begin{pmatrix} (I - e^{-\lambda_k^\mu} A_{-1})g \\ e^{\lambda_k^\mu \theta} g \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\lambda_k^\mu \theta} g \end{pmatrix},$$

where  $\lambda_k^\mu = \log |\mu| + i(\arg \mu + 2\pi k)$ ,  $k \in \mathbb{Z}$  is the eigenvalue corresponding to  $\tilde{g}_k$ .

*P r o o f.* Let  $\begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix}$  be an arbitrary eigenvector of  $\tilde{A}_1$  and  $\lambda$  be the corresponding eigenvalue. Taking into account (9), we have  $\frac{d}{d\theta} \varphi(\theta) = \lambda \varphi(\theta)$  and  $(A_0 + BP_0)q = \lambda q$ . From the first equality we obtain  $\varphi(\theta) = e^{\lambda \theta} c$ ,  $c \in \mathbb{C}^n$ ,  $c \neq 0$ . Since  $q = \varphi(0) - A_{-1} \varphi(1)$ , then the second equality yields  $(A_0 + BP_0)(c - A_{-1} e^{-\lambda} c) - \lambda(c - A_{-1} e^{-\lambda} c) = (A_0 + BP_0 - \lambda I)(I - A_{-1} e^{-\lambda})c = 0$ . Therefore, either  $(I - A_{-1} e^{-\lambda})c$  is an eigenvector for  $A_0 + BP_0$  corresponding to  $\lambda$  or  $c$  is an eigenvector of  $A_{-1}$  corresponding to  $e^\lambda$ . This alternative completes the proof. ■

Using (a3) one can choose  $P_0 \in \mathbb{R}^{(r \times n)}$  in such a way that the spectrum  $\sigma(A_0 + BP_0)$  consists of  $n$  **distinct negative eigenvalues which do not belong to the set**  $\log \Sigma$ . Let further  $P_0^0$  be such a matrix and  $P_1^0 = P_1(P_0^0)$  and  $\tilde{A}_1^0$  be the operator (6) corresponding to the choice  $P_0 = P_0^0$ ,  $P_1 = P_1^0$ . Then, by Prop. 3.1, the spectrum  $\sigma(\tilde{A}_1^0)$  belongs to the semiplane  $\{\lambda : \text{Re}(\lambda) \leq 0\}$ . Our next goal is to prove that the system

$$\frac{d}{dt} \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix} = \tilde{A}_1^0 \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix} + \mathcal{B}v(t), \tag{10}$$

is strongly stabilizable by linear bounded controls. To show that, we first prove dissipativity of the operator  $\tilde{A}_1^0$  in some equivalent norm in  $Z$ .

Let  $d_j, j = 1, \dots, n$  be the eigenvectors of  $A_0 + BP_0^0$  corresponding to  $\lambda_j < 0$ . Denote by  $D$  the nonsingular matrix  $D = (d_1 d_2 \dots d_n)$ . Now observe that, due to (a1) – (a2), the matrix  $A_{-1}$  can be represented in the form

$$A_{-1} = GJG^{-1}, \tag{11}$$

where  $G$  is a nonsingular matrix and  $J$  is a contraction,  $\|J\| \leq 1$ . As the matrix  $J$  one can take, for example, a block diagonal form of  $A_{-1}$  which blocks are

$$J_k = \begin{pmatrix} \mu_k & \nu_k & 0 & \dots & 0 \\ 0 & \mu_k & \nu_k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_k \end{pmatrix},$$

where  $|\nu_k| \leq 1 - |\mu_k|$ ,  $k = 1, \dots, \ell$  (note that all the eigenvalues  $\mu_k$  of  $A_{-1}$  such that  $|\mu_k| = 1$  are simple (a2)). Finally let us introduce the linear bounded operator  $F : \mathbb{C}^n \rightarrow L_2[(-1, 0), \mathbb{C}^n]$  defined by

$$Fq = F \left( \sum_{j=1}^n q_j d_j \right) = -G^{-1} \sum_{j=1}^n q_j \left( I - A_{-1} e^{-\lambda_j} \right)^{-1} e^{\lambda_j \theta} d_j. \quad (12)$$

If we denote by  $\Delta(\theta)$  the matrix with column  $(I - A_{-1} e^{-\lambda_j})^{-1} e^{\lambda_j \theta} d_j$ , then  $F$  can be written as

$$Fq = -G^{-1} \Delta(\theta) D^{-1} q. \quad (13)$$

Consider now a linear bounded operator  $T : Z \rightarrow Z$  given by

$$Tz = T \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} D^{-1} & 0 \\ F & G^{-1} \end{pmatrix} \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} D^{-1} q \\ (Fq)(\theta) + G^{-1} \varphi(\theta) \end{pmatrix}, \quad (14)$$

the corresponding inner product  $\langle \cdot, \cdot \rangle_T = \langle T \cdot, T \cdot \rangle$  and the equivalent Hilbert norm  $\| \cdot \|_T$  in  $Z$  defined by

$$\left\| \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\|_T = \left\| T \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \right\| = \left( \|D^{-1} q\|^2 + \int_{-1}^0 \|(Fq)(\theta) + G^{-1} \varphi(\theta)\|^2 d\theta \right)^{\frac{1}{2}}.$$

This new norm allows to get the dissipativity of the operator  $\tilde{\mathcal{A}}_1^0$ .

**Proposition 3.2.** *Operator  $\tilde{\mathcal{A}}_1^0$  is dissipative in the norm  $\| \cdot \|_T$ , i.e., for all  $z = \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \in \mathcal{D}(\tilde{\mathcal{A}}_1^0) = \mathcal{D}(\mathcal{A})$  we have  $\operatorname{Re} \langle \tilde{\mathcal{A}}_1^0 z, z \rangle_T \leq 0$ .*

*P r o o f.* Let  $z = \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \in \mathcal{D}(\tilde{\mathcal{A}}_1^0)$ , then we have

$$\langle \tilde{\mathcal{A}}_1^0 z, z \rangle_T = \langle D^{-1}(A_0 + BP_0^0)q, D^{-1}q \rangle + R, \quad (15)$$

where

$$R = \int_{-1}^0 \left\langle (F(A_0 + BP_0^0)q)(\theta) + G^{-1} \frac{d}{d\theta} \varphi(\theta), (Fq)(\theta) + G^{-1} \varphi(\theta) \right\rangle d\theta. \quad (16)$$

The vector  $q$  may be decomposed as  $q = \sum_{j=1}^n q_j d_j$ . Then

$$\langle D^{-1}(A_0 + BP_0^0)q, D^{-1}q \rangle = \sum_{j=1}^n \lambda_j \|d_j\|^2$$



and, therefore,

$$\operatorname{Re} \left( \langle D^{-1}(A_0 + BP_0^0)q, D^{-1}q \rangle \right) = \sum_{j=1}^n \operatorname{Re}(\lambda_j) \|d_j\|^2 \leq 0. \quad (17)$$

Taking into account (12), we have

$$(F(A_0 + BP_0^0)q)(\theta) = -G^{-1} \sum_{j=1}^n q_j \left( I - A_{-1}e^{-\lambda_j} \right)^{-1} \lambda_j e^{\lambda_j \theta} d_j = \frac{d}{d\theta} (Fq)(\theta).$$

Therefore the term  $R$  given in (16) may be written as

$$\begin{aligned} R &= \int_{-1}^0 \left\langle \frac{d}{d\theta} [(Fq)(\theta) + G^{-1}\varphi(\theta)], (Fq)(\theta) + G^{-1}\varphi(\theta) \right\rangle d\theta \\ &= \|(Fq)(\theta) + G^{-1}\varphi(\theta)\|^2 \Big|_{-1}^0 \\ &\quad - \int_{-1}^0 \left\langle (Fq)(\theta) + G^{-1}\varphi(\theta), \frac{d}{d\theta} [(Fq)(\theta) + G^{-1}\varphi(\theta)] \right\rangle d\theta \end{aligned}$$

and this gives

$$\operatorname{Re}(R) = \frac{1}{2} (\|(Fq)(0) + G^{-1}\varphi(0)\|^2 - \|(Fq)(-1) + G^{-1}\varphi(-1)\|^2). \quad (18)$$

Note that, as  $z \in \mathcal{D}(\mathcal{A})$ , we have

$$\varphi(0) - A_{-1}\varphi(-1) = q = \sum_{j=1}^n q_j d_j. \quad (19)$$

Let us put  $\psi(\theta) = \varphi(\theta) - \sum_{j=1}^n q_j (I - A_{-1}e^{-\lambda_j})^{-1} e^{\lambda_j \theta} d_j$ . Then one can easily check that (19) implies

$$\psi(0) = A_{-1}\psi(-1). \quad (20)$$

From (12) we obtain

$$\begin{aligned} (Fq)(\theta) + G^{-1}\varphi(\theta) &= -G^{-1} \left( \sum_{j=1}^n q_j (I - A_{-1}e^{-\lambda_j})^{-1} e^{\lambda_j \theta} d_j - \varphi(\theta) \right) \\ &= G^{-1}\psi(\theta). \end{aligned}$$

Hence, taking into account (20), relation (18) can be rewritten as

$$\operatorname{Re}(R) = \frac{1}{2} (\|G^{-1}\psi(0)\|^2 - \|G^{-1}\psi(-1)\|^2)$$

$$= \frac{1}{2}(\|G^{-1}A_{-1}\psi(-1)\|^2 - \|G^{-1}\psi(-1)\|^2).$$

Let us substitute  $G^{-1}\psi(-1) = w$  and make use of (11). That yields

$$\operatorname{Re}(R) = \frac{1}{2}(\|Jw\|^2 - \|w\|^2) \leq 0. \quad (21)$$

Comparison of (15), (17) and (21) completes the proof.  $\blacksquare$

**Corollary 3.3.** *It follows from Prop. 3.2 that the semigroup  $\{e^{\tilde{\mathcal{A}}_1^0 t}\}_{t \geq 0}$  is contractive in the norm  $\|\cdot\|_T$ . In fact,*

$$\frac{d}{dt}\|e^{\tilde{\mathcal{A}}_1^0 t} z\|_T^2 = 2\operatorname{Re}\left(\left\langle \tilde{\mathcal{A}}_1^0 e^{\tilde{\mathcal{A}}_1^0 t} z, e^{\tilde{\mathcal{A}}_1^0 t} z \right\rangle\right) \leq 0.$$

This means that (10) is a contractive system in the space  $Z$  with norm  $\|\cdot\|_T$  (see [11]).

#### 4. The strong stabilizability

In order to analyze strong stabilizability of (10) we make use of the following theorem on the strong stabilizability of contractive systems [11, Theorem 5]:

Consider a system of the form

$$\frac{d}{dt}z = \mathcal{A}z + \mathcal{B}u, \quad z \in H, u \in U,$$

where  $H$  and  $U$  are Hilbert spaces, the operator  $\mathcal{A}$  generates a strongly continuous contractive semigroup  $\{e^{\mathcal{A}t}\}_{t \geq 0}$  and  $\mathcal{B}$  a linear bounded operator from  $H$  to  $U$ . If there exists  $t_0 > 0$  such that the set

$$\sigma(e^{\mathcal{A}t_0}) \cap \{w \in \mathbb{C} : |w| = 1\}$$

is at most countable, then the system is strongly stabilizable (with the aid of linear bounded control law) if and only if there does not exist an eigenvector  $z_0$  of the operator  $\mathcal{A}$  corresponding to an eigenvalue  $\lambda_0$ ,  $\operatorname{Re}(\lambda_0) = 0$ , such that  $z_0 \in \operatorname{Ker} \mathcal{B}^*$ . If this condition holds then the strong stabilizing control can be chosen as  $u = -\mathcal{B}^* z$ .

It has been shown that the semigroup  $\{e^{\tilde{\mathcal{A}}_1^0 t}\}_{t \geq 0}$  is contractive in the space  $Z$  with norm  $\|\cdot\|_T$ . It is known [5] that

$$\sigma(e^{\tilde{\mathcal{A}}_1^0 t_0}) \subset \overline{\exp(t_0 \sigma(\tilde{\mathcal{A}}_1^0))}$$

( $\bar{S}$  means the closure of  $S$ ). From Proposition 3.1 we have for  $t_0 = 1$ :

$$\exp(\sigma(\tilde{\mathcal{A}}_1^0)) = \exp(\sigma(A^0 + BP_0^0)) \cup \sum$$

and, therefore, this set is finite. Hence the set

$$\sigma(e^{\tilde{\mathcal{A}}_1^0}) \cap \{w \in \mathbb{C} : |w| = 1\} \subset \sigma(e^{\tilde{\mathcal{A}}_1^0}) \subset \exp(\sigma(\tilde{\mathcal{A}}_1^0))$$

is also finite. Thus, on the basis of [11, Theorem 5] we conclude that the system (10) is strongly stabilizable (notice that stabilizability in norms  $\|\cdot\|$  and  $\|\cdot\|_T$  is equivalent) iff there exists no eigenvector  $z_0$  of the operator  $\tilde{\mathcal{A}}_1^0$  corresponding to a pure imaginary eigenvalue such that

$$z_0 \in \text{Ker } \mathcal{B}_T^*, \tag{22}$$

where  $\mathcal{B}_T^* : (Z, \|\cdot\|_T) \rightarrow \mathbb{C}^r$  is the adjoint operator to  $\mathcal{B}$  in the norm  $\|\cdot\|_T$ . In this case the strong stabilizing control can be chosen in the form

$$v(t) = -\mathcal{B}_T^* \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix}. \tag{23}$$

In order to compute the feedback law, we need the expression of the operator  $\mathcal{B}_T^*$ .

Let  $u \in \mathbb{C}^r$  and  $z = \begin{pmatrix} q \\ \varphi(\cdot) \end{pmatrix} \in Z$ . Then, taking in account the form of the operator  $T$  in (14), we have

$$\langle u, \mathcal{B}_T^* z \rangle = \langle \mathcal{B}u, z \rangle_T = \langle T\mathcal{B}u, Tz \rangle = \left\langle \begin{pmatrix} D^{-1}Bu \\ F\mathcal{B}u \end{pmatrix}, \begin{pmatrix} D^{-1}q \\ Fq + G^{-1}\varphi \end{pmatrix} \right\rangle.$$

And a simple computation gives

$$\mathcal{B}_T^* z = (B^* D^{-1*} D^{-1} + B^* F^* F)q + B^* F^* G^{-1} \varphi. \tag{24}$$

Using the expressions (12) and (13) of  $F$ , we get

$$F^* \psi = - \int_{-1}^0 D^{-1*} \Delta^*(\theta) G^{-1*} \psi(\theta) d\theta = -D^{-1*} \int_{-1}^0 \tilde{Q}(\theta) \psi(\theta) d\theta,$$

where  $\tilde{Q}(\theta) = \Delta^*(\theta) G^{-1*}$ , and then, putting  $Q = \int_{-1}^0 \tilde{Q}(\theta) \tilde{Q}^*(\theta) d\theta$ , we obtain

$$F^* F = \int_{-1}^0 D^{-1*} \Delta^*(\theta) G^{-1*} G^{-1} \Delta(\theta) D^{-1} d\theta = D^{-1*} Q D^{-1}.$$

Finally, with  $\widehat{Q}(\theta) = \widetilde{Q}(\theta)G^{-1}$ , the feedback may be written as

$$v(t) = -\mathcal{B}_T^* z(t) = -B^* D^{-1*} \left( (I + Q)D^{-1}y(t) - \int_{-1}^0 \widehat{Q}(\theta)x_t(\theta)d\theta \right). \quad (25)$$

Now let us analyze the condition (22). Among all the eigenvalues of  $\widetilde{\mathcal{A}}_1^0$  the pure imaginary ones are (see Prop. 3.1)

$$\lambda_k^\mu = \log |\mu| + i(\arg \mu + 2\pi k), \quad k \in \mathbb{Z}, \quad (26)$$

for  $\mu \in \Sigma$  such that  $|\mu| = 1$ . For the corresponding eigenvectors  $\widetilde{g}_k = \begin{pmatrix} 0 \\ e^{\lambda_k^\mu \theta} g \end{pmatrix}$ ,  $k \in \mathbb{Z}$ , we have

$$\begin{aligned} \mathcal{B}_T^* \widetilde{g}_k &= B^* D^{-1*} \int_{-1}^0 \widehat{Q}(\theta) e^{\lambda_k^\mu \theta} g d\theta \\ &= B^* D^{-1*} \begin{pmatrix} d_1^* (I - A_{-1}^* e^{-\lambda_1})^{-1} \frac{1 - e^{-\lambda_1 - \lambda_k^\mu}}{\lambda_1 + \lambda_k^\mu} \\ \dots \\ d_n^* (I - A_{-1}^* e^{-\lambda_n})^{-1} \frac{1 - e^{-\lambda_n - \lambda_k^\mu}}{\lambda_n + \lambda_k^\mu} \end{pmatrix} G^{-1*} G^{-1} g. \end{aligned} \quad (27)$$

In (27)  $g$  is an eigenvector of  $A_{-1}$  corresponding to eigenvalue  $\mu$ . This implies that  $G^{-1*} G^{-1} g$  is an eigenvector of  $A_{-1}^*$  corresponding to complex conjugate eigenvalue  $\bar{\mu}$ . Indeed, taking in account (11), we get

$$\begin{aligned} \langle J^* G^{-1} g, G^{-1} g \rangle &= \langle G^* A_{-1}^* G^{-1*} G^{-1} g, G^{-1} g \rangle \\ &= \langle G^{-1*} G^{-1} g, A_{-1} g \rangle \\ &= \bar{\mu} \langle G^{-1} g, G^{-1} g \rangle. \end{aligned} \quad (28)$$

Since the adjoint operator  $J^*$  is also a contraction then (28), with Cauchy-Buniakovski inequality, yields  $J^* G^{-1} g = \bar{\mu} G^{-1} g$ . From this and (11) we get  $A_{-1}^* G^{-1*} G^{-1} g = \bar{\mu} G^{-1*} G^{-1} g$ . This fact and the observation that  $e^{-\lambda_k^\mu} = \bar{\mu}$ ,  $\overline{\lambda_k^\mu} = -\lambda_k^\mu$ ,  $k \in \mathbb{Z}$  allow to rewrite (27) as

$$\mathcal{B}_T^* \widetilde{g}_k = B^* D^{-1*} \begin{pmatrix} \frac{1}{\lambda_1 + \lambda_k^\mu} d_1^* \\ \dots \\ \frac{1}{\lambda_n + \lambda_k^\mu} d_n^* \end{pmatrix} g = B^* R_{\lambda_k^\mu}^* (A + B P_0^0) g, \quad (29)$$

where  $R_\lambda(A_0 + B P_0^0) = (A_0 + B P_0^0 - \lambda I)^{-1}$  is the resolvent of the matrix  $A_0 + B P_0^0$ . With respect to formulas (25) and (29) the necessary and sufficient conditions of the strong stabilizability for the system (10) take the following form:

**Theorem 4.1.** *System (10) is strongly stabilizable (with the aid of the of bounded controls) iff there exists no eigenvector  $g$  of matrix  $A_{-1}$  corresponding to an eigenvalue  $\mu \in \Sigma$ ,  $|\mu| = 1$  and  $k \in \mathbb{Z}$  such that*

$$B^* R_{\lambda_k^\mu}^*(A_0 + BP_0^0)g = 0,$$

where  $\lambda_k^\mu$  is given by (26). Under this condition a stabilizing control is given by (25).

**Remark 4.2.** *Let  $P_0$  be a  $(r \times n)$ -matrix and let  $\lambda \in \mathbb{C}$  be such that  $\lambda \notin \sigma(A_0 + BP_0^0) \cup \sigma(A_0 + BP_0)$ . Let us precise that*

$$\begin{aligned} R_\lambda^*(A_0 + BP_0^0) - R_\lambda^*(A_0 + BP_0) &= R_\lambda^*(A_0 + BP_0)(P_0^* - P_0^{0*})B^* R_\lambda^*(A_0 + BP_0^0), \\ &= R_\lambda^*(A_0 + BP_0^0)(P_0^* - P_0^{0*})B^* R_\lambda^*(A_0 + BP_0). \end{aligned}$$

From this identity one can easily conclude that for given  $\mu, g, \lambda_k^\mu$  the relation  $B^* R_{\lambda_k^\mu}^*(A_0 + BP_0^0)g = 0$  holds if and only if  $B^* R_{\lambda_k^\mu}^*(A_0 + BP_0)g = 0$  for an arbitrary  $P_0$  such that  $\lambda_k^\mu \notin \sigma(A + BP_0)$ .

The following theorem is the main result of the paper.

**Theorem 4.3.** *Let a of the form system (2) satisfy the assumptions (a1)–(a4). Then this system is strongly stabilizable by a feedback control of class 2 if and only if for an arbitrarily chosen matrix  $P_0$  such that*

$$\sigma(A_0 + BP_0) \cap \log(\Sigma) \cap (i\mathbb{R}) = \emptyset$$

there does not exist an eigenvector  $g$  of  $A_{-1}$  corresponding to an eigenvalue  $\mu \in \Sigma$ ,  $|\mu| = 1$  and  $k \in \mathbb{Z}$  such that

$$B^* R_{\lambda_k^\mu}^*(A_0 + BP_0)g = 0, \tag{30}$$

where  $\lambda_k^\mu$  is given by (26). Under this condition the strong stabilization can be achieved by the choice of control:

$$u = P_0^0 x(t) + P_1^0 x(t - 1) + v,$$

where  $P_0^0$  and  $P_1^0$  are defined in Sect. 3 and  $v$  is given by (25):

$$v = -B^* D^{-1*} \left( ((I + Q)D^{-1}(x(t) - A_{-1}x(t - 1))) - \int_{-1}^0 \widehat{Q}(\theta)x(\theta)d\theta \right).$$

P r o o f. Sufficiency follows directly from Theorem 4.1 and the Remark 4.2 to this theorem.

Let us prove the necessity. Assume that there exists a control  $u = \mathcal{P} \begin{pmatrix} y(t) \\ x_t(\cdot) \end{pmatrix}$  of class 2 which strongly stabilizes system (3). This means that the operator  $\tilde{\mathcal{A}} = \mathcal{A} + \mathcal{BP}$  with  $\mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(A)$  is infinitesimal and the semigroup  $\{e^{\tilde{\mathcal{A}}t}\}_{t \geq 0}$  is strongly asymptotic stable. Then  $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_1^0 + \mathcal{BP}_1$ , where  $\mathcal{BP}_1$  is an operator bounded with respect to  $\mathcal{A}$ . If (30) does not hold then (see Remark 4.2) there exist an eigenvector  $g$  of  $A_{-1}$  corresponding to an eigenvalue  $\mu \in \Sigma$ ,  $|\mu| = 1$  and  $k \in \mathbb{Z}$  such that  $B^* R_{\lambda_k^\mu}^*(A_0 + \mathcal{BP}_1^0)g = 0$ . This implies (see (29)) that the eigenvector  $\tilde{g}_k$  of  $\tilde{\mathcal{A}}_1^0$ , i.e.  $\tilde{\mathcal{A}}_1^0 \tilde{g}_k = \lambda_k^\mu \tilde{g}_k$  belongs to  $\text{Ker } \mathcal{B}_T^*$ .

Let us show that  $\tilde{g}_k$  is an eigenvector of the operator  $(\tilde{\mathcal{A}}_1^0)_T^*$ , adjoint of  $\tilde{\mathcal{A}}_1^0$  in the norm  $\|\cdot\|_T$ , and the corresponding to  $\tilde{g}_k$  eigenvalue of  $(\tilde{\mathcal{A}}_1^0)_T^*$  equals  $\overline{\lambda_k^\mu} = -\lambda_k^\mu$ . Let  $f \in \mathcal{D}(\tilde{\mathcal{A}}_1^0) = \mathcal{D}(\mathcal{A})$  and  $w \in \mathbb{C}$ . Then using the dissipativity of  $\tilde{\mathcal{A}}_1^0$  (see Prop. 3.2) we have

$$\begin{aligned} 0 &\geq \text{Re} \left\langle (\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)(\tilde{g}_k + wf), (\tilde{g}_k + wf) \right\rangle_T \\ &\geq w \text{Re} \left\langle (\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k \right\rangle_T + |w|^2 \text{Re} \left\langle (\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, f \right\rangle_T. \end{aligned}$$

Let us put  $w = \alpha \overline{\left\langle (\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k \right\rangle_T}$ ,  $\alpha \in \mathbb{R}$ . This leads to the inequality

$$\alpha \left| \left\langle (\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k \right\rangle_T \right|^2 (1 + \alpha \text{Re} \langle (A - \lambda_k^\mu I)f, f \rangle) \leq 0,$$

which holds for all  $\alpha \in \mathbb{R}$ . From this follows that

$$\left\langle (\tilde{\mathcal{A}}_1^0 - \lambda_k^\mu I)f, \tilde{g}_k \right\rangle_T = 0, \text{ for all } f \in \mathcal{D}(\tilde{\mathcal{A}}_1^0).$$

The later relation means that  $\tilde{g}_k \in \mathcal{D}((\tilde{\mathcal{A}}_1^0)_T^*)$  and

$$(\tilde{\mathcal{A}}_1^0)_T^* \tilde{g}_k = \overline{\lambda_k^\mu} \tilde{g}_k = -\lambda_k^\mu \tilde{g}_k.$$

Since in addition

$$\tilde{g}_k \in \text{Ker } \mathcal{B}_T^* \subset \text{Ker } (\mathcal{BP}_1)_T^*$$

then  $\tilde{g}_k \in \mathcal{D}(\tilde{\mathcal{A}}_T^*)$  and

$$\tilde{\mathcal{A}}_T^* \tilde{g}_k = (\tilde{\mathcal{A}}_1^0)_T^* \tilde{g}_k + (\mathcal{BP}_1)_T^* \tilde{g}_k = -\lambda_k^\mu \tilde{g}_k.$$

Hence,

$$\left( e^{\tilde{\mathcal{A}}t} \right)_T^* \tilde{g}_k = e^{-\lambda_k^\mu t} \tilde{g}_k$$

and, as a consequence,

$$\left\langle e^{\tilde{A}t} \tilde{g}_k, \tilde{g}_k \right\rangle_T = \left\langle \tilde{g}_k, \left( e^{\tilde{A}t} \right)_T^* \tilde{g}_k \right\rangle_T = e^{\lambda_k^* t} \|\tilde{g}_k\|_T^2, \quad t \geq 0.$$

Thus,

$$\|e^{\tilde{A}t} \tilde{g}_k\|_T \geq \|\tilde{g}_k\|_T \not\rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

This contradiction completes the proof. ■

**Remark 4.4.** Assume that  $\text{rank } B = n$ . In this case one can easily observe that assumptions (a3)–(a4) are satisfied automatically. Besides, the condition (30) from Theorem 4.3 is also always satisfied. So any system (2) with  $\text{rank } B = n$  and (a1)–(a2) is strongly stabilizable.

**E x a m p l e.** Consider the following one-dimensional system

$$\dot{x}(t) = -x(t) + x(t-1) + \dot{x}(t-1) + u(t). \quad (31)$$

It is shown in [15] that this system is not exponentially stabilizable by a feedback of class 2, because only a finite part of the spectrum of the closed loop system can be moved to a semiplane  $\{\lambda : \text{Re}(\lambda) \leq \alpha < 0\}$ .

Now observe that (31) is strongly stabilizable due to Theorem 4.3. In fact, for this system we have  $n = 1, A_0 = -1, A_1 = A_{-1} = 1, B = 1, \Sigma = \{1\}$  which is a simple eigenvalue. Since  $\text{rank } B = 1 = n$  and (a1)–(a2) are satisfied the (31) is strongly stabilizable. Let us find a stabilizing control. Since  $A_1 + A_0 A_{-1} = 0$  and  $\sigma(A_0) = \{-1\}$  is real negative we can put  $P_0^0 = P_1^0 = 0$ . Some simple calculations give

$$G = 1, \quad D = 1, \quad (Fq)(\theta) = -(1 - e)^{-1} e^{-\theta} q$$

and, therefore,

$$Q = \frac{1}{2} \left( \frac{e+1}{e-1} \right), \quad \hat{Q}(\theta) = \frac{e^{-\theta}}{e-1}.$$

Thus, a stabilizing control from Theorem 4.3 for system (31) takes the form

$$u(t) = - \left( 1 + \frac{1}{2} \frac{e+1}{e-1} \right) (x(t) - x(t-1)) - \frac{1}{e-1} \int_{-1}^0 e^{-\theta} x(t+\theta) d\theta.$$

## 5. Conclusion

For linear systems of neutral type we gave a characterization of a class of strongly stabilizable systems by relatively bounded feedback laws. No derivative of the state is needed in the feedback. The counterpart is that the stabilizability

is not exponential. As a perspective, one can expect that this technique may be used for more general systems with delay of neutral type, using the same infinite dimensional abstract framework.

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