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Finite difference operators with a finite band spectrum

M. Shapiro[1]

Department of Mathematics, Michigan State University
East Lansing, MI 48824

E-mail:mshapiro@math.msu.edu

V. Vinnikov

Department of Mathematics, Ben-Gurion University
P.O. Box 653, Beer-Sheva, 84105, Israel

E-mail:vinnikov@cs.bgu.ac.il

P. Yuditskii[2]

Institute for Analysis Johannes Kepler University of Linz
A–4040 Linz, Austria

E-mail:yuditski@math.msu.edu

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We discuss a functional model for multidiagonal selfadjoint operators with almost periodic coefficients that generalizes the well known model for finite band Jacobi matrices. It give us an opportunity to construct examples of almost periodic operators with different spectral properties. Main result deals with an exact condition for the uniqueness of the model of the given type.

1. Ergodic finite difference operators and associated Riemann surfaces

The standard (three-diagonal) finite band Jacobi matrices [2, 4] can be defined as almost periodic or even ergodic Jacobi matrices with absolutely continuous spectrum that consists of a finite system of intervals. We wish to find a natural

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extension of this class of finite difference operators onto multidiagonal case. First let us recall what is ergodic operator [1, 3], see also [5].

Let $(\Omega, \mathfrak{A}, d\chi)$ be a separable probability space and let $T : \Omega \to \Omega$ be an invertible ergodic transformation, i.e., T is measurable, it preserves $d\chi$, and every measurable T-invariant set has measure 0 or 1. Let $\{q^{(k)}\}_{k=0}^{d}$ be functions from $L^\infty_{d\chi}$, more $q^{(d)}$ is positive–valued and $q^{(0)}$ is real-valued.

Then with almost every $\omega \in \Omega$ we associate selfadjoint $2d+1$-diagonal operator $J(\omega)$ as follows:

$$(J(\omega)x)_n = \sum_{k=-d}^{d} \overline{q_n^{(k)}(\omega)} x_{n+k}, \quad x = \{x_n\}_{n=-\infty}^{\infty} \in l^2(\mathbb{Z}), \tag{1}$$

where $q_n^{(k)}(\omega) := q^{(k)}(T^n\omega)$ and $q^{(-k)}(\omega) := \overline{q^{(k)}(T^{-k}\omega)}$.

Note that the structure of $J(\omega)$ is described by the following identity:

$$J(\omega)S = SJ(T\omega), \tag{2}$$

where S is the shift operator in $l^2(\mathbb{Z})$. The last relation indicates strongly that one can associate with the family of matrices $\{J(\omega)\}_{\omega\in\Omega}$ a natural pair of commuting operators.

Namely, let $L^2_{d\chi}(l^2(\mathbb{Z}))$ be the space of $l^2(\mathbb{Z})$–valued vector functions, $x(\omega) \in l^2(\mathbb{Z})$, with the norm

$$||x||^2 = \int_\Omega ||x(\omega)||^2 \, d\chi.$$

Define

$$(\widehat{J}x)(\omega) = J(\omega)x(\omega), \quad (\widehat{S}x)(\omega) = Sx(T\omega), \quad x \in L^2_{d\chi}(l^2(\mathbb{Z})).$$

Then (2) implies

$$(\widehat{J}\widehat{S}x)(\omega) = J(\omega)Sx(T\omega) = SJ(T\omega)x(T\omega) = (\widehat{S}\widehat{J}x)(\omega).$$

Further, $\widehat{S}$ is a unitary operator and $\widehat{J}$ is selfadjoint. The space $L^2_{d\chi}(l^2(\mathbb{Z}_+))$ is an invariant subspace for $\widehat{S}$. It is not invariant with respect to $\widehat{J}$ but it is invariant with respect to $\widehat{J}\widehat{S}^d$. Put

$$\widehat{S}_+ = \widehat{S}|L^2_{d\chi}(l^2(\mathbb{Z}_+)), \quad (\widehat{J}\widehat{S}^d)_+ = \widehat{J}\widehat{S}^d|L^2_{d\chi}(l^2(\mathbb{Z}_+)).$$

Definition 1.1 (local functional model). *We say that a pair of commuting operators $A_1 : H \to H$ and $A_2 : H \to H$ has a (local) functional model if there is a unitary embedding $i : H \to H_O$ in a space H_O of holomorphic in some domain O functions $F(\zeta)$, $\zeta \in O$, with a reproducing kernel ($F \mapsto F(\zeta_0)$, $\zeta_0 \in O$, is a* *bounded functional in H_O) such that operators $i_\star A_1$ and $i_\star A_2$ become a pair of operators of multiplication by holomorphic functions, say*

$$A_1 x \mapsto a_1(\zeta)F(\zeta), \quad A_2 x \mapsto a_2(\zeta)F(\zeta).$$

Existence of a local functional model implies a number of quite strong consequences. In what follows $b(\zeta)$ and $\lambda(\zeta)$ denote the functions (symbols) related to operators $\widehat{S}_+$ and $(\widehat{J}\widehat{S}^d)_+$. Let k_ζ be the reproducing kernel in H_O and let $\hat{k}_\zeta$ be its preimage $i^{-1}k_\zeta$ in $L^2_{d\chi}(l^2(\mathbb{Z}_+))$. Then

$$\langle \widehat{S}_+^* \hat{k}_\zeta, x\rangle = \langle \hat{k}_\zeta, \widehat{S}_+ x\rangle = \langle k_\zeta, bF\rangle. \tag{3}$$

By the reproducing property

$$\langle k_\zeta, bF\rangle = \overline{b(\zeta)F(\zeta)} = \langle \overline{b(\zeta)} k_\zeta, F\rangle.$$

Hence,

$$\langle \widehat{S}_+^* \hat{k}_\zeta, x\rangle = \langle \overline{b(\zeta)} \hat{k}_\zeta, x\rangle.$$

That is $\hat{k}_\zeta$ is an eigenvector of $\widehat{S}_+^*$ with the eigenvalue $\overline{b(\zeta)}$. In the same way, $\hat{k}_\zeta$ is an eigenvector of $(\widehat{J}\widehat{S})_+^*$ with the eigenvalue $\overline{\lambda(\zeta)}$.

Thus, if a functional model exists then the spectral problem

$$\begin{cases} \widehat{S}_+^* \hat{k}_\zeta & = \overline{b(\zeta)} \hat{k}_\zeta \\ (\widehat{J}\widehat{S}^d)_+^* \hat{k}_\zeta & = \overline{\lambda(\zeta)} \hat{k}_\zeta \end{cases}, \tag{4}$$

has a solution $\hat{k}_\zeta$ antiholomorphic in ζ. Moreover, linear combinations of all $\hat{k}_\zeta$ are dense in $L^2_{d\chi}(l^2(\mathbb{Z}_+))$. Vice versa, if (4) has a solution of such kind then we define

$$F(\zeta) := \langle x, \hat{k}_\zeta\rangle, \quad ||F||^2 := ||x||^2.$$

This provides a local functional model for the pair $\widehat{S}_+$, $(\widehat{J}\widehat{S})_+$.

The following lemma is evident.

Lemma 1.2. *Let $U : L^2_{d\chi} \to L^2_{d\chi}$ be the unitary operator associated with the ergodic transformation T: $(Uc)(\omega) = c(T\omega)$, $c \in L^2_{d\chi}$. We denote by the same letter q both a function $q \in L^\infty_{d\chi}$ and the multiplication operator by q (e.g., $(qc)(\omega) := q(\omega)c(\omega)$).*

Problem (4) is equivalent to the following spectral problem:

$$\left\{ \sum_{k=-d}^{d} U^k \overline{q^{(k)} b^k(\zeta)} \right\} c_\zeta = \overline{z(\zeta)} c_\zeta, \tag{5}$$

where $z(\zeta) := \lambda(\zeta)/b^d(\zeta)$ and c_ζ is an antiholomorphic $L^2_{d\chi}$-valued vector function. Moreover, $\{c_\zeta\}$ is complete in $L^2_{d\chi}$ if and only if $\{\hat{k}_\zeta\}$ is complete in $L^2_{d\chi}(l^2(\mathbb{Z}_+))$.

We may hope to glue a *global* functional model on a Riemann surface $X_0 = \mathbb{D}/\Gamma_0$ formed by functions (z, b). Of course, this model does not necessarily exist (even existence of a local model requires some additional assumptions on the ergodic map and the coefficients functions).

The surface X_0 is in generic case of infinite genus. However we can reduce it because X_0 possesses a family of automorphisms. Let $e_{\{\gamma\}}$ be an eigenvector of U with an eigenvalue $\bar{\mu}_{\{\gamma\}}$. The systems of eigenfunctions and eigenvalues form both Abelian groups with respect to multiplication. Using (5), we get immediately that $\{\gamma\} : (z, b) \mapsto (z, \mu_{\{\gamma\}} b)$ is an automorphism of X_0. Taking a quotient of X_0 with respect to these automorphisms, we obtain a much smaller surface $X = \mathbb{D}/\Gamma$, $\Gamma = \{\gamma\Gamma_0\}$. Note that z is still a function on X but b becomes a character automorphic function. Finally, using z we may glue boundary of X and to get in this way a compact Riemann surface X_c, such that $X = X_c \setminus E$. The simplest assumption is that the boundary E is a finite system of cuts on X_c.

Thus, in a certain case we may expect that the triple $\{X_c, z, E\}$ characterizes spectrum of a finite difference operator. If so we call it a finite band operator. We point out that in fact every triple of this kind gives rise to a family of ergodic finite difference operators. More precise and detailed characteristic of such operators is given in the next section.

Now we would like to describe all triples of given type up to a natural equivalence relation.

Definition 1.3. *We say that two triples (X_{c1}, z_1, E_1) and (X_{c2}, z_2, E_2) are equivalent if there exists a holomorphic homeomorphism $h : X_{c1} \to X_{c2}$ such that $z_1 = h^*(z_2)$ and $E_2 = h(E_1)$.*

Note, that for any triple (X_c, z, E) the holomorphic function $z : X_c \to \mathbb{C}P^1$ is a ramified coverings of $\mathbb{C}P^1$. It is quite convenient to describe equivalence classes of ramified coverings in terms of branching divisor. Namely, a point $p \in X_c$ such that $\frac{dz}{d\zeta}|_p = 0$ where ζ is a local holomorphic coordinate in a neighborhood of p is called *ramification point (or, critical point)*. Its image $z(p)$ is called *a branching point (or, critical value)*. The set of all branching points of function z form a branching divisor $B(z)$ of z.

Clearly, branching divisors of equivalent functions are the same. Moreover, the compact holomorphic curve X_c is also uniquely determined by the branching divisor and some additional ramification data (of combinatorial type). Namely, assume that z has degree d and let $z_i \in \mathbb{C}P^1$, $i = 1, \dots, N$, be branching points of function z, $w \in \mathbb{C}P^1$ be a non branching point. Fix a system of nonintersecting paths $\gamma = \{\gamma_i\}_{i=1,\dots,N}$. The ith path γ_i connects w and z_i. We want to construct a system of loops $l_i \subset \mathbb{C}P^1$. To construct l_i we start from w and follow first γ_i almost to z_i, then encircle z_i counterclockwise along a small circle and finally go back to w along $-\gamma_i$. Using l_i, we associate with each branching point an element of permutation group $\sigma_i \in \Sigma_d$.

The point w has exactly d preimages. Let us label them by integers $\{1, \dots, d\}$. Let us follow the loop l_i and lift this loop to X_c starting from each of the preimages of w. The monodromy along path l_i gives us a permutation $\sigma_i \in \Sigma_d$ of preimages. Note that the product $\sigma_1 \cdot \ldots \sigma_d$ is the identity operator Id. Therefore, the function z determines N branching points and N permutations. These permutations are not uniquely defined, they depend on the labeling of preimages of w. Therefore, they are determined up to a conjugacy by the elements of Σ_d.

Given a set of branching points $B(z) = \{z_i\}_{i=1,\dots,N} \subset \mathbb{C}P^1$ and a system of permutations $\sigma(z) = (\sigma_1, \dots, \sigma_N) \in \Sigma_d \times \dots \times \Sigma_d/\Sigma_d$, such that $\sigma_1 \cdot \dots \cdot \sigma_d = Id$, where the last quotient is taken with respect to diagonal conjugation, we can restore by Riemann theorem the surface X_c and function z.

Hence, the triple (X_c, z, E) is equivalent to the quadruple $(B(z), \gamma, \sigma(z), E)$.

2. The global functional model

Let $\pi(\zeta) : \mathbb{D} \to X$ be a uniformization of the surface $X = X_c \setminus E$. Thus, there exists a discrete subgroup Γ of the group $SU(1,1)$ consisting of elements of the form

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \ \gamma_{11} = \overline{\gamma_{22}}, \ \gamma_{12} = \overline{\gamma_{21}}, \ \det \gamma = 1,$$

such that $\pi(\zeta)$ is automorphic with respect to Γ, i.e., $\pi(\gamma(\zeta)) = \pi(\zeta)$, $\forall \gamma \in \Gamma$, and any two preimages of $P_0 \in X$ are Γ-equivalent. We normalize $Z(\zeta) := (z \circ \pi)(\zeta)$ by the conditions $Z(0) = \infty$, $(\zeta^d Z)(0) > 0$.

Note that Γ acts dissipatively on $\mathbb{T}$ with respect to the Lebesgue measure dm, that is there exists a measurable (fundamental) set $\mathbb{E}$, which does not contain any two Γ-equivalent points, and the union $\cup_{\gamma \in \Gamma} \gamma(\mathbb{E})$ is a set of full measure. In fact $\mathbb{E}$ can be chosen as a finite union of intervals. For the space of square summable functions on $\mathbb{E}$ (with respect to dm), we use the notation $L^2_{dm|\mathbb{E}}$.

A character of Γ is a complex–valued function $\alpha : \Gamma \to \mathbb{T}$, satisfying

$$\alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma.$$

The characters form an Abelian compact group denoted by Γ^*.

Let f be an analytic function in $\mathbb{D}$, $\gamma \in \Gamma$. Then we put

$$f|[\gamma]_k = \frac{f(\gamma(\zeta))}{(\gamma_{21}\zeta + \gamma_{22})^k} \quad k = 1, 2.$$

Notice that $f|[\gamma]_2 = f$ for all $\gamma \in \Gamma$, means that the form $f(\zeta)d\zeta$ is invariant with respect to the substitutions $\zeta \to \gamma(\zeta)$ ($f(\zeta)d\zeta$ is an Abelian integral on $\mathbb{D}/\Gamma$).

Analogously, $f|[\gamma] = \alpha(\gamma)f$ for all $\gamma \in \Gamma$, $\alpha \in \Gamma^*$, means that the form $|f(\zeta)|^2\,|d\zeta|$ is invariant with respect to these substitutions.

We recall, that a function $f(\zeta)$ is of Smirnov class, if it can be represented as a ratio of two functions from H^∞ with an outer denominator. The following spaces related to the Riemann surface $\mathbb{D}/\Gamma$ are counterparts of the standard Hardy spaces H^2 (H^1) on the unit disk.

Definition 2.1. *The space $A_1^2(\Gamma, \alpha)$ ($A_2^1(\Gamma, \alpha)$) is formed by functions f, which are analytic on $\mathbb{D}$ and satisfy the following three conditions:*

$$1) f \text{ is of Smirnov class,}$$
$$2) f|[\gamma] = \alpha(\gamma)f \quad (f|[\gamma]_2 = \alpha(\gamma)f) \quad \forall\gamma \in \Gamma,$$
$$3) \int_{\mathbb{E}} |f|^2\,dm < \infty \quad (\int_{\mathbb{E}} |f|\,dm < \infty).$$

$A_1^2(\Gamma, \alpha)$ is a Hilbert space with the reproducing kernel $k^\alpha(\zeta, \zeta_0)$, moreover,

$$0 < \inf_{\alpha\in\Gamma^*} k^\alpha(\zeta_0, \zeta_0) \le \sup_{\alpha\in\Gamma^*} k^\alpha(\zeta_0, \zeta_0) < \infty. \tag{6}$$

Put

$$k^\alpha(\zeta) = k^\alpha(\zeta, 0) \quad \text{and} \quad K^\alpha(\zeta) = \overline{K_\zeta^\alpha(0)} = \frac{k^\alpha(\zeta)}{\sqrt{k^\alpha(0)}}.$$

We need one more special function. The Blaschke product

$$b(\zeta) = \zeta \prod_{\gamma\in\Gamma, \gamma\neq 1_2} \frac{\gamma(0) - \zeta}{1 - \overline{\gamma(0)}\zeta} \frac{|\gamma(0)|}{\gamma(0)}$$

is called the *Green's function* of Γ with respect to the origin. It is a character–automorphic function, i.e., there exists $\mu \in \Gamma^*$ such that $b(\gamma(\zeta)) = \mu(\gamma)b(\zeta)$. Note, if $G(P) = G(P, \infty)$ denotes the Green's function of the surface X, then

$$G(\pi(\zeta)) = -\log|b(\zeta)|.$$

Let $\Gamma_0 := \ker\mu$, that is $\Gamma_0 = \{\gamma \in \Gamma : \mu(\gamma) = 1\}$. Evidently, $b(\zeta)$ and $(Zb^d)(\zeta)$ are holomorphic functions on the surface $X_0 = \mathbb{D}/\Gamma_0$.

Now, assume that $\alpha_0 \in \Gamma_0$ can be extended to a character on Γ, i.e.,

$$\Omega_{\alpha_0} = \{\alpha \in \Gamma^* : \alpha|\Gamma_0 = \alpha_0\} \neq \emptyset.$$

Note that the set of characters

$$\Omega_\iota = \{\alpha \in \Gamma^* : \alpha|\Gamma_0 = \iota\},$$

where $\iota(\gamma) = 1$ for all $\gamma \in \Gamma_0$ is isomorphic to the set $(\Gamma/\Gamma_0)^*$.

Let us fix an element $\hat{\alpha}_0 \in \Omega_{\alpha_0}$. Since

$$\{\alpha \in \Gamma^* : \alpha|\Gamma_0 = \alpha_0\} = \{\hat{\alpha}_0\beta : \beta \in \Gamma^* : \beta|\Gamma_0 = \iota\}$$

we can define a measure $d\chi_{\alpha_0}(\alpha)$ on Ω_{α_0} by the relation

$$d\chi_{\alpha_0}(\alpha) = d\chi_{\alpha_0}(\hat{\alpha}_0\beta) = d\chi_\iota(\beta),$$

where $d\chi_\iota(\beta)$ is the Haar measure on $(\Gamma/\Gamma_0)^*$ (the measure $d\chi_{\alpha_0}(\alpha)$ does not depend on a choice of the element $\hat{\alpha}_0$).

Obviously, $T\alpha := \mu^{-1}\alpha$ is an invertible ergodic measure–preserving transformation on $\Omega = \Omega_{\alpha_0}$ with respect to the measure $d\chi = d\chi_{\alpha_0}$.

The following theorem is a slightly modified version of Theorem 2.2 from [5].

Theorem 2.2. *Given $\alpha \in \Gamma^*$, the system of functions $\{b^n K^{\alpha\mu^{-n}}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis in $L^2_{dm|\mathbb{E}}$. With respect to this basis, the multiplication operator by Z is a $2d+1$–diagonal ergodic finite difference operator with $\Omega = \Omega_{\alpha_0}$, $d\chi = d\chi_{\alpha_0}$, $T\alpha := \mu^{-1}\alpha$ and $\alpha_0 = \alpha|\Gamma_0$.*

Moreover, the operators $\widehat{S}_+$ and $(\widehat{J}\widehat{S}^d)_+$ are unitary equivalent to multiplication by b and $(b^d Z)$ in $A^2_1(\Gamma_0, \alpha_0)$, respectively. This unitary map is given by the formula

$$\sum_{\{\gamma\}\in\Gamma/\Gamma_0} f|[\gamma]\alpha^{-1}(\gamma) = \sum_{n\in\mathbb{Z}_+} x_n(\alpha) b^n K^{\alpha\mu^{-n}},$$

where $f \in A^2_1(\Gamma_0, \alpha_0)$ and the vector function $x(\alpha) = \{x_n(\alpha)\}$ belongs to $L^2_{d\chi}(l^2(\mathbb{Z}_+))$.

3. Uniqueness theorem

Theorem 3.1. *Assume that a finite difference ergodic operator has a finite band functional model that is there exist a triple $\{X_c, \tilde{z}, E\}$, a character $\alpha_0 \in \Gamma^*$ and a map F from Ω to $\tilde{\Omega} := \Omega_{\alpha_0}$ such that $FT\omega = \mu^{-1}F\omega$, $\chi(F^{-1}(A)) = \tilde{\chi}(A)$, $A \subset \tilde{\Omega}$, with $d\tilde{\chi} := d\chi_{\alpha_0}$, here μ is the character of the Green function $\tilde{b}$ on $X_c \backslash E$. Moreover, $q^{(k)}(\omega) = \tilde{q}^{(k)}(F\omega)$, where the coefficients $\tilde{q}^{(k)}(\alpha)$ are generated by the multiplication operator $\tilde{z}$ with respect to the orthonormal basis $\{\tilde{b}^n K^{\alpha\mu^{-n}}\}_{n\in\mathbb{Z}}$.*

If the functions $\tilde{z}$ and $\{d\log\tilde{b}/d\tilde{z}\}$ separate points on $X_c \setminus E$ then any local functional model is generated by one of the branches of the function $\tilde{b}$.

P r o o f. Put

$$A(z;\omega)=\begin{bmatrix} 0 & 1 & & \dots & 0\\ \vdots & & \ddots & & \vdots\\ \vdots & & & \ddots & \vdots\\ 0 & 0 & & \dots & 1\\ -\overline{\left(\frac{q^{(-d)}(T^{-d}\omega)}{q^{(d)}(T^d\omega)}\right)} & \dots & \overline{\left(\frac{z(\zeta)-q^{(0)}(\omega)}{q^{(d)}(T^d\omega)}\right)} & \dots & -\overline{\left(\frac{q^{(d-1)}(T^{d-1}\omega)}{q^{(d)}(T^d\omega)}\right)}\end{bmatrix}. \quad (7)$$

According to (5)

$$A(z;\omega)f_\zeta(\omega)=\overline{b(\zeta)}f_\zeta(T\omega) \quad (8)$$

with

$$f_\zeta(\omega)=\begin{bmatrix} c_\zeta(T^{-d}\omega)\overline{b^{-d}(\zeta)}\\ \vdots\\ c_\zeta(T^{d-1}\omega)\overline{b^{d-1}(\zeta)}\end{bmatrix}.$$

Considering (if necessary) a subdomain $\tilde{O}\subset O$ let us introduce one-to-one map $\zeta\mapsto z$. Then, let us put $\tilde{z}=z$ assuming that $\{\tilde{z}, d\log\tilde{b}/d\tilde{z}\}$ is a point on $X_c\setminus E$. Actually there are exactly d different preimages $\{\tilde{b}_l\}_1^d$ with the given $\tilde{z}$ and different $d\log\tilde{b}_l/d\tilde{z}$. Finally, since a pair $(\tilde{z},\tilde{b}_l)$ determines a point on $X_c\setminus E$ we can put $K(\alpha;\tilde{z},\tilde{b}_l):=K^{\alpha}_{\tilde{\zeta}}(0)$ choosing one of preimages $\tilde{\zeta}$ on the universal covering.

Using this notation and identities $\tilde{z}=z$, $q^{(k)}(\omega)=\tilde{q}^{(k)}(F\omega)$, we write

$$A(z;\omega)\mathbb{K}(z;F\omega)=\mathbb{K}(z;\mu^{-1}F\omega)\begin{bmatrix}\overline{\tilde{b}_1} & & \\ & \ddots & \\ & & \overline{\tilde{b}_d^{-1}}\end{bmatrix}, \quad (9)$$

where the matrix $\mathbb{K}$ is constructed from the reproducing kernels

$$\mathbb{K}(z,\alpha)=\begin{bmatrix} K(\mu^d\alpha;z,\tilde{b}_1)\overline{\tilde{b}_1^{-d}} & \dots & K(\mu^d\alpha;z,\tilde{b}_d^{-1})\overline{\tilde{b}_d^{d}}\\ \vdots & & \vdots\\ K(\mu^{-d+1}\alpha;z,\tilde{b}_1)\overline{\tilde{b}_1^{-d}} & \dots & K(\mu^{-d+1}\alpha;z,\tilde{b}_d^{-1})\overline{\tilde{b}_d^{d}}\end{bmatrix}.$$

Combining (8) and (9), we get

$$\begin{bmatrix}\overline{\tilde{b}_1(\zeta)}h_1(\omega)\\ \vdots\\ \overline{\tilde{b}_d^{-1}(\zeta)}h_{2d}(\omega)\end{bmatrix}=\overline{b(\zeta)}\begin{bmatrix}h_1(T\omega)\\ \vdots\\ h_{2d}(T\omega)\end{bmatrix}$$

with

$$\begin{bmatrix} h_1(\omega) \\ \vdots \\ h_{2d}(\omega) \end{bmatrix} := \mathbb{K}(z,\alpha)^{-1} f_\zeta(\omega).$$

First of all, $h_{d+l}(\omega) = 0$ for $l \ge 1$ because $|\tilde{b}_l(\zeta)b(\zeta)| < 1$. Since the spectrum of U is discreet and $\tilde{b}_l(\zeta)$ as well as $b(\zeta)$ are holomorphic the ratio $\tilde{b}_l(\zeta)/b(\zeta)$ should be a constant if only $h_l(\omega) \neq 0$. Making use of the assumption $(\log \tilde{b}_l(\zeta))' \neq (\log \tilde{b}_m(\zeta))'$, $l \neq m$, we obtain that only one of entries $h_{l_0}(\omega)$ is different from zero.

Therefore, $c_\zeta(\omega) = e_{\{\gamma_0\}}(\omega)K(F\omega; z, \tilde{b}_{l_0})$ with $Ue_{\{\gamma_0\}} = \bar{\mu}_{\{\gamma_0\}}e_{\{\gamma_0\}}$; and hence $Af := e_{\{\gamma_0\}}(\omega)f(F\omega)$ is not only isometric but a unitary map from $L^2_{d\tilde{\chi}}$ to $L^2_{d\chi}$ such that $A\tilde{U} = UA$. This means that $\bar{\mu}_{\{\gamma_0\}}$ is an eigenvalue of $\tilde{U}$ and the local model is given by one of the branches of the function $\tilde{b}$, $c_\zeta(\omega) = K(F\omega; z, \mu_\gamma \tilde{b}_{l_0})$ with a certain γ. ■

The following example shows that in the case when the functions $\tilde{z}$ and $\{d \log \tilde{b}/d\tilde{z}\}$ do not separate points on $X_c \setminus E$ one can give different *global* functional realizations for the same ergodic operator.

E x a m p l e. Let $J = S^d + S^{-d}$. There exist a "trivial" functional model with $X_c \setminus E \sim \mathbb{D}$. In this case J is the multiplication operator by $z = \zeta^d + \zeta^{-d}$ with respect to the standard basis $\{\zeta^l\}$ in $L^2_{\mathbb{T}}$. Note that $b = \zeta$, thus

$$w := \frac{d \log b}{dz} = \frac{1}{\zeta^d - \zeta^{-d}} \frac{1}{d},$$

that is $z^2 + (wd)^{-2} = 4$, $|(wd)^{-1} + z| < 2$.

On the other hand, let us fix any polynomial $T(u)$, $\deg T = d$, with real critical values on $\mathbb{R} \setminus [-2,2]$ and define $X_c \setminus E = T^{-1}(\overline{\mathbb{C}} \setminus [-2,2]) \sim \overline{\mathbb{C}} \setminus T^{-1}[-2,2]$. As it well known the last set is the resolvent set for a d–periodic Jacobi matrix [4], say J_0. Moreover, $T(J_0) = J$, and $-\log|b|$ is just the Green function of this domain in the complex plain. So, using the standard functional model for J_0 with the symbols u and b we get a functional model for J with $z = T(u)$ and the same b. Note that as before $z^2 + (wd)^{-2} = 4$, $|(wd)^{-1} + z| < 2$ with $w := \frac{d \log b}{dz}$. ■

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