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Full description of totally geodesic unit vector fields on 2-dimensional Riemannian manifolds

A. Yampolsky

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61103, Ukraine
E-mail:yamp@univer.kharkov.ua

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We give a full geometrical description of local totally geodesic unit vector field on Riemannian 2-manifold, considering the field as a local imbedding of the manifold into its unit tangent bundle with the Sasaki metric.

Introduction

Let (M, g) be an $(n+1)$-dimensional Riemannian manifold with metric g. A vector field ξ on it is called *holonomic* if ξ is a field of normals of some family of regular hypersurfaces in M and *nonholonomic* otherwise. The foundation of the classical geometry of unit vector fields was proposed by A. Voss at the end of the nineteenth century. The theory includes the *Gaussian* and *the mean curvature* of a vector field and their generalizations (see [1] for details).

Recently, the geometry of vector fields has been considered from another point of view. Let T_1M be a unit tangent bundle of M endowed with the Sasaki metric [14]. If ξ is a unit vector field on M, then one may consider ξ as a mapping $\xi : M \to T_1M$. The image $\xi(M)$ is a submanifold in T_1M with metric induced from T_1M and one may apply the methods from the study of the geometry of submanifolds to determine geometrical characteristics of a unit vector field. A unit vector field ξ is said to be *minimal* if $\xi(M)$ is a minimal submanifold in T_1M. A unit vector field on S^3 tangent to the fibers of the Hopf fibration $S^3 \xrightarrow{S^1} S^2$ is a unique unit vector field with globally minimal volume [10]. This result fails in higher dimensions. A lower volume is achieved by a vector field with one singular point, namely the inverse image under stereographic projection inverse

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image of a parallel vector field on E^n [13]. The lowest volume is reached for the *North–South* vector field with two singular points.

A local approach to minimality of unit vector fields was developed in [6]. A number of examples of locally minimal unit vector fields was found [2–4, 6–9, 11–13, 15–17] on various manifolds. In [18] the author presented *an explicit expression* for the second fundamental form of $\xi(M)$ and found some examples of vector fields with constant mean curvature. This expression is the key to solving a problem about *totally geodesic vector fields* on a given Riemannian manifold. Originally, the problem of a full description of *all totally geodesic submanifolds* in the tangent (sphere) bundle of spaces of constant curvature was posed by A. Borisenko in [5]. The totally geodesic vector fields form a special class of such submanifolds. In [19] this problem was solved in the case of 2-manifolds of constant curvature. In [21] an example of a totally geodesic unit vector field was found on a surface of revolution with nonconstant but sign-preserving Gaussian curvature.

In this paper, we completely determine the Riemannian 2-manifolds which admit a unit vector field ξ such that $\xi(M)$ is a totally geodesic submanifold in T_1M. Moreover, we explicitly determine the vector field. Under some restrictions, we find an isometric immersion of the metric into Euclidean 3-space which gives a surface with the necessary properties.

1. The main result

Let ξ be a unit vector field on a Riemannian manifold (M^n, g). Then ξ can be considered as a mapping $\xi : M^n \to T_1M^n$. In this way one can use geometrical properties of the submanifold $\xi(M^n)$ to determine the geometrical characteristics of the vector field.

Definition 1.1. *A unit vector field on Riemannian manifold M^n is said to be totally geodesic, if the submanifold $\xi(M^n) \subset T_1M^n$ is totally geodesic in the unit tangent bundle with the Sasaki metric.*

Definition 1.2. *A point $q \in M^n$ is said to be stationary for the vector field ξ if $\nabla_X \xi\,|_q = 0$ for all $X \in T_qM^n$.*

If stationary points fills a domain $D \subset M^n$, then locally $M^n = M^{n-k} \times E^k$, where E^k is a Euclidean factor of dimension $k \geq 1$. In the case $n = 2$, the manifold is then flat in D. If the manifold is of sign-preserving Gaussian curvature, then we can always restrict our considerations to the domain with no stationary points of a given unit vector field. The main result of the paper is the following theorem.

Theorem 1.1. *Let M^2 be a Riemannian manifold with sign-preserving Gaussian curvature K. Then , on some open subset U of M, there exists a unit totally geodesic vector field ξ if and only if:*

(a) the metric g on U is locally of the form

$$ds^2 = du^2 + \sin^2\alpha(u)\, dv^2,$$

where $\alpha(u)$ solves the differential equation $\dfrac{d\alpha}{du} = 1 - \dfrac{a+1}{\cos\alpha}$;

(b) the totally geodesic unit vector field ξ is of the form

$$\xi = \cos(av + \omega_0)\,\partial_u + \frac{\sin(av+\omega_0)}{\sin\alpha(u)}\,\partial_v,$$

where $a, \omega_0 = const.$

R e m a r k. The Gaussian curvature K of the metric is

$$K = \frac{d\alpha}{du}. \tag{1}$$

Therefore, $\alpha(u)$ is the total curvature of the manifold along the meridian of the metric. The vector field is parallel along meridians and bends along parallels with constant angle speed a with respect to the coordinate frame.

P r o o f. Let ξ be a given unit vector field on Riemannian manifold M^n. For dimension reasons, the kernel of the linear operator $\nabla_X\xi : TM^n \to \xi^\perp$ is not empty. Therefore, there is a nonzero vector field e_0 such that $\nabla_{e_0}\xi = 0$. In the case $n = 2$, the field e_0 can be found explicitly. Denote by η a unit vector field on M^2 which is orthogonal to ξ. Set

$$\nabla_\xi\xi = k\,\eta, \quad \nabla_\eta\eta = \varkappa\,\xi,$$

where k and $\varkappa$ are the signed geodesic curvatures of the integral trajectories of the fields ξ and η respectively. Introduce an orthonormal frame

$$e_0 = \frac{\varkappa}{\lambda}\,\xi + \frac{k}{\lambda}\,\eta, \quad e_1 = \frac{k}{\lambda}\,\xi - \frac{\varkappa}{\lambda}\,\eta, \quad \lambda = \sqrt{k^2+\varkappa^2}.$$

The fields e_0 and e_1 are correctly defined on an open subset $U \subset M^2$ where the field ξ has no stationary points, i.e., points where $\lambda = 0$. Restrict ourselves to this open part. It is elementary to check that

$$\nabla_{e_0}\xi = 0, \quad \nabla_{e_1}\xi = \lambda\eta. \tag{2}$$

Denote by ω the angle function between ξ and e_0. Then

$$k = \lambda\sin\omega, \quad \varkappa = \lambda\cos\omega, \tag{3}$$

and we can set

$$\begin{aligned} \xi &= \cos\omega\, e_0 + \sin\omega\, e_1, \\ \eta &= \sin\omega\, e_0 - \cos\omega\, e_1. \end{aligned} \tag{4}$$

Denote by μ and σ the *signed* geodesic curvatures of the integral curves of the fields e_0 and e_1 respectively. Then

$$\nabla_{e_0} e_0 = \mu\, e_1, \quad \nabla_{e_1} e_1 = \sigma\, e_0.$$

In these terms, the second fundamental form of the submanifold $\xi(M) \subset T_1M$ can be expressed as [19]

$$\Omega = \begin{bmatrix} -\mu \dfrac{\lambda}{\sqrt{1+\lambda^2}} & \dfrac{1}{2}\left(\sigma\,\lambda + \dfrac{1-\lambda^2}{1+\lambda^2} e_0(\lambda)\right) \\ \dfrac{1}{2}\left(\sigma\,\lambda + \dfrac{1-\lambda^2}{1+\lambda^2} e_0(\lambda)\right) & e_1\left(\dfrac{\lambda}{\sqrt{1+\lambda^2}}\right) \end{bmatrix}. \tag{5}$$

Set

$$\cos(\alpha/2) = \frac{1}{\sqrt{1+\lambda^2}}.$$

Then we have

$$\begin{aligned} &\frac{\lambda}{\sqrt{1+\lambda^2}} = \sin(\alpha/2), \quad \frac{1-\lambda^2}{1+\lambda^2} = \cos\alpha, \\ &e_0(\lambda) = \frac{e_0(\alpha)}{2\cos^2(\alpha/2)}, \quad e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) = \frac{1}{2}\cos(\alpha/2)\, e_1(\alpha). \end{aligned}$$

After these simplifications

$$\Omega = \frac{1}{2}\begin{bmatrix} -2\mu\, \sin(\alpha/2) & \dfrac{\sigma\, \sin\alpha + e_0(\alpha)\cos\alpha}{2\cos^2(\alpha/2)} \\ \dfrac{\sigma\, \sin\alpha + e_0(\alpha)\cos\alpha}{2\cos^2(\alpha/2)} & \cos(\alpha/2)\, e_1(\alpha) \end{bmatrix}.$$

Set $\Omega \equiv 0$. Then $\mu \equiv 0$, since $\sin(\alpha/2) \equiv 0$ implies $\lambda \equiv 0$, which contradicts the hypothesis. Therefore, if a totally geodesic vector field exists, then the *integral trajectories of the field e_0 are geodesics.*

Since $\cos(\alpha/2) \neq 0$, then

$$e_1(\alpha) \equiv 0. \tag{6}$$

Introduce a local semigeodesic coordinate system (u, v) such that

$$\partial_u = e_0, \;\; \partial_v = f(u, v)\, e_1,$$

where $f(u,v)$ is some nonzero function. Then the line element of M^2 can be written as

$$ds^2 = du^2 + f^2 dv^2.$$

The condition (6) implies $\partial_v \alpha = 0$, which means that $\alpha = \alpha(u)$.

Consider now the last condition

$$\sigma \sin\alpha + e_0(\alpha)\cos\alpha = 0.$$

If $\cos\alpha \equiv 0$, then $\sin\alpha \equiv 1$ and hence $\sigma \equiv 0$. This means that e_0 is a parallel vector field on M^2 and hence $K = 0$ again. Set

$$\sigma \tan\alpha + e_0(\alpha) = 0.$$

With respect to the chosen semigeodesic coordinate system, $\sigma = -\partial_u f/f$ and we come to the following relation

$$\frac{\partial_u f}{f} = \cot\alpha\ \partial_u \alpha.$$

Because of (6), we have $\alpha = \alpha(u)$ and the equation above has an evident solution

$$f(u,v) = C(v)\sin\alpha,$$

where $C(v) \neq 0$ is a constant of integration. Making a v-parameter change one can always set $C(v) \equiv 1$. Therefore, the line element of a 2-manifold M which admits a totally geodesic vector unit field is necessarily of the form

$$ds^2 = du^2 + \sin^2\alpha(u)\, dv^2. \tag{7}$$

Turn now to the vector field. A direct computation yields

$$\nabla_{e_0}\xi = \nabla_{e_0}(\cos\omega\, e_0 + \sin\omega\, e_1) = (-e_0(\omega) - \mu)\,\eta,$$
$$\nabla_{e_1}\xi = \nabla_{e_1}(\cos\omega\, e_0 + \sin\omega\, e_1) = (-e_1(\omega) + \sigma)\,\eta.$$

Since $\mu = 0$ and $\nabla_{e_0}\xi = 0$, we see that $\partial_u \omega = 0$ and hence $\omega = \omega(v)$. The second equality means, that

$$-e_1(\omega) + \sigma = \tan(\alpha/2).$$

With respect to a chosen coordinate system, we have

$$\sigma = -\cot\alpha\ \partial_u \alpha$$

and hence

$$\partial_v \omega = \sin\alpha\,(\sigma - \tan(\alpha/2) = -\cos\alpha\ \partial_u\alpha - 2\sin^2(\alpha/2).$$

The right hand side does not depend on the v-parameter and therefore $\partial^2_{vv}\omega = 0$ which means that

$$\omega = av + \omega_0, \quad (a, \omega_0 = const).$$

As a consequence, we come to the following differential equation for the function $\alpha(u)$:

$$\cos\alpha\ \partial_u\alpha + 2\sin^2(\alpha/2) = -a$$

or equivalently

$$\frac{d\alpha}{du} = 1 - \frac{a+1}{\cos\alpha}. \tag{8}$$

The proof is complete. ■

R e m a r k. A direct computation shows that if α is a solution of (8), then Gaussian curvature of the metric (7) takes the form (1). Since it is supposed that K is sign-preserving, the relation (1) allows to choose α as a new parameter on u-curves. With respect to the parameter α we have

$$du = \frac{d\alpha}{K} = -\frac{\cos\alpha}{a+1-\cos\alpha}\,d\alpha$$

and the line element (7) takes the form

$$ds^2 = \left(\frac{\cos\alpha}{a+1-\cos\alpha}\right)^2 d\alpha^2 + \sin^2\alpha\, dv^2. \tag{9}$$

R e m a r k. If ξ is a unit vector field on the Riemannian manifold M^n, then the induced metric on $\xi(M^n)$ is $d\tilde{s}^2 = g_{ik}du^i du^k + \big\langle\nabla_i\xi, \nabla_k\xi\big\rangle du^i du^k$. If ξ is a totally geodesic vector field on M^2, then the metric of M^2 has the standard form (7) and $\nabla_{\partial_u}\xi = \nabla_{e_0}\xi = 0$, $\nabla_{\partial_v}\xi = \sin\alpha\nabla_{e_1}\xi = \sin\alpha\lambda\,\eta = 2\sin^2(\alpha/2)\,\eta$. Thus, we have

$$d\tilde{s}^2 = du^2 + \sin^2\alpha\, dv^2 + 4\sin^4(\alpha/2)dv^2 = du^2 + 4\sin^2(\alpha/2)dv^2.$$

Taking into account (1), we can easily find the Gaussian curvature of the totally geodesic submanifold $\xi(M^2)$, namely

$$\tilde{K} = \frac{1}{4}K(K - 2\cot(\alpha/2)K'_\alpha),$$

where $K(\alpha)$ is the Gaussian curvature of M^2 given by relations (1) and (8).

The equations (8) and (1) completely determine the class of Riemannian 2-dimensional manifolds admitting a totally geodesic unit vector field.

Proposition 1.1. *Let M^2 be a Riemannian manifold with a line element of the form*

$$ds^2 = du^2 + \sin^2\alpha(u)dv^2.$$

Denote by K the Gaussian curvature of M^2. Then $K = \dfrac{d\alpha}{du}$ if and only if the function $\alpha(u)$ satisfies

$$\frac{d\alpha}{du} = 1 + \frac{m}{\cos\alpha}, \quad m = const.$$

P r o o f. The sufficient part is already proved. Suppose now that

$$\frac{d\alpha}{du} = K(\neq 0).$$

Then we have

$$\alpha' = K = -\frac{\partial_{uu}(\sin\alpha)}{\sin\alpha} = (\alpha')^2 - \cot\alpha\,\alpha''.$$

Therefore, $\alpha'' = -\alpha'(1-\alpha')\tan\alpha$, or $\dfrac{\alpha''}{\alpha'-1} = \alpha'\tan\alpha$, or

$$(\ln|\alpha'-1|)' = -(\ln|\cos\alpha|)'.$$

Evidently, now $|\alpha'-1| = \dfrac{|m|}{|\cos\alpha|}$ where $m = const$ is a constant of integration. Finally, $\dfrac{d\alpha}{du} = 1 + \dfrac{m}{\cos\alpha}$. ■

Corollary 1.1. *Let M^2 be a Riemannian manifold of constant curvature $c \neq 0$. Then M^2 admits a totally geodesic unit vector field if and only if $c = 1$. This vector field is parallel along meridians and moves along parallels with unit angle speed.*

P r o o f. If $K = c = const$, then (1) can be satisfied if and only if $c = 1$, $a = -1$. ■

The equation (1) implies an elementary nonexistence result.

Corollary 1.2. *Let M^2 be a Riemannian manifold with Gaussian curvature K. Then M^2 does not admit a totally geodesic unit vector field ξ with angle speed a if $|K-1| < |a+1|$.*

P r o o f. Indeed, one can easily see that $\cos\alpha = \dfrac{a+1}{1-K}$. If $|a+1| > |K-1|$, then we come to a contradiction. ■

2. Integral trajectories of the totally geodesic vector field

The integral trajectories of the totally geodesic vector field ξ can be found easily as follows. Let $\gamma = \{u(s), v(s)\}$ be an integral trajectory. Since

$$\xi = \cos\omega\, e_0 + \sin\omega\, e_1 = \cos\omega\, \partial_u + \frac{\sin\omega}{\sin\alpha}\, \partial_v,$$

we can set

$$\frac{du}{ds} = \cos\omega, \quad \frac{dv}{ds} = \frac{\sin\omega}{\sin\alpha}$$

and then

$$\frac{du}{dv} = \cot\omega \sin\alpha.$$

Since $\alpha = \alpha(u)$ and $\omega = av + \omega_0$, we come to the equation with separable variables

$$\frac{du}{\sin\alpha} = \cot\omega\, dv.$$

Using (8), we can find

$$\frac{du}{d\alpha} = \frac{\cos\alpha}{-a - 1 + \cos\alpha}$$

and make a parameter change in the left hand side of the equation above. Then we come to the equation

$$\frac{\cos\alpha\, d\alpha}{\sin\alpha(-a - 1 + \cos\alpha)} = \cot\omega\, dv.$$

Taking primitives, we have

$$\begin{aligned}
&\tan(\alpha/2)\sin(av + \omega_0) = c\,(a + (a+2)\tan^2(\alpha/2))^{\frac{a+1}{a+2}} && \text{for } a \neq 0, -2,\\
&\tan(\alpha/2)\sin(-2v + \omega_0) = c\, e^{\frac{1}{2}\tan^2(\alpha/2)} && \text{for } a = -2,\\
&\tfrac{1}{2}\tan\omega_0 \left(\tfrac{1}{1-\cos\alpha} + \ln\big|\tan(\alpha/2)\big|\right) = v - c && \text{for } a = 0.
\end{aligned}$$

Taking into account (3), we remark that $\tan(\alpha/2)\sin\omega = k$ and $\tan^2(\alpha/2) = k^2 + \varkappa^2$. Therefore, we have an intrinsic equation on the integral curves of the totally geodesic vector field

$$\begin{aligned}
&k = c\left[a + (a+2)(k^2 + \varkappa^2)\right]^{\frac{a+1}{a+2}} && \text{for } a \neq 0, -2,\\
&k = c\, e^{\frac{1}{2}(k^2 + \varkappa^2)} && \text{for } a = -2,\\
&k = \sin\omega_0 \, \exp\left[2\cot\omega_0 (v - c) - \tfrac{1}{2}\tfrac{1 + k^2 + \varkappa^2}{k^2 + \varkappa^2}\right] && \text{for } a = 0,
\end{aligned}$$

where c is a constant of integration.

Moreover, in any case

$$\begin{aligned}\xi(k) &= \cos\omega\,\partial_u[\tan(\alpha/2)\sin\omega] + \frac{\sin\omega}{\sin\alpha}\partial_v[\tan(\alpha/2)\sin\omega] \\ &= \frac{\cos\omega\sin\omega\,\alpha'_u}{2\cos^2(\alpha/2)} + \frac{a\,\sin\omega\cos\omega\tan(\alpha/2)}{\sin\alpha} = \frac{\cos\omega\sin\omega}{2\cos^2(\alpha/2)}(\alpha'_u + a).\end{aligned}$$

The equation (8) yields

$$\xi(k) = \frac{(a+1)\cos\omega\sin\omega}{2\cos^2(\alpha/2)}\left(1 - \frac{1}{\cos\alpha}\right).$$

Thus, if $a = -1$, then the integral trajectories of the field ξ form a family of circles. The metric of M^2 is

$$ds^2 = du^2 + \sin^2 u\,dv^2,$$

and we are dealing with the unit sphere parameterized by

$$r = \big\{\sin u\cos v, \sin u\sin v, \cos u\big\}.$$

These circles satisfy

$$\tan(u/2)\sin v = c. \tag{10}$$

Let (ρ,φ) be polar coordinates in a Cartesian plane which passes through the center of the sphere such that $(0,0,1)$ is the *north* pole on the sphere. Then the paramesters (ρ,φ) and (u,v) are connected via stereographic projection from the *south* pole as

$$\begin{aligned}\rho &= \tan(u/2),\\ \varphi &= v.\end{aligned}$$

Therefore, the equation (10) defines a family of parallel straight lines on the Cartesian plane. *The family of integral curves of a totally geodesic vector field on the unit sphere can be obtained as inverse images under stereogrphic projection of this family.*

An explicit equation of this family is

$$r(v) = \left\{\frac{2c\sin v\cos v}{c^2+\sin^2 v}, \frac{2c\sin^2 v}{c^2+\sin^2 v}, -\frac{c^2-\sin^2 v}{c^2+\sin^2 v}\right\},$$

where c is the geodesic curvature of the corresponding circle. All of these circles pass through the south pole $(0,0,-1)$ when $v = 0,\pi$. We can find this by using the expression $\tan(u/2) = c/\sin v$ and trigonometric expressions for $\sin u$ and $\cos u$ via $\tan(u/2)$.

The unit sphere is not the unique surface that realizes the metric (9). Let (x, y, z) be standard Cartesian coordinates in E^3. We can find an isometric immersion of the metric (9) into E^3 in a class of a surfaces of revolution. To do this, set

$$x(\alpha) = \sin\alpha,$$
$$(x'_\alpha)^2 + (z'_\alpha)^2 = \left(\frac{\cos\alpha}{a+1-\cos\alpha}\right)^2,$$

and we easily find

$$x(\alpha) = \sin\alpha,$$
$$z(\alpha) = \int\limits_{\alpha_0}^{\alpha} \frac{\cos t}{a+1-\cos t}\sqrt{1-(a+1-\cos t)^2}\, dt,$$

where the interval of integration is limited by the restrictions

$$\begin{cases} 1+a < \cos\alpha < 2+a, \\ -2 < a < -1, \end{cases} \quad \text{or} \quad \begin{cases} a < \cos\alpha < 1+a, \\ -1 < a < 0. \end{cases}$$

The restrictions mean that if $|a+1| \geq 1$, then the metric (9) does not admit an isometric immersion into E^3 in a class of surfaces of revolution.

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