Matematicheskaya fizika, analiz, geometriya
2004, v. 11, No. 4, p. 375–379

# A sharp inequality for the order of the minimal positive harmonic function in T-homogeneous domain

V. Azarin and A. Gol'dberg

*Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel*
E-mail:azarin@macs.biu.ac.il

Received February 2, 2004

Let $G$ be a simply connected domain in $\mathbb{C}$ which is $T$-homoheneous, i.e., $TG = G$ for some $T > 0$. Let $\rho(G)$ be the order of the minimal positive harmonic function in $G$. We prove that a kind of symmetrization of $G$ and prove that it does not increase $\rho(G)$. This implies a sharp lower bound for $\rho(G)$ in terms of conformal modulus of a quadrilateral naturally connected with $G$.

*To Iossif Vladimirovich Ostrovskii on the occasion of his 70-th birthday*

## 1. Introduction and main result

A domain $G \subset \mathbb{C}$ is called *$T$-homogeneous* if $TG := \{Tz : z \in G\} = G$ for some $T > 0$. A circle $C_R := \{|z| = R,\ R > 0\}$ is called *separating* for $G$ if $C_R \cap G$ is one nonempty arc. Let $G$ be a $T$-homogeneous, simply connected domain, having the separating circle $C_1$. Then every circle $C_{T^n},\ n \in \mathbb{Z}$, is separating.

Consider the following construction: set $G_n := G \cap \{T^n < |z| < T^{n+1}\},\ n \in \mathbb{Z}$. They are simply connected. Replace every $G_{2k-1},\ k \in \mathbb{Z}$, by $G^*_{2k}$ that is symmetric to $G_{2k}$ with respect to the circle $\{|z| = T^{2k}\}$. Thus obtained domain $G^S$ is also $T$-homogeneous, simply connected and has a separating circle. It is a kind of symmetrization of $G$.

Let $H(z) \not\equiv 0$ be a positive harmonic function in $G$, satisfying the condition $H(z_0) = 1$, that is locally bounded and equal to zero at every regular point of the boundary $\partial G$. We also suppose that $H$ is equal to zero outside $\overline{G}$. Set $M(r, H) = \sup_{|z|=r} H(z)$.

As it is proved in [1, Theorem 0.5] for every $T$-homogeneous domain there exists an $H$ of the form

$$H(e^z) = q(z)e^{\rho(G)x},\ z = x + iy,$$

Mathematics Subject Classification 2000: 31A05, 30C75.

where $q(z)$ is bounded.

Thus

$$\rho(G) = \lim_{r\to+\infty} \frac{\log M(r,H)}{\log r}. \tag{1}$$

It is called *order of minimal positive function* $H$, but actually it depends only on $G$.

Let us also note that the property of $G$ to be connected and $T$-homogeneous, and the value of $\rho(G)$ are preserved under homotheties of $G$ and rotations, i.e., replacing $G$ for $te^{i\phi}G$, $t > 0, \phi \in [0; 2\pi)$. So we have $\rho(te^{i\phi}G) = \rho(G)$. Thus in all the assertions, concerning $\rho(G)$, we can replace the separating circle $C_1$ by arbitrary circle $C_R$.

Recall that if a quadrilateral ABCD is conformally mapped onto a rectangle $\{z : 0 < x < 1,\ 0 < y < M\}$ such that $A \mapsto 0, B \mapsto 1, C \mapsto 1 + iM, D \mapsto iM$ then $\operatorname{mod} ABCD(AB, CD) := M$ is the *conformal modulus* of quadrilateral $ABCD$ with *marked sides* $AB$ and $CD$.

In [1, Theorem 4.7, Corollary 4.9] the following assertion was proved

**Theorem ADP.** *Let $G$ be a $T$-homogeneous domain with a separating circle then*

$$\rho(G^S) = \frac{\pi}{\log T} \quad \operatorname{mod} G_0(J_0, J_1), \tag{2}$$

*where $J_0 = G \cap \{|z| = 1\}$, $J_1 = TJ_0$, and $\operatorname{mod} G_0(J_0, J_1)$ is the conformal modulus of quadrilateral $G_0$ with marked sides $J_0, J_1$.*

*If $G$ is symmetric with respect to a separating circle then $G^S$ can be replaced by $G$.*

We recall the proof of Theorem ADP because we need some details from it.

We are going to prove

**Theorem 1.** *The following inequality holds*

$$\rho(G) \geq \rho(G^S).$$

*Equality is attained on $G'$s that are symmetric with respect to $C_1$.*

Theorem ADP and Theorem 1 immediately imply

**Corollary.** *The following sharp inequality holds*

$$\rho(G) \geq \frac{\pi}{\log T} \quad \operatorname{mod} G_0(J_0, J_1).$$

We are grateful to Prof. I.V. Ostrovskii for very valuable discussion and suggestions.

## 2. Proofs

Set $P := \log T$ and make the transformation of $\mathbb{C}\backslash 0$ by $z \mapsto \log z$. Since $G$ does not include zero and is simply connected it is possible to select a single-valued branch of logarithm that maps $G$ onto a $P$-periodic domain $D$. Then $G_n \mapsto D_n$ such that $D_n \subset \{nP < \Re z < (n+1)P\}$ and $G^*_{2k} \mapsto D^*_{2k}$, where $D^*_{2k}$ is symmetric to $D_{2k}$ with respect to $\{\Re z = 2kP\}$. Thus the image of $G^S$ is the domain $D^S$ which is obtained from $D$ by replacing $D_{2k-1}$ for $D^*_{2k}$.

P r o o f o f T h e o r e m A D P. Let $f^S(z)$ be a function that maps conformally the quadrilateral $D_0$ with marked sides $I_0 := D \cap \{\Re z = 0\}$ and $I_1 := D \cap \{\Re z = P\}$ to the rectangle $R = (0, c) \times (0, \pi)$ such that the marked sides are mapped to the vertical sides of the rectangle. Then $c = \pi \mod D_0(I_0, I_1) = \pi \mod G_0(J_0, J_1)$.

The function $f^S$ can be continued analytically to $D^*_0$ and further to the $D^S$ such that it maps $D^S$ onto the strip $\{0 < \Im z < \pi\}$. Since $D^S$ is $2P$-periodic, we have $f^S(z + 2P) = f^S(z) + 2c$, and

$$\Re f^S(z + 2P) = \Re f^S(z) + 2c. \tag{3}$$

Now consider the harmonic function

$$H(z) = \Im e^{f^S(\log z)} = e^{\Re f^S(\log z)} \sin \Im f^S(\log z). \tag{4}$$

This function is positive within $G$ and is equal to zero at every regular point of boundary. It has a finite order. Actually, from (3) we have

$$\Re f^S(z + 2Pn) \le C + 2cn \tag{5}$$

and $C := \sup_{z \in D_0} |\Re f^S(z)| < \infty$, and we can replace $n$ for $\Re z/2P$ by maximum principle. Thus

$$M(r, H) \le C_1 r^{\alpha},$$

where $C_1$ and $\alpha$ can be expressed in terms of $C$ and $c$. Using (3), (4) and (1) for $r = e^{2nP}$, we obtain (2). ■

For the proof of Theorem 1 we need the following assertion (lemma of Grötzsch) (see [2, p. 142], also see [3, Ch. 2, §D, Th. 4])

**Lemma G.** *Let a quadrilateral $ABCD$ with the marked sides $AB$ and $CD$ be divided by a curve $EF \subset ABCD$, $E \in BC$, $F \in AD$ onto two quadrilaterals $ABEF$ with the marked sides $AB$ and $EF$, and $FECD$ with the marked sides $EF$ and $CD$. Then*

$$\mod ABEF + \mod FECD \le \mod ABCD. \tag{6}$$

*When ABCD is a rectangle the equality in (6) holds iff ABEF and FECD are rectangles.*

One can see that if $EF$ is an arc of a circle or a segment of a line and $ABEF$ and $FECD$ are symmetric with respect $EF$ then $\mod ABEF = \mod FECD$ and equality holds in (6).

We will also use [1, Prop. 4.10] that can be reformulated in the following form:

**Theorem P.** *Let $D$ be a domain corresponding to $G$ by the mapping $z \mapsto \log z$. Set $D^{2n} := D \cap \{0 < \Re z < 2Pn\}$. Then following holds*

$$\rho(G) = \frac{\pi}{P} \lim_{n\to\infty} \frac{\mod D^{2n}(I_0, I_{2n})}{2n}.$$

We give the proof for the sake of completeness.

P r o o f. We use the following assertion of A. Eremenko (see [4, Remark 3], see also [5, Lemma 6.4] that can be formulated in the same way).

**Theorem E.** *Let $f(z)$ conformally map $D$ onto the strip $\{0 < \Im w < \pi\}$ such that $0 \mapsto -\infty$ and $\infty \mapsto +\infty$. Then*

$$f(z + P) = f(z) + c_1, \tag{7}$$

*where $c_1 > 0$.*

Set

$$J_k := f(I_k),\ m_k := \min_{J_k} \Re\zeta,\ M_k := \max_{J_k} \Re\zeta.$$

The numbers $m_k, M_k$ depend on $k$ in following way

$$m_k = m_0 + kc_1,\ M_k = M_0 + kc_1,\ k = 0, \pm 1, \ldots$$

because of (7).

Denote by

$$R_{a,b} := \{\zeta : 0 < \Im\zeta < \pi,\ a < \Re\zeta < b\}$$

a rectangle. Since

$$R_{m_0,M_{2n}} \subset f(D^{2n}) \subset R_{M_0,m_{2n}}$$

we obtain

$$2nc_1 + m_0 - M_0 < \pi \mod D^{2n}(I_0, I_{2n}) < 2nc_1 + M_0 - m_0.$$

From this we obtain the equality

$$c_1 = \pi \lim_{n\to\infty} \frac{\mod D^{2n}(I_0, I_{2n})}{2n}.$$

Using arguments from the proof of Theorem ADP with (7) instead of (3), we obtain that $c_1 = P\rho(G)$. This completes the proof of Theorem P. ∎

P r o o f   o f   T h e o r e m   1. We apply Lemma G to the quadrilateral $D^2 := D \cap \{-P < \Re z < P\}$ with the naturally marked sides and obtain that

$$\operatorname{mod} D_{-1}(I_{-1}, I_0) + \operatorname{mod} D_1(I_0, I_1) \le \operatorname{mod} D^2(I_{-1}, I_1).$$

Obviously $\operatorname{mod} D_{-1}(I_{-1}, I_0) = \operatorname{mod} D_1(I_0, I_1)$. Thus

$$2\mu := \operatorname{mod} D^2(I_{-1}, I_1) \ge 2 \cdot \operatorname{mod} D_1(I_0, I_1) =: 2M.$$

By the same Lemma G, used repeatedly, we obtain the inequality

$$\operatorname{mod} D^{2n}(I_0, I_{2n}) \ge 2n\mu \ge 2nM.$$

Thus from Theorems ADP and P we obtain the inequality of Theorem 1. It is clear that the equality is attained iff $\mu = M$, i.e., when $D_0$ is symmetric to $D_{-1}$ and hence $D$ is symmetric with respect to $I_0$. ■

## References


[1] *V. Azarin, D. Drasin, and P. Poggi-Corradini,* A generalization of trigonometric convexity and its relation to positive harmonic functions in homogeneous domains. — *J. d'Anal. Math.* (To appear)

[2] *L.I. Volkovyskii,* Quasiconformal mappings. L'vov (1954). (Russian: Kvazikonformnye otobrazheniya)

[3] *L. Ahlfors,* Lectures on quasiconformal mappings. D. van Nostrand Co. Inc., Princeton, New Jersy (1966).

[4] *A.A. Gol'dberg,* Estimates of conformal maps of curvilinear strips. — *Mat. Fiz., Anal., Geom.* (2002), v. 9, No. 1, p. 1–4.

[5] *P. Poggi-Corradini,* The Hardy class of geometric models and the essential spectral radius of composition operators. — *J. Funct. Anal.* (1997), v. 143, p. 129–156.