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A probabilistic approach to q-polynomial coefficients, Euler and Stirling numbers. I

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It is known that Bernoulli scheme of independent trials with two outcomes is connected with the binomial coefficients. The aim of this paper is to indicate stochastic processes which are connected with the q-polynomial coefficients (in particular, with the q-binomial coefficients, or the Gaussian polynomials), Stirling numbers of the first and the second kind, and Euler numbers in a natural way. A probabilistic approach allows us to give very simple proofs of some identities for these coefficients.

To my teacher Professor Iossif Vladimirovich Ostrovskii
on the occasion of his 70-th birthday

1. Introduction

The connection of binomial coefficients $\binom{n}{k}$ with Bernoulli scheme of independent trials is well known. To be more specific, let $x \in (0,1)$, n be a positive integer, $\mathcal{E}$ be a trial with two outcomes 0 and 1 with probabilities in a single trial being equal to $1-x$ and x, respectively. Let $\mathcal{E}$ be repeated n times under the condition that every outcome of any trial is independent of outcomes of all other trials. Let $\mathbf{1}\binom{n}{k}$ denote an event such that the outcome 1 has happened k times in n repetitions of the trial $\mathcal{E}$. Then the probability $P(\mathbf{1}\binom{n}{k})$ of this event equals

$$P(\mathbf{1}\tbinom{n}{k}) = \binom{n}{k} x^k (1-x)^{n-k}\,.$$

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The aim of this paper is to construct random processes which are connected with q-polynomial coefficients

$$\begin{bmatrix} i_1+i_2+\ldots+i_m \\ i_1,i_2,\ldots,i_m \end{bmatrix}_q := \frac{(q)_{i_1+i_2+\ldots+i_m}}{(q)_{i_1}(q)_{i_2}\ldots(q)_{i_m}}, \tag{1.1}$$

where $(q)_j := (1-q)(1-q^2)\ldots(1-q^j)$ for $j \in \mathbf{N}$, $(q)_0 := 1$, in particular, q-binomial coefficients (or Gaussian polynomials). The probabilistic approach gives very simple proofs of some identities for q-polynomial coefficients.

We apply an analogues approach to Stirling numbers of the first and the second kind $\begin{bmatrix} n \\ k \end{bmatrix}$, $\begin{Bmatrix} n \\ k \end{Bmatrix}$ and Euler numbers $\left\langle {n \atop k} \right\rangle$ (we follow notation used in [3, Ch. 6]).

In this paper, we introduce numbers associated with Euler and Stirling numbers and obtain some identities for them.

As usual, $\mathbf{N} = \{1,2,3,\ldots\}$ and $\mathbf{N_0} = \{0,1,2,3,\ldots\}$ denote the sets of positive integers and nonnegative integers, respectively. We use the following notation (see [1, Ch. 3]):

$$\begin{aligned} (x;q)_n &:= (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for } n \in \mathbf{N}, \quad (x;q)_0 := 1; \\ (q)_n &:= (q;q)_n . \end{aligned}$$

The identity

$$(x;q)_m \cdot (q^m x;q)_n = (x;q)_{m+n} \tag{1.2}$$

will be useful in the sequel.

We recall that Stirling numbers of the first and the second kind $\begin{bmatrix} n \\ k \end{bmatrix}$, $\begin{Bmatrix} n \\ k \end{Bmatrix}$, and Euler numbers $\left\langle {n \atop k} \right\rangle$ may be defined for $n \in \mathbf{N_0}$ and integer k such that $0 \le k \le n$ as numbers which equal 1 if $n = k = 0$, and 0, if $k < 0$ or $k > n$, and satisfy the following recurrence identities (see [3, Sect. 6.1]):

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}, \\ \begin{Bmatrix} n \\ k \end{Bmatrix} &= \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}, \\ \left\langle {n \atop k} \right\rangle &= (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle, \end{aligned} \tag{1.3}$$

respectively.

We remind some elementary probabilistic concepts which we use in this paper (see, for example, [2, Chs. 1, 2]). A *probability space* is a pair (Ω, p) where Ω is a finite set (it is called a *sample space*) and p is a function on Ω (it is called a *probability*) such that $p(\omega) \ge 0$ for all $\omega \in \Omega$ and $\sum_{\omega\in\Omega} p(\omega) = 1$. Every set $A \subset \Omega$ is said to be an *event*. The *probability of an event A* is defined by

$P(A) = \sum_{\omega \in A} p(\omega)$. If $A, B \subset \Omega$ and $P(B) \neq 0$, then $P(A|B) := P(A \cap B)/P(B)$ is said to be the *conditional probability* of A given B. The formula $P(A \cap B) = P(B)P(A|B)$ is called *the multiplication theorem of probability*. We say that events A_l, $l = 1, 2, \dots L$, form a *partition* of a sample space Ω if A_l are mutually exclusive and $\cup_{l=1}^{L} A_l = \Omega$. In that case, *the formula of total probability* is valid:

$$P(B) = \sum_{l=1}^{L} P(B|\, A_l) P(A_l)\,. \tag{1.4}$$

We also use the following notation: $i_{(k)} := \underbrace{i, i, \dots, i}_{k}$ for $k \in \mathbf{N}$.

The paper is organized as follows. In Section 2, we consider q-binomial coefficients. We consider this special case of q-polynomial coefficients separately for the convenience of readers.

Section 3 is devoted to q-polynomial coefficients. We consider the case $m = 3$ in (1.1) only. The general case is obvious after that.

In Sections 4–6, we consider Stirling numbers of the second and the first kind and Euler numbers, respectively.

In the beginning of each section we define a probability space or a space equipped with a weight which is connected with corresponding coefficients in a natural way (see Theorems 2.1, 3.1, 4.3, 5.2, 6.2). Using probabilistic arguments we give very simple proofs of some identities for these coefficients (see Theorems 2.2, 2.3, 3.2, 3.3, 4.4, 5.3, 6.3). We also introduce a notion of coefficients associated with Stirling and Euler numbers and deduce some identities for them (see Theorems 4.5, 5.4, 6.4).

The paper is divided in two parts. The first part contains Sect. 1–3. The second part contains Sect. 4–6.

This study has been stimulated by the paper [4]. When this work has been already written, I have learned about the paper [6] which contains other processes which lead to the q-binomial coefficients.

2. q-binomial coefficients (Gaussian polynomials)

Let x and q be arbitrary real numbers from the interval $(0, 1)$, n be a positive integer. Let $\mathcal{E}$ be a trial with two outcomes 0 and 1. We consider a sequence of n trials $\mathcal{E}$. We take the probability of outcomes 0 and 1 in the first trial $\mathcal{E}$ to be equal $1 - x$ and x, respectively. Assume that the trial $\mathcal{E}$ is repeated m times and the outcome 0 has occurred j times where $0 \le j \le m$. Then we take the probability of 0 and 1 in the $(m+1)^{th}$ repetition of $\mathcal{E}$ to be $1 - q^j x$ and $q^j x$, respectively.

A mathematical model of this sequence of n trials $\mathcal{E}$ is the probability space

$(\Omega_n, p_{x,q,n})$ where

$$\Omega_n := \{\omega = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) : \varepsilon_k = 0 \text{ or } 1, k = 1, 2, \dots, n\},$$

the probability $p_{x,q,n}(\omega)$ of an elementary event $\omega = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is equal to

$$p_{x,q,n}(\omega) = p_{x,q,n}((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)) = f_1 \cdot f_2 \cdot \dots \cdot f_n, \tag{2.1}$$

where $f_1 = x$, if $\varepsilon_1 = 1$, $f_1 = 1 - x$, if $\varepsilon_1 = 0$, and for every integer m $(1 \le m \le n-1)$,

$$f_{m+1} = \begin{cases} q^j x, & \text{if } \varepsilon_{m+1} = 1, \\ 1 - q^j x, & \text{if } \varepsilon_{m+1} = 0, \end{cases} \tag{2.2}$$

where $j = \#\{l : 1 \le l \le m, \varepsilon_l = 0\}$. The probability of an event $A \subset \Omega_n$ is defined by

$$P_{x,q,n}(A) := \sum_{\omega \in A} p_{x,q,n}(\omega). \tag{2.3}$$

For the sake of brevity, sometimes we write P instead of $P_{x,q,n}$. It is not difficult to see that $P_{x,q,m}(A) = P_{x,q,n}(A \times \{0,1\}^{n-m})$ if $m < n$ and $A \subset \Omega_m$. (The sign $\times$ means the Cartesian product. Therefore $\omega = (\varepsilon_1, \dots, \varepsilon_m, \varepsilon_{m+1}, \dots, \varepsilon_n) \in A \times \{0,1\}^{n-m}$ if and only if $(\varepsilon_1, \dots, \varepsilon_m) \in A$ and $\varepsilon_{m+1}, \dots, \varepsilon_n = 0$ or 1. In other words, the event $A \times \{0,1\}^{n-m} \subset \Omega_n$ does not depend on the $(m+1)^{th}$, $(m+2)^{th}, \dots, n^{th}$repetitions of the trial $\mathcal{E}$.)

For $i = 0, 1$ and integers $l, m,$ and k $(1 \le l \le n,\ 1 \le m \le n,\ 0 \le k \le m)$ we define

$$\mathbf{i}^{(l)} := \{\omega = (\varepsilon_1, \varepsilon_2, \dots \varepsilon_n) \in \Omega_n : \varepsilon_l = i\}, \tag{2.4}$$

$$\mathbf{i}\tbinom{m}{k} := \{\omega = (\varepsilon_1, \varepsilon_2, \dots \varepsilon_n) \in \Omega_n : \#\{l : 1 \le l \le m, \varepsilon_l = i\} = k\}. \tag{2.5}$$

Using this notation, we may write

$$P_{x,q,m+1}(\mathbf{1}^{(m+1)} | \mathbf{1}\tbinom{m}{k}) = q^{m-k}x, \quad P_{x,q,m+1}(\mathbf{0}^{(m+1)} | \mathbf{1}\tbinom{m}{k}) = 1 - q^{m-k}x$$

for every $m \ge 1$. It should be pointed out that the exponent of the power q^{m-k} is equal to the quantity of all trials with outcome 0 among the first m trials.

The following theorem is a key theorem of this section.

Theorem 2.1. *Let $n, k \in \mathbf{N}_0$, $0 \le k \le n$. Then*

$$P(\mathbf{1}\tbinom{n}{k}) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}. \tag{2.6}$$

P r o o f. It is not difficult to see that $p(\omega) = x^k (x;q)_{n-k} q^r$ for every $\omega \in \mathbf{1}\tbinom{n}{k}$, where r is an integer such that $0 \le r \le k(n-k)$. For example,

$$\begin{aligned} p((1_{(k)}, 0_{(n-k)})) &= x^k (x;q)_{n-k}, \\ p((0_{(n-k)}, 1_{(k)})) &= x^k (x;q)_{n-k} q^{k(n-k)}. \end{aligned}$$

Therefore, we may write for all $n \in \mathbf{N}$ and integer k such that $0 \le k \le n$:

$$P(\mathbf{1}\tbinom{n}{k}) = x^k (x;q)_{n-k} \gamma\tbinom{n}{k}\,, \tag{2.7}$$

where $\gamma\binom{n}{k}$ is a polynomial in q. We define also $\gamma\binom{0}{0} = 1$ and $\gamma\binom{n}{k} = 0$ for $k > n$ and $k < 0$.

We need prove that $\gamma\binom{n}{k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$, where $\gamma\binom{n}{k}$ is a polynomial defined by (2.7). First, we prove that the polynomials $\gamma\binom{n}{k}$ satisfy the following recurrence identity:

$$\gamma\tbinom{n}{k} = \gamma\tbinom{n-1}{k} + q^{n-k}\gamma\tbinom{n-1}{k-1}. \tag{2.8}$$

We apply the formula of total probability. The events $\mathbf{1}\binom{n-1}{j}$ $(j = 0, 1, \ldots, n-1)$ are disjoint with union Ω_n. Since $P(\mathbf{1}\binom{n}{k} |\, \mathbf{1}\binom{n-1}{j}) = 0$ for all j but $j = k-1$ and $j = k$, we may write

$$\begin{aligned}
&\gamma\tbinom{n}{k} x^k (x;q)_{n-k} = P(\mathbf{1}\tbinom{n}{k}) \\
&= P(\mathbf{1}\tbinom{n}{k} |\, \mathbf{1}\tbinom{n-1}{k}) P(\mathbf{1}\tbinom{n-1}{k}) + P(\mathbf{1}\tbinom{n}{k} |\, \mathbf{1}\tbinom{n-1}{k-1}) P(\mathbf{1}\tbinom{n-1}{k-1}) \\
&= P(\mathbf{0}^{(n)} |\, \mathbf{1}\tbinom{n-1}{k}) P(\mathbf{1}\tbinom{n-1}{k}) + P(\mathbf{1}^{(n)} |\, \mathbf{1}\tbinom{n-1}{k-1}) P(\mathbf{1}\tbinom{n-1}{k-1}) \\
&= (1 - q^{n-k-1}x)\gamma\tbinom{n-1}{k} x^k (x;q)_{n-1-k} + x q^{n-k} \gamma\tbinom{n-1}{k-1} x^{k-1} (x;q)_{(n-1)-(k-1)}\,.
\end{aligned}$$

Using (1.2) and dividing by $x^k(x;q)_{n-k}$, we conclude that (2.8) is true. It is known that the Gaussian polynomials satisfy the same recurrence identity:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

Next, (2.7) yields that $\gamma\binom{n}{0} = 1$ and $\gamma\binom{n}{n} = 1$ for all $n \in \mathbf{N}$. Therefore polynomials $\gamma\binom{i}{j}$ and Gaussian polynomials satisfy the same recurrence identity and boundary conditions. This completes the proof of Theorem 2.1. ■

We apply Theorem 2.1 to the proof of two known identities which contain q-binomial coefficients (see, for example, [1, Formulas (3.3.9), (3.3.10)]).

Theorem 2.2. 1) *If $a, b, m \in \mathbf{N}_0$, then*

$$\begin{bmatrix} a+b \\ m \end{bmatrix}_q = \frac{1}{2} \sum_{\substack{j,k \ge 0 \\ j+k=m}} \begin{bmatrix} a \\ j \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{-jk} \left(q^{ak} + q^{bj} \right) . \tag{2.9}$$

2) *If $m, n \in \mathbf{N}_0$, then*

$$\begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}_q = \sum_{j=0}^{n} q^j \begin{bmatrix} m+j \\ m \end{bmatrix}_q . \tag{2.10}$$

P r o o f. 1) The events $\mathbf{1}\binom{a}{0}, \mathbf{1}\binom{a}{1}, \mathbf{1}\binom{a}{2}, \ldots, \mathbf{1}\binom{a}{a}$ form a partition of the sample space Ω_n. The formula of total probability yields

$$P(\mathbf{1}\tbinom{a+b}{m}) = \sum_{j=0}^{a} P(\mathbf{1}\tbinom{a+b}{m} | \, \mathbf{1}\tbinom{a}{j}) P(\mathbf{1}\tbinom{a}{j}). \tag{2.11}$$

Obviously, $P(\mathbf{1}\binom{a+b}{m} | \, \mathbf{1}\binom{a}{j}) = 0$, if $j > m$. To calculate $P(\mathbf{1}\binom{a+b}{m} | \, \mathbf{1}\binom{a}{j})$ for $j \leq m$ we need the following

Lemma. *If $a, b, m \in \mathbf{N}_0$ and $0 \leq j \leq m$, then*

$$P_{x,q,a+b}(\mathbf{1}\tbinom{a+b}{m} | \, \mathbf{1}\tbinom{a}{j}) = P_{xq^{a-j},q,b}(\mathbf{1}\tbinom{b}{m-j}) \,. \tag{2.12}$$

The proof of this lemma follows from the definition of conditional probability and from the definitions (2.1)–(2.3)of probability in the considered probability space. It is elementary and may be omitted.

By (2.12) and (1.2), we see that the j^{th} term at the right-hand side of (2.11) equals

$$\begin{aligned} &\begin{bmatrix} b \\ m-j \end{bmatrix}_q x^{m-j} q^{(a-j)(m-j)} (xq^{a-j}; q)_{b-m+j} \cdot \begin{bmatrix} a \\ j \end{bmatrix}_q x^j (x;q)_{a-j} \\ &= \begin{bmatrix} b \\ m-j \end{bmatrix}_q \begin{bmatrix} a \\ j \end{bmatrix}_q x^m q^{(a-j)(m-j)} (x;q)_{a+b-m} \,. \end{aligned} \tag{2.13}$$

Substituting (2.6) and (2.13) into (2.11) and dividing by $x^m (x;q)_{a+b-m}$, we get

$$\begin{bmatrix} a+b \\ m \end{bmatrix}_q = \sum_{j=0}^{m} \begin{bmatrix} b \\ m-j \end{bmatrix}_q \begin{bmatrix} a \\ j \end{bmatrix}_q q^{(a-j)(m-j)} \,.$$

Replacing $m - j$ with k shows that

$$\begin{bmatrix} a+b \\ m \end{bmatrix}_q = \sum_{\substack{j,k \geq 0 \\ j+k=m}} \begin{bmatrix} a \\ j \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{(a-j)k} \,. \tag{2.14}$$

If we first replace in (2.14) a with b and b with a, and then j with k and k with j, we get

$$\begin{bmatrix} a+b \\ m \end{bmatrix}_q = \sum_{\substack{j,k \geq 0 \\ j+k=m}} \begin{bmatrix} a \\ j \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{j(b-k)} \,. \tag{2.15}$$

Summing (2.14) and (2.15), we get (2.9).

2) Let us calculate the probability $P(\mathbf{1}\binom{n+m+1}{m+1})$ in two ways. We have by (2.6)

$$P_{x,q,n+m+1}(\mathbf{1}\tbinom{n+m+1}{m+1}) = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}_q x^{m+1}(x;q)_n\,. \tag{2.16}$$

Now we calculate this probability in a different way. For $j = 0, 1, 2, \ldots, n$, we introduce the following events:

$$C_j := \mathbf{1}\tbinom{n+m+1}{m+1} \cap \{ \quad (\varepsilon_1, \ldots, \varepsilon_{n+m+1}) \in \Omega_{n+m+1} : \\ \varepsilon_{m+j+1} = 1, \varepsilon_{m+j+2} = \ldots = \varepsilon_{n+m+1} = 0\}\,.$$

(We distinguished the last outcome 1 in the sequence of $n+m+1$ trials $\mathcal{E}$.) Any element ω of C_j can be written as $\omega = (\omega', 1, 0_{(n-j)})$, where $\omega' \in \mathbf{1}\binom{m+j}{m}$. The events C_j are mutually exclusive and $\cup_{j=0}^{n} C_j = \mathbf{1}\binom{n+m+1}{m+1}$. Therefore:

$$P(\mathbf{1}\tbinom{n+m+1}{m+1}) = \sum_{j=0}^{n} P(C_j)\,. \tag{2.17}$$

Using (2.6) and (1.2), we obtain

$$\begin{aligned} &P_{x,q,n+m+1}(C_j) \\ &= P_{x,q,m+j}(\mathbf{1}\tbinom{m+j}{m}) \cdot (xq^j) \cdot (1-xq^j)(1-xq^{j+1}) \cdots (1-xq^{n-1}) \\ &= \begin{bmatrix} m+j \\ m \end{bmatrix}_q x^m (x;q)_j \cdot xq^j \cdot (xq^j;q)_{n-j} = q^j \begin{bmatrix} m+j \\ m \end{bmatrix}_q x^{m+1}(x;q)_n\,. \end{aligned} \tag{2.18}$$

Substituting (2.16) and (2.18) into (2.17) and dividing by $x^{m+1}(x;q)_n$, we obtain (2.10). ∎

We may distinguish the first rather than the last - or the first and the last-outcome 1 in the sequence of trials. This leads to the following theorem.

Theorem 2.3. *If $m, n \in \mathbf{N}_0$, then*

$$\begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}_q = \sum_{j=0}^{n} q^{(m+1)j} \begin{bmatrix} n+m-j \\ m \end{bmatrix}_q\,, \tag{2.19}$$

$$\begin{bmatrix} n+m+2 \\ m+2 \end{bmatrix}_q = \frac{1}{2} \sum_{\substack{j,k \geq 0 \\ j+k \leq n}} \begin{bmatrix} n+m-j-k \\ m \end{bmatrix}_q q^{n-(j+k)} \left(q^{(m+2)j} + q^{(m+2)k} \right)\,. \tag{2.20}$$

P r o o f. 1) For $j = 0, 1, 2, \ldots, n$, we introduce the following events:

$D_j := \mathbf{1}\binom{n+m+1}{m+1} \cap \{(\varepsilon_1, \ldots, \varepsilon_{n+m+1}) \in \Omega_{n+m+1} : \varepsilon_1 = \ldots = \varepsilon_j = 0, \varepsilon_{j+1} = 1\}$.

D_j is formed by elements $\omega = (0_{(j)}, 1, \omega')$ where $\omega' \in \mathbf{1}\binom{n+m+1}{m}$. Therefore

$$\begin{aligned} P(D_j) &= (x;q)_j \cdot (xq^j) \cdot P_{xq^j,q,n+m-j}(\mathbf{1}\tbinom{n+m-j}{m}) \\ &= (x;q)_j \cdot \begin{bmatrix} n-j+m \\ m \end{bmatrix}_q \cdot (xq^j)^m (xq^j;q)_{n-j} \\ &= \begin{bmatrix} n-j+m \\ m \end{bmatrix}_q x^{m+1}(x;q)_n q^{(m+1)j}. \end{aligned} \tag{2.21}$$

The events D_j are mutually exclusive and $\cup_{j=0}^n D_j = \mathbf{1}\binom{n+m+1}{m+1}$. Therefore, by the additive property of probability, $P(\mathbf{1}\binom{n+m+1}{m+1}) = \sum_{j=0}^n P(D_j)$. Inserting (2.16) and (2.21) into this formula and dividing by $x^{m+1}(x;q)_n$, we get (2.19).

2) For $j \geq 0$ i $k \geq 0$ such that $j + k \leq n$, we consider events

$$\begin{aligned} E_{j,k} := \mathbf{1}\tbinom{n+m+2}{m+2} \cap \{ \quad & (\varepsilon_1, \ldots, \varepsilon_{n+m+2}) \in \Omega_{n+m+2} : \\ & \varepsilon_1 = \ldots \varepsilon_j = 0, \varepsilon_{j+1} = 1, \\ & \varepsilon_{n+m+3-k} = 1, \varepsilon_{n+m+4-k} = \ldots = \varepsilon_{n+m+2} = 0\}. \end{aligned}$$

The event $E_{j,k}$ is formed by elements $\omega = (0_{(j)}, 1, \omega', 1, 0_{(k)})$, where $\omega' \in \mathbf{1}\binom{n+m-j-k}{m}$. We calculate the probability of the events $E_{j,k}$ using (2.1)–(2.3):

$$\begin{aligned} &P_{x,q,n+m+2}(E_{j,k}) \\ &= (x;q)_j \cdot xq^j \cdot P_{q^j x,q,n+m-j-k}(\mathbf{1}\tbinom{n+m-j-k}{m}) \cdot xq^{j+(n-j-k)} \cdot (xq^{n-k};q)_k . \end{aligned}$$

Since

$$P_{q^j x,q,n+m-j-k}(\mathbf{1}\tbinom{n+m-j-k}{m}) = \begin{bmatrix} n+m-j-k \\ m \end{bmatrix}_q (xq^j)^m (xq^j, q)_{n-(j+k)},$$

it follows by (1.2) that

$$P(E_{j,k}) = \begin{bmatrix} n+m-j-k \\ m \end{bmatrix}_q q^{mj+j+n-k} x^{m+2} (x;q)_n . \tag{2.22}$$

Since $E_{j,k}$ are mutually exclusive and $\mathbf{1}\binom{n+m+2}{m+2} = \cup_{j,k} E_{j,k}$, the following equality holds: $P(\mathbf{1}\binom{n+m+2}{m+2}) = \sum_{j,k} P(E_{j,k})$. Substituting (2.22) into this equality, using (2.6), and dividing by $x^{m+2}(x;q)_n$, we get

$$\begin{bmatrix} n+m+2 \\ m+2 \end{bmatrix}_q = \sum_{\substack{j,k \geq 0 \\ j+k \leq n}} \begin{bmatrix} n+m-j-k \\ m \end{bmatrix}_q q^{mj+n+j-k} . \tag{2.23}$$

Replacing j in (2.23) by k, and k by j, we obtain

$$\begin{bmatrix} n+m+2 \\ m+2 \end{bmatrix}_q = \sum_{\substack{j,k\geq 0 \\ j+k\leq n}} \begin{bmatrix} n+m-j-k \\ m \end{bmatrix}_q q^{mk+n+k-j} . \tag{2.24}$$

Summing (2.23) and (2.24), we get (2.20). ∎

3. q-polynomial coefficients

Let $n \in \mathbf{N}$, $u_1 > 0$, $u_2 > 0$, $U := u_1 + u_2 < 1$, $0 < q < 1$. Let $\mathcal{E}$ be a trial with four outcomes $0, 1, 2, *$. We consider a special random process satisfying the following conditions:

1) The trial $\mathcal{E}$ is repeated n times.

2) The probabilities of the outcomes $0, 1, 2$ in the first trial are equal to

$$1 - u_1 - u_2 = 1 - U , \quad u_1 , \quad u_2 , \tag{3.1}$$

respectively. Consequently, the probability of the outcome $*$ in the first trial is equal to zero.

3) Let m be a positive integer, $m < n$. Suppose that in the first m trials the outcomes $0, 1, 2$ have happened i_0, i_1, i_2 times, respectively, with $i_0 + i_1 + i_2 = m$ (so, there is no outcome $*$ in the first m trials). Then the probabilities of outcomes $0, 1, 2$ in the $(m+1)^{th}$ trial are equal

$$1 - q^{i_0} U , \quad q^{i_0} u_1 , \quad q^{i_0+i_1} u_2 , \tag{3.2}$$

respectively. Consequently, the outcome $*$ happens in the $(m+1)^{th}$ trial with the probability

$$1 - q^{i_0} u_1 - q^{i_0+i_1} u_2 - (1 - q^{i_0} u_1 - q^{i_0} u_2) = q^{i_0} u_2 (1 - q^{i_1}) . \tag{3.3}$$

4) If the outcome $*$ happens in the k^{th} trial, then $*$ will happen in the $(k+1)^{th}$ trial with the probability 1.

We construct a probability space corresponding to the random process described above. The sample space Ω_n consists of all sequences $\omega = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of the length n, such that its elements ε_j are equal to $0, 1, 2, *$, and the following condition is valid: if $\varepsilon_k = *$ for some k $(1 < k < n)$, then $\varepsilon_l = *$ for any $l = k+1, k+2, \ldots, n$. We define the probability $p_{u_1,u_2,U,q,n}(\omega)$ of the elementary event $\omega = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ as

$$p_{u_1,u_2,U,q,n}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)) = f_1 \cdot f_2 \cdot \ldots \cdot f_n , \tag{3.4}$$

where, according to (3.1),

$$f_1 = \begin{cases} 1-U\,, & \text{if } \ \varepsilon_1 = 0\,, \\ u_1\,, & \text{if } \ \varepsilon_1 = 1\,, \\ u_2\,, & \text{if } \ \varepsilon_1 = 2\,, \\ 0\,, & \text{if } \ \varepsilon_1 = *\,, \end{cases} \tag{3.5}$$

and according to (3.2) and (3.3), for any m, $1 \le m \le n-1$,

$$f_{m+1} = \begin{cases} 1-q^{i_0}U\,, & \text{if } \ \varepsilon_{m+1} = 0\,, \\ q^{i_0}u_1\,, & \text{if } \ \varepsilon_{m+1} = 1\,, \\ q^{i_0+i_1}u_2\,, & \text{if } \ \varepsilon_{m+1} = 2\,, \\ q^{i_0}u_2(1-q^{i_1})\,, & \text{if } \ \varepsilon_{m+1} = *\,, \end{cases} \tag{3.6}$$

if $i_0 + i_1 + i_2 = m$ and $\#\{j : 1 \le j \le m, \varepsilon_j = s\} = i_s\,, \quad s = 0\,,1\,,2\,.$ We define also

$$\varepsilon_{m+1} = * \quad \text{and} \quad f_{m+1} = 1, \qquad \text{if } \varepsilon_m = *\,. \tag{3.7}$$

We define the probability of an event $A \subset \Omega_n$ as follows

$$P_{u_1,u_2,U,q,n}(A) := \sum_{\omega \in A} p_{u_1,u_2,U,q,n}(\omega)\,. \tag{3.8}$$

Although $U = u_1 + u_2$, it is useful to write U as a subscript of the probability sign P. Sometimes we write P instead of $P_{u_1,u_2,U,q,n}$. As in Section 2, we see that $P_{u_1,u_2,U,q,m}(A) = P_{u_1,u_2,U,q,n}(A)$, if $m < n$ and the event $A \subset \Omega_n$ does not depend on outcomes of the trial $\mathcal{E}$ with the numbers $m+1, m+2, \ldots, n$.

Let us introduce notation for some events of the sample space Ω_n. For $i \in \{0, 1, 2, *\}$ and integer m, $1 \le m \le n$, and k, $0 \le k \le n$, we use definitions (2.4), (2.5) from Sect. 2. For $m \in \mathbf{N}$, $j \in \{0, 1, 2\}$, $i_0, i_1, i_2 \in \mathbf{N}_0$ such that $i_0 + i_1 + i_2 = m$, we define

$$A\binom{m}{i_0,i_1,i_2} := \mathbf{0}\binom{m}{i_0} \cap \mathbf{1}\binom{m}{i_1} \cap \mathbf{2}\binom{m}{i_2}\,.$$

The event $A\binom{m}{i_0,i_1,i_2}$ can be described as follows: outcomes $0, 1, 2$ happen i_0, i_1, i_2 times, respectively, in the repetitions of the trial $\mathcal{E}$ with the numbers $1, 2, \ldots, m$. We set $A\binom{m}{i_0,i_1,i_2} = \emptyset$ if $i_j < 0$ or $i_j > m$ for some $j \in \{0, 1, 2\}$.

We may write (3.1), (3.2), (3.3) in the introduced notation as follows ($P = P_{u_1,u_2,U,q,n}$):

$$P(\mathbf{0}^{(1)}) = 1 - u_1 - u_2\,, \quad P(\mathbf{1}^{(1)}) = u_1\,, \quad P(\mathbf{2}^{(1)}) = u_2\,, \quad P(*^{(1)}) = 0\,; \tag{3.9}$$

if $m \in \mathbf{N}$, $m < n$, $i_0 + i_1 + i_2 = m$, then

$$P\left(\mathbf{0}^{(m+1)} | \, A\binom{m}{i_0,i_1,i_2}\right) = 1 - q^{i_0}U\,,\ P\left(\mathbf{1}^{(m+1)} | \, A\binom{m}{i_0,i_1,i_2}\right) = q^{i_0}u_1\,,$$
$$P\left(\mathbf{2}^{(m+1)} | \, A\binom{m}{i_0,i_1,i_2}\right) = q^{i_0+i_1}u_2\,,\ P\left(*^{(m+1)} | \, A\binom{m}{i_0,i_1,i_2}\right) = q^{i_0}(1-q^{i_1})u_2\,. \tag{3.10}$$

Formula (3.7) means that $P(\mathbf{j}^{(l+1)}|*^{(l)}) = 0$ for $j = 0,1,2$; $P(*^{(l+1)}|*^{(l)}) = 1$ for all $l = 1,2,\ldots,n-1$.

The following theorem is an analogue of Theorem 2.1. It is a key theorem of this section.

Theorem 3.1. *Let $n \in \mathbf{N}_0$, $i_0,i_1,i_2 \in \mathbf{N}_0$, $i_0+i_1+i_2 = n$. Then*

$$P\left(A\binom{n}{i_0,i_1,i_2}\right) = u_1^{i_1}u_2^{i_2}(U;q)_{i_0}\begin{bmatrix} n \\ i_0,i_1,i_2 \end{bmatrix}_q . \tag{3.11}$$

P r o o f. It can be readily seen that the probability of every elementary event $\omega \in A\binom{n}{i_0,i_1,i_2}$ $(i_0+i_1+i_2 = n)$ equals $u_1^{i_1}u_2^{i_2}(U;q)_{i_0}q^r$, where r is an integer such that $0 \le r \le i_0i_1 + i_0i_2 + i_1i_2$. For example:

$$p((1_{(i_1)},2_{(i_2)},0_{(i_0)})) = u_1^{i_1}u_2^{i_2}(U;q)_{i_0}q^{i_1i_2},$$
$$p((2_{(i_2)},1_{(i_1)},0_{(i_0)})) = u_1^{i_1}u_2^{i_2}(U;q)_{i_0},$$
$$p((0_{(i_0)},1_{(i_1)},2_{(i_2)})) = u_1^{i_1}u_2^{i_2}(U;q)_{i_0}q^{i_0u_1+(i_0+i_1)i_2}.$$

Therefore,

$$P\left(A\binom{n}{i_0,i_1,i_2}\right) = u_1^{i_1}u_2^{i_2}(U;q)_{i_0}\, c\binom{n}{i_0,i_1,i_2}, \tag{3.12}$$

where $c\binom{n}{i_0,i_1,i_2} = c(n,i_0,i_1,i_2;q)$ is a polynomial in q. We set $c\binom{0}{0,0,0} = 1$ and $c\binom{n}{i_0,i_1,i_2} = 0$ if one of the numbers i_0,i_1,i_2 is negative.

We need to prove that $c\binom{n}{i_0,i_1,i_2} = \begin{bmatrix} n \\ i_0,i_1,i_2 \end{bmatrix}_q$, where $c\binom{n}{i_0,i_1,i_2}$ is a polynomial in q defined by (3.12). First we prove a recurrence relation for coefficients $c\binom{n}{i_0,i_1,i_2}$. We show that if $n \ge 1$, $i_0,i_1,i_2 \ge 0$, $i_0+i_1+i_2 = n$, then

$$c\binom{n}{i_0,i_1,i_2} = c\binom{n-1}{i_0-1,i_1,i_2} + q^{i_0}\, c\binom{n-1}{i_0,i_1-1,i_2} + q^{i_0+i_1}\, c\binom{n-1}{i_0,i_1,i_2-1}. \tag{3.13}$$

Since the events $A\binom{n-1}{j_0,j_1,j_2}$ $(j_0,j_1,j_2 \ge 0,\ j_0+j_1+j_2 = n-1)$ and $B_{n-1} := \{(\varepsilon_1,\ldots,\varepsilon_n) \in \Omega_n : \varepsilon_k = *\ (\exists k, 1 \le k \le n-1)\}$ form a partition of the sample space Ω_n and since $P\left(A\binom{n}{i_0,i_1,i_2}|B_{n-1}\right) = 0$, it follows by the formula of total probability that

$$P\left(A\binom{n}{i_0,i_1,i_2}\right) = (1-q^{i_0-1}U)P\left(A\binom{n-1}{i_0-1,i_1,i_2}\right)$$
$$+q^{i_0}u_1P\left(A\binom{n-1}{i_0,i_1-1,i_2}\right) + q^{i_0+i_1}u_2P\left(A\binom{n-1}{i_0,i_1,i_2-1}\right). \tag{3.14}$$

Substituting (3.12) into (3.14), using the formula $(1-q^{i_0-1}U)(U;q)_{i_0-1} = (U;q)_{i_0}$, and dividing by $u_1u_2(U;q)_{i_0}$, we obtain (3.13).

Now we prove the result by induction on n. By definition $c\binom{0}{0,0,0} = \begin{bmatrix} 0 \\ 0,0,0 \end{bmatrix}_q$. We assume that the equality $c\binom{n-1}{j_0,j_1,j_2} = \begin{bmatrix} n-1 \\ j_0,j_1,j_2 \end{bmatrix}_q$ holds for all $j_0,j_1,j_2 \ge 0$

such that $j_0 + j_1 + j_2 = n-1$. We will show that $c\binom{n}{i_0,i_1,i_2} = \begin{bmatrix} n \\ i_0, i_1, i_2 \end{bmatrix}_q$ for all $i_0, i_1, i_2 \geq 0$ such that $i_0 + i_1 + i_2 = n$. Applying (3.13), we get:

$$\begin{aligned} c\binom{n}{i_0,i_1,i_2} &= c\binom{n-1}{i_0-1,i_1,i_2} + q^{i_0}\, c\binom{n-1}{i_0,i_1-1,i_2} + q^{i_0+i_1}\, c\binom{n-1}{i_0,i_1,i_2-1} \\ &= \frac{(q)_{n-1}}{(q)_{i_0}(q)_{i_1}(q)_{i_2}}\left((1-q^{i_0}) + q^{i_0}(1-q^{i_1}) + q^{i_0+i_1}(1-q^{i_2})\right) \\ &= \frac{(q)_{n-1}}{(q)_{i_0}(q)_{i_1}(q)_{i_2}}(1-q^n) = \frac{(q)_n}{(q)_{i_0}(q)_{i_1}(q)_{i_2}}, \end{aligned}$$

and the proof of the theorem is complete. ■

The next theorem contains an analogue of the Vandermond formula.

Theorem 3.2. *If $n, k \in \mathbf{N}_0$, $0 \leq k \leq n$, $i_0, i_1, i_2 \geq 0$, $i_0 + i_1 + i_2 = n$, then*

$$\begin{aligned} &\begin{bmatrix} n \\ i_0, i_1, i_2 \end{bmatrix}_q \\ &= \sum_{\substack{j_0,j_1,j_2 \geq 0 \\ j_0+j_1+j_2=k}} \begin{bmatrix} k \\ j_0, j_1, j_2 \end{bmatrix}_q \begin{bmatrix} n-k \\ i_0-j_0, i_1-j_1, i_2-j_2 \end{bmatrix}_q q^{j_0(i_1-j_1)+(j_0+j_1)(i_2-j_2)}. \end{aligned} \tag{3.15}$$

P r o o f. The events $A\binom{k}{j_0,j_1,j_2}$ (k is fixed, $j_0, j_1, j_2 \geq 0$, $j_0 + j_1 + j_2 = k$) and $B_k := \{(\varepsilon_1, \ldots, \varepsilon_n) \in \Omega_n : \varepsilon_l = * \ (\exists l, 1 \leq l \leq k)\}$ form a partition of Ω_n. Since $P\left(A\binom{n}{i_0,i_1,i_2} | B_k\right) = 0$, it follows by the formula of total probability, that

$$P\left(A\binom{n}{i_0,i_1,i_2}\right) = \sum_{\substack{j_0,j_1,j_2 \geq 0 \\ j_0+j_1+j_2=k}} P\left(A\binom{n}{i_0,i_1,i_2} \mid A\binom{k}{j_0,j_1,j_2}\right) P\left(A\binom{k}{j_0,j_1,j_2}\right). \tag{3.16}$$

Let us first find the conditional probabilities at the right-hand side of (3.16). If the event $A\binom{k}{j_0,j_1,j_2}$ has occurred, then the probabilities of outcomes $0, 1, 2$ in the $(k+1)^{th}$ trial $\mathcal{E}$ are equal to $1 - q^{j_0}U$, $q^{j_0}u_1$, and $q^{j_0+j_1}u_2$, respectively. In $n-k$ trials with numbers $k+1, k+2, \ldots, n$ there must be $i_0 - j_0$ outcomes 0, $i_1 - j_1$ outcomes 1, and $i_2 - j_2$ outcomes 2. Therefore:

$$\begin{aligned} &P\left(A\binom{n}{i_0,i_1,i_2} \mid A\binom{k}{j_0,j_1,j_2}\right) \\ &= P_{q^{j_0}u_1, q^{j_0+j_1}u_2, q^{j_0}U, q, n-k}\left(A\binom{n-k}{i_0-j_0, i_1-j_1, i_2-j_2}\right) \\ &= \left(q^{j_0}u_1\right)^{i_1-j_1}\left(q^{j_0+j_1}u_2\right)^{i_2-j_2}(q^{j_0}U, q)_{i_0-j_0} \begin{bmatrix} n-k \\ i_0-j_0, i_1-j_1, i_2-j_2 \end{bmatrix}_q. \end{aligned} \tag{3.17}$$

Substituting (3.11) and (3.17) into (3.16), we get

$$(U;q)_{i_0} u_1^{i_1} u_2^{i_2} \begin{bmatrix} n \\ i_0, i_1, i_2 \end{bmatrix}_q = \sum_{(j_0,j_1,j_2)} \left(q^{j_0} u_1\right)^{i_1-j_1} \left(q^{j_0+j_1} u_2\right)^{i_2-j_2} \times$$

$$\times (q^{j_0}U, q))_{i_0-j_0} \begin{bmatrix} n-k \\ i_0-j_0, i_1-j_1, i_2-j_2 \end{bmatrix}_q (U;q)_{j_0} u_1^{j_1} u_2^{j_2} \begin{bmatrix} k \\ j_0, j_1, j_2 \end{bmatrix}_q .$$

Using (1.2) and dividing by $(U;q)_{i_0} u_1^{i_1} u_2^{i_2}$, we obtain (3.15). ∎

The following theorem contains results that give analogues of Theorems 2.2 and 2.3.

Theorem 3.3. *For all i_0, $i_1, i_2 \in \mathbf{N}_0$ the following identities are valid:*

$$\begin{bmatrix} i_0+i_1+i_2 \\ i_0, i_1, i_2 \end{bmatrix}_q = \sum_{k=0}^{i_0} \begin{bmatrix} k+i_1+i_2 \\ k, i_1, i_2 \end{bmatrix}_q \left(1 - \frac{1-q^k}{1-q^{k+i_1+i_2}}\right), \tag{3.18}$$

$$\begin{bmatrix} i_0+i_1+i_2 \\ i_0, i_1, i_2 \end{bmatrix}_q = \left(1-q^{i_1+i_2}\right) \sum_{k=0}^{i_0} \begin{bmatrix} k+i_1+i_2 \\ k, i_1, i_2 \end{bmatrix}_q \frac{q^{(i_1+i_2)(i_0-k)}}{1-q^{k+i_1+i_2}}, \tag{3.19}$$

$$\begin{bmatrix} i_0+i_1+i_2 \\ i_0, i_1, i_2 \end{bmatrix}_q$$

$$= \left(1-q^{i_1+i_2-1}\right) \sum_{k=0}^{i_0} \begin{bmatrix} k+i_1+i_2 \\ k, i_1, i_2 \end{bmatrix}_q \frac{q^k \left(1-q^{(i_1+i_2)(i_0-k+1)}\right)}{\left(1-q^{k+i_1+i_2-1}\right)\left(1-q^{k+i_1+i_2}\right)} \tag{3.20}$$

R e m a r k 3.1. Identities (2.10), (2.19), (2.20) follow from (3.18), (3.19), (3.20), respectively, if we take $i_2 = 0$.

P r o o f. 1) For $r = 1, 2$ and $j = i_1+i_2, i_1+i_2+1, \ldots, i_0+i_1+i_2$ we consider the events

$$C_j^r := A\binom{i_0+i_1+i_2}{i_0,i_1,i_2} \cap \{(\varepsilon_1, \ldots, \varepsilon_{i_0+i_1+i_2}) : \varepsilon_j = r, \varepsilon_{j+1} = \ldots = \varepsilon_{i_0+i_1+i_2} = 0\}.$$

(For example, the event C_j^1 is formed by elements $\omega = (\omega', 1, 0_{(i_0+i_1+i_2-j)})$ where $\omega' \in A\binom{j-1}{j-i_1-i_2, i_1-1, i_2}$.) By the additive property of probability, we have

$$P\left(A\binom{i_0+i_1+i_2}{i_0,i_1,i_2}\right) = \sum_{j=i_1+i_2}^{i_0+i_1+i_2} P\left(C_j^1\right) + \sum_{j=i_1+i_2}^{i_0+i_1+i_2} P\left(C_j^2\right). \tag{3.21}$$

It is readily seen that

$$\begin{aligned} P(C_j^1) &= (U;q)_{i_0} u_1^{i_1} u_2^{i_2} q^{j-i_1-i_2} \begin{bmatrix} j-1 \\ j-i_1-i_2, i_1-1, i_2 \end{bmatrix}_q , \\ P(C_j^2) &= (U;q)_{i_0} u_1^{i_1} u_2^{i_2} q^{j-i_2} \begin{bmatrix} j-1 \\ j-i_1-i_2, i_1, i_2-1 \end{bmatrix}_q . \end{aligned} \tag{3.22}$$

Substituting (3.11) and (3.22) into (3.21) and dividing by $(U;q)_{i_0} u_1^{i_1} u_2^{i_2}$, we get

$$\begin{aligned} \begin{bmatrix} i_0+i_1+i_2 \\ i_0, i_1, i_2 \end{bmatrix}_q &= \sum_{j=i_1+i_2}^{i_0+i_1+i_2} \begin{bmatrix} j-1 \\ j-i_1-i_2, i_1-1, i_2 \end{bmatrix}_q q^{j-i_1-i_2} \\ &+ \sum_{j=i_1+i_2}^{i_0+i_1+i_2} \begin{bmatrix} j-1 \\ j-i_1-i_2, i_1, i_2-1 \end{bmatrix}_q q^{j-i_2} . \end{aligned} \tag{3.23}$$

Replacing j in (3.23) by $k = j - i_1 - i_2$ and using the identities

$$\begin{aligned} \begin{bmatrix} a+b+c-1 \\ a, b-1, c \end{bmatrix}_q &= \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \frac{1-q^b}{1-q^{a+b+c}} , \\ \begin{bmatrix} a+b+c-1 \\ a, b, c-1 \end{bmatrix}_q &= \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \frac{1-q^c}{1-q^{a+b+c}} , \end{aligned} \tag{3.24}$$

we see that the righthand side of (3.23) is equal to

$$\begin{aligned} &\sum_{k=0}^{i_0} \begin{bmatrix} k+i_1+i_2-1 \\ k, i_1-1, i_2 \end{bmatrix}_q q^k + \sum_{k=0}^{i_0} \begin{bmatrix} k+i_1+i_2-1 \\ k, i_1, i_2-1 \end{bmatrix}_q q^{k+i_1} \\ = &\sum_{k=0}^{i_0} \begin{bmatrix} k+i_1+i_2-1 \\ k, i_1, i_2 \end{bmatrix}_q \frac{1}{1-q^{k+i_1+i_2}} \Big((1-q^{i_1}) q^k + (1-q^{i_2}) q^{k+i_1} \Big) . \end{aligned}$$

This gives (3.18).

2) The proof of(3.19) is similar to that of (3.18). We only need take the events

$$D_j^r := A\binom{i_0+i_1+i_2}{i_0, i_1, i_2} \cap \{(\varepsilon_1, \ldots, \varepsilon_{i_0+i_1+i_2}) : \varepsilon_1 = \ldots = \varepsilon_j = 0, \varepsilon_{j+1} = r\}$$

for $r = 1, 2$ and $j = 0, 1, \ldots, i_0$ instead of C_j^r.

3) The proof of (3.20) is similar to those of (3.18) and (3.19). For $j, k \geq 0$, $j + k \leq i_0$, $a, b = 1, 2$ we consider the events

$$\begin{aligned} E_{jk}^{ab} = A\binom{i_0+i_1+i_2}{i_0, i_1, i_2} \cap \{ \ &(\varepsilon_1, \ldots, \varepsilon_{i_0+i_1+i_2}) \in \Omega_{i_0+i_1+i_2} : \varepsilon_1 = \ldots = \varepsilon_j = 0, \varepsilon_{j+1} = a, \\ &\varepsilon_{i_0+i_1+i_2-k} = b, \varepsilon_{i_0+i_1+i_2-k+1} = \ldots = \varepsilon_{i_0+i_1+i_2} = 0\} . \end{aligned}$$

instead of C_j^r. ■

R e m a r k 3.2. It is clear that the arguments of Section 3 are perfectly general. In order to generalize our arguments to the case of q-polynomial coefficients $\begin{bmatrix} i_0+\ldots+i_k \\ i_0,\ldots,i_k \end{bmatrix}_q$, $k \geq 3$, we can take a trial $\mathcal{E}$ with $k+2$ outcomes $0,1,2,\ldots,k,*$. Analogously to (3.1), we assume that the probabilities of the outcomes $0,1,2,\ldots,k$ in the first trial are equal to

$$1-U\,,\ u_1\,,\ u_2\,,\ \ldots\,,\ u_k\,,$$

respectively, where $u_i > 0$ $(i = 1,2,\ldots,k)$, $U := u_1+u_2+\ldots+u_k < 1$. If m is a positive integer, $i_0, i_1,\ldots,i_k \geq 0$, $i_0+i_1+\ldots+i_k = m$, and if we know that in the first m trials the outcomes $0,1,2,\ldots,k$ have happened $i_0, i_1, i_2,\ldots,i_k$ times, respectively, we assume that the probabilities of the outcomes $0,1,2,\ldots,k$ in the $(m+1)^{th}$ trial are equal to

$$1-q^{i_0}U\,,\ q^{i_0}u_1\,,\ q^{i_0+i_1}u_2\,,\ \ldots\,,\ q^{i_0+i_1+\ldots+i_{k-1}}u_k\,,$$

respectively.

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