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Coarse equidistribution of the argument of entire functions of finite order

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Given a sector $S \subset \mathbb{C}$ and an entire function f of order ρ, we estimate from below the relative area of the preimage $f^{-1}S$. We show that there exist arbitrarily large r such that for any sector S of opening α, the relative area of the set $\{f^{-1}S\} \cap r\mathbb{D}$ is bounded from below by $\alpha \cdot \kappa(\rho)$, where $\kappa(\rho) > 0$ depends only on ρ, and $\kappa(\rho) \sim \mathrm{const}\, \rho^{-1}$ for $\rho \to \infty$.

To Paul Koosis and Iossif Ostrovskii on the occasion of their birthdays

1. Main results

There is an impressive amount of classical results on the asymptotic behaviour of the absolute value of entire functions (Phragmén–Lindelöf-type theorems, minimum modulus theorems, and their numerous ramifications). All of them are based on subharmonicity of the function $\log |f|$. At the same time, very little is known about the argument of entire functions. Here we present several results (motivated by [5, Theorem 2.2]) that show somewhat surprising equidistribution patterns in the asymptotic behaviour of the argument.

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Given a sector $S = \{w\colon 0 < |w| < \infty,\ \theta_1 < \arg w < \theta_2\}$ of opening $\alpha = \theta_2 - \theta_1$ and an entire function f (everywhere below it will be assumed nonconstant), consider the relative area of the preimage $f^{-1}S$:

$$A(r, S, f) = \frac{\text{Area}(f^{-1}S \cap r\mathbb{D})}{\text{Area}(r\mathbb{D})}\,,$$

where $r\mathbb{D} = \{z\colon |z| < r\}$.

Theorem 1.1. *Let f be an entire function of finite positive order ρ. Then there exist arbitrarily large r such that for every sector S of opening α, we have*

$$A(r, S, f) \geq \frac{\alpha \cdot \kappa(\rho)}{2\pi}\,,$$

where $\kappa(\rho) > 0$ depends only on ρ, and $\kappa(\rho) \sim const\,\rho^{-1}$ for $\rho \to \infty$.

Plausibly, the statement of Theorem 1.1 can be complemented by $\kappa(\rho) \to 1$ as $\rho \to 0$. For entire functions of order zero the equiditstribution pattern is more visible since they behave like monomials z^n on a sequence of wide annuli.

Theorem 1.2. *Let f be an entire function of order zero. Then, given $\varepsilon > 0$, there exist arbitrary large values r such that for every sector S of opening α,*

$$\frac{\alpha - \varepsilon}{2\pi} \leq A(r, S, f) \leq \frac{\alpha + \varepsilon}{2\pi}\,. \tag{1.1}$$

The upper bound in (1.1) follows from the lower bound applied to the complementary sector $\mathbb{C} \setminus S$.

Corollary 1.4. *Suppose f is an entire function of order zero. Then for any sector S of opening α,*

$$\liminf_{r\to\infty} A(r, S, f) \leq \frac{\alpha}{2\pi} \leq \limsup_{r\to\infty} A(r, S, f)\,.$$

Note that for functions of positive order the lower limit

$$\liminf_{r\to\infty} A(r, S, f)$$

is not obliged to be small when α is small. If the order $\rho > 1/2$, consider an entire function f that tends to 1 as $z \to \infty$ uniformly within some angle, and take $S = \{w\colon -\alpha/2 < \arg(w) < \alpha/2\}$. If the order $\rho \leq 1/2$, the following example was suggested by Alexander Fryntov: take

$$f(z) = 1 + \prod_{k=1}^{\infty} \left(1 - \frac{z}{T^k}\right)^{[T^{k\rho}]}\,,$$

where T is a sufficiently large positive number.

The next result gives a nonasymptotic version of the coarse equidistribution principle. Define the *doubling exponent* $\beta(D, f)$ of an analytic function f on the disc D as

$$\beta(D, f) = \log \frac{\sup_D |f|}{\max_{\frac{1}{2}D} |f|}\,.$$

It measures a certain complexity of the function f, cf [6, 7]. For example, if f is a polynomial of degree d, then it is not difficult to see that $\beta(D, f) \le Cd$ for any disc D in $\mathbb{C}$.

Theorem 1.5. *Let f be a nonzero analytic function in the unit disc $\mathbb{D}$, $f(0) = 0$. Then for any sector S of opening α (with vertex at the origin)*

$$A(r, S, f) \ge \frac{c\alpha}{\log \beta^*(\mathbb{D}, f)}\,,$$

where c is a positive numerical constant, and $\beta^* = \max(\beta, 2)$.

A special case of Theorem 1.5 with $S = \{w\colon \mathrm{Re}\, w > 0\}$ appeared in our recent work with L. Polterovich [5, Theorem 2.2]. It was preceded by a qualitative compactness lemma proved by N. Nadirashvili in [4].

Concluding this introduction, we mention a curious resemblance between Theorem 1.5 and a result of D. Marshall and W. Smith [3] that says that for any univalent analytic function f in $\mathbb{D}$ and any sector S of opening α

$$\iint\limits_{f^{-1}S} |f|\, d\,\mathrm{Area} \ge \kappa(\alpha) \iint\limits_{\mathbb{D}} |f|\, d\,\mathrm{Area}\,, \tag{1.2}$$

where κ depends only on α. It remains an open question whether (1.2) persists for arbitrary analytic functions f in $\mathbb{D}$ vanishing at the origin.

Organization of the paper. In Section 2, we introduce a characteristic $\Omega(r, f)$ which measures oscillation of $\arg f$ on concentric circles and increases with r. Then we formulate our Main Lemma and prove Theorems 1.1 and 1.5. In Section 3, we prove the Main Lemma. In Section 4, we prove Theorem 1.2. This part is independent from the previous sections.

Convention. Notation $A \lesssim B$ means that $A \le cB$ where c is a positive numerical constant.

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2. Oscillation of argument and the Main Lemma

Suppose that the function f is analytic in the disc $R\mathbb{D}$ and does not vanish on the circle $r\mathbb{T}$, $0 < r < R$. Consider all arcs $L \subset r\mathbb{T}$ travelled counter-clockwise (including the entire circumference $r\mathbb{T}$ viewed as an arc whose end and beginning coincide). Put

$$\omega(r,f) := \max_{L\subset r\mathbb{T}} \Delta_L \arg f,$$

where $\Delta_L \arg f$ is the increment of the argument of f over L. The function $r \mapsto \omega(r,f)$ is not necessarily monotone. To fix this drawback, we slightly modify the definition and define a monotone function $\Omega(r,f)$ that is close to $\omega(r,f)$. By $\mathcal{Z}_f$ we denote the zero set of f (the zeroes are counted with their multiplicities). Let $n(r,f) = \#\mathcal{Z}_f \cap r\overline{\mathbb{D}}$, $|\mathcal{Z}_f| = \{|\zeta| \colon \zeta \in \mathcal{Z}_f\}$.

Given $r \in (0,R) \setminus |\mathcal{Z}_f|$, consider the factorization

$$f(z) = e^{g_r(z)} \prod_{\zeta\in\mathcal{Z}_f\cap r\mathbb{D}} (z-\zeta)$$

and take

$$\Omega(r,f) = 2\pi n(r,f) + \operatorname*{osc}_{r\mathbb{T}}(\mathrm{Im} g_r)\,,$$

where

$$\operatorname*{osc}_{r\mathbb{T}}(h) = \max_{r\mathbb{T}} h - \min_{r\mathbb{T}} h$$

is the oscillation of the function h on the circle $r\mathbb{T}$. Below we present several properties of the characteristic $\Omega(r,f)$.

1. *The function $\Omega(r,f)$ increases with r.* Indeed, if $r_1 < r_2$ and there are no zeroes of f in the annulus $\{r_1 \le |z| \le r_2\}$, then $\mathrm{Im} g_{r_1}$ and $\mathrm{Im} g_{r_1}$ are traces of the same harmonic function on different circles, and by the maximum principle, the oscillation of any harmonic function on the circle increases when the radius increases. Now, let us see what happens when r runs through r_0 such that f vanishes on $r_0\mathbb{T}$. If $\zeta_0 \in r_0\mathbb{T}$ is a zero of f of multiplicity m, then we need to add m to $n(r,f)$, and subtract $m \arg(z-\zeta_0)$ from $\mathrm{Im} g_r$. Since

$$\lim_{\varepsilon\to 0} \operatorname*{osc}_{(r_0-\varepsilon)\mathbb{T}} \arg(z-\zeta_0) = \pi\,,$$

we see that in this case

$$\operatorname*{osc}_{(r_0+\varepsilon)\mathbb{T}}(\mathrm{Im}\, g_{r_0+\varepsilon}) - \operatorname*{osc}_{(r_0-\varepsilon)\mathbb{T}}(\mathrm{Im}\, g_{r_0-\varepsilon}) \ge -m\pi\,.$$

Thus $\Omega(r_0+0,f) > \Omega(r_0-0,f)$.

2. *For all* $r \in (0,R) \setminus |\mathcal{Z}_f|$,

$$\frac{1}{2}\Omega(r,f) \le \omega(r,f) \le \Omega(r,f)\,.$$

The upper bound follows by definitions of ω and Ω. The lower bound is also easy: if $2\pi n(r,f) \ge \frac{1}{2}\Omega(r,f)$, then

$$\omega(r,f) \ge 2\pi n(r,f) \ge \frac{1}{2}\Omega(r,f)\,.$$

If $2\pi n(r,f) < \frac{1}{2}\Omega(r,f)$, then the oscillation of $\mathrm{Im} g_r$ on $r\mathbb{T}$ is larger than $\frac{1}{2}\Omega(r,f)$. Consider the arc on $r\mathbb{T}$ that runs counterclockwise from the minimum to the maximum of $\mathrm{Im} g_r$. The increment of $\arg f$ on this arc cannot be smaller than the oscillation of $\mathrm{Im} g_r$; i.e., than $\frac{1}{2}\Omega(r,f)$.

3. *Let*

$$\beta(r,f) = \log M(r,f) - \log M(\tfrac{1}{2}r,f)\,.$$

Then

$$\Omega(\tfrac{1}{2}r,f) \lesssim \beta^*(r,f) \qquad \textit{and} \qquad \beta(\tfrac{1}{2}r,f) \lesssim \Omega^*(r,f)\,,$$

where $a^* := \max(a,2)$.

These inequalities go back to A. Gelfond [1] and S. Hellerstein–J. Korevaar [2]. We shall use only the first bound whose proof can be found in [5]. This bound immediately yields the following property

4. *Suppose* f *is an entire function. Then*

$$\Omega(r,f) \lesssim \log M(2r,f)\,, \qquad r \ge r_0(f)\,.$$

For $L \subset [0,\infty)$, we set $K_L = \{z\colon |z| \in L\}$. The following lemma plays a central role:

Lemma 2.1. *Suppose that the analytic function* f *on* $\overline{\mathbb{D}}$ *and* $t \in (0,1)$ *are such that*

$$\inf_{[t,1]} \omega(r,f) \ge 2\pi\,,$$

and

$$\Omega(t,f) \ge \frac{1}{2}\Omega(1,f)\,.$$

Then

$$\mathrm{Area}(f^{-1}S \cap K_{[t,1]}) \gtrsim \alpha(1-t)^2\,.$$

Now, we deduce Theorems 1.1 and 1.5 from this lemma. The lemma will be proven in the next section.

P r o o f o f T h e o r e m 1.1. Since the function f has order ρ, we can find arbitrarily large r such that

$$\Omega(2r,f) \le 2^{2\rho}\Omega(r,f)$$

and

$$\omega(t,f) > 2\pi$$

for $t \ge r$. Assume that $\rho \ge 2$ and split the interval $[r,2r]$ into $[5\rho]$ equal parts by points $r_0 = r$, $r_1, \dots, r_{[5\rho]} = 2r$. Note that the inequality

$$\Omega(r_{j+1},f) \le 2^4\Omega(r_j,f)$$

holds for at least half of the indices $j = 1,\dots,[5\rho]$. For these j, by Lemma 2.1, the relative area of the set $f^{-1}S \cap K_{[r_j,r_{j+1}]}$ with respect to $K_{[r_j,r_{j+1}]}$ is $\gtrsim \alpha\rho^{-1}$. Hence,

$$A(r,S,f) \gtrsim \frac{\alpha}{\rho},$$

and we are done. ■

P r o o f o f T h e o r e m 1.5. Choose $k \in \mathbb{N}$ such that

$$2^k\pi \le \Omega(r,f) < 2^{k+1}\pi$$

(recall that $f(0) = 0$, thus $\Omega(1,f) \ge 2\pi$). Choose $0 = r_0 < r_1 < \cdots < r_k \le 1$ so that $\Omega(r_j - 0,f) \le 2^j\pi \le \Omega(r_j + 0,f)$, and set $r_{k+1} = 1$. Applying (properly scaled) Lemma 2.1 to the annuli $K_j = K_{[r_j,r_{j+1}]}$, we get

$$\mathrm{Area}(f^{-1}S \cap K_j) \gtrsim \alpha(r_{j+1}-r_j)^2 .$$

By Cauchy's inequality,

$$\sum_{j=0}^{k}(r_{j+1}-r_j)^2 \ge \frac{1}{k+1} .$$

Therefore,

$$\mathrm{Area}(f^{-1}S \cap \mathbb{D}) \ge \sum_{j=1}^{k}\mathrm{Area}(f^{-1}S \cap K_j) \gtrsim \frac{\alpha}{k},$$

completing the proof. ■

3. Proof of the Main Lemma

Let, as above, $S = \{w\colon 0 < |w| < \infty, \theta_1 < \arg w < \theta_2\}$, $\theta_2 - \theta_1 = \alpha$. Fix $r \in [t, 1] \setminus |\mathcal{Z}_f|$, and introduce two types of 'traversing arcs' on $r\mathbb{T}$: T-arcs and S-arcs. An open arc $J \subset r\mathbb{T}$ is called a T-arc, if a continuous branch of $\arg f$ maps J onto an interval $(\theta_1 + 2\pi m, \theta_1 + 2\pi(m+1))$ for some $m \in \mathbb{Z}$. Each T-arc J contains a traversing S-arc I which is mapped by the same branch of $\arg f$ onto $(\theta_1 + 2\pi m, \theta_1 + \alpha + 2\pi m)$. For each $r \in [t, 1] \setminus |\mathcal{Z}_f|$, there are at least

$$M = \left[\frac{1}{2\pi} \inf_{[t,1]\setminus|\mathcal{Z}_f|} \omega(r, f)\right] \geq 1$$

disjoint traversing T-arcs. We choose M of them and discard the rest.

Let $t_1 = \frac{1}{2}(1+t)$, $K = K_{[t,t_1]}$, and let E be the union of all S-arcs in K. We need to estimate from below the area of E. Start with the argument used in [5]. For each S-arc $I \subset r\mathbb{T}$,

$$\int_I |\nabla \arg f|\, |dz| \geq \alpha\,.$$

Therefore,

$$\int_{E\cap r\mathbb{T}} |\nabla \arg f|\, |dz| \geq \alpha M\,.$$

Integrating by $r \in [t, t_1]$, we get

$$\iint_E |\nabla \arg f|\, d\,\mathrm{Area} \geq \frac{\alpha(1-t)M}{2}\,. \tag{3.1}$$

Now we shall try to estimate from above the double integral on the left-hand side.

Factoring

$$f(z) = e^{g(z)} \prod_{\zeta\in\mathcal{Z}_f} (z-\zeta)\,,$$

we get

$$|\nabla \arg f| \leq |\nabla \mathrm{Im}\, g| + \sum_{\zeta\in\mathcal{Z}_f} \frac{1}{|z-\zeta|}\,,$$

and

$$\iint_E |\nabla \arg f|\, d\,\mathrm{Area} \leq \max_{t_1\overline{\mathbb{D}}} |\nabla \mathrm{Im}\, g| \cdot \mathrm{Area}(E) + \#\mathcal{Z}_f \cdot \sup_{\zeta\in\mathcal{Z}_f} \iint_E \frac{d\,\mathrm{Area}(z)}{|z-\zeta|}\,.$$

Estimate the terms on the right-hand side. We have $\Omega(1, f) \leq 16\pi M$,

$$\#\mathcal{Z}_f \leq \frac{1}{2\pi}\Omega(1, f) \leq 8M\,,$$

and

$$\operatorname*{osc}_{\mathbb{T}}(\operatorname{Im} g) \le \Omega(1, f) \le 16\pi M\,.$$

Since the function $\operatorname{Im} g$ is harmonic in $\mathbb{D}$, we obtain

$$\max_{t_1\overline{\mathbb{D}}} |\nabla \operatorname{Im} g| \le \frac{1}{1-t_1} \cdot \operatorname*{osc}_{\mathbb{T}}(\operatorname{Im} g) \le \frac{32\pi M}{1-t}\,,$$

and finally

$$\iint\limits_E |\nabla \arg f|\, d\operatorname{Area} \lesssim M \left(\frac{\operatorname{Area}(E)}{1-t} + \sup_{\zeta\in\mathcal{Z}_f} \iint\limits_E \frac{d\operatorname{Area}(z)}{|z-\zeta|} \right)\,. \tag{3.2}$$

It remains to estimate the double integral on the right-hand side. Till that point we followed the strategy from [5]. A straightforward bound

$$\iint\limits_E \frac{d\operatorname{Area}(z)}{|z-\zeta|} \le 2\sqrt{\pi \operatorname{Area}(E)} \tag{3.3}$$

used in there is not sufficient anymore:* it leads only to the estimate

$$\operatorname{Area}(f^{-1}S \cap K) \gtrsim \alpha^2 (1-t)^2\,.$$

We try to get something better taking into account the structure of the set E (recall that $E \cap r\mathbb{T}$ is always a union of M disjoint S-arcs). For this purpose, we reduce the general case to the one when all S-arcs are short, the T-arcs containing them are not very short (that is, the S-arcs are 'well separated'), and the zero set $\mathcal{Z}_f$ is not too close to E.

First, we sort the S-arcs. We call an S-arc I a *short* one, if

$$|I| \le \frac{\alpha\eta(1-t)}{M}\,, \tag{3.4}$$

where a small positive numerical constant η will be chosen later. Otherwise, we say that I is not short. By $M_s(r)$ we denote the number of short S-arcs on $r\mathbb{T}$. Let $E_{n.s.}$ be the union of all nonshort arcs in K. Clearly,

$$\operatorname{Area}(E_{n.s.}) \ge \frac{\alpha\eta(1-t)}{M} \int\limits_t^{t_1} (M - M_s(r))\, dr\,. \tag{3.5}$$

Now consider the short S-arcs in K. In fact, we do not need all of them. Let E_s^* be the union of all S-arcs I in K satisfying the following three conditions:

* However, it will be employed below during an auxiliary step.

(a) I is short (i.e., (3.4) holds);

(b) the corresponding T-arc $J \supset I$ is not very short:

$$|J| \geq \frac{\delta(1-t)}{M},$$

where a small positive numerical constant δ will be chosen later;

(c) if $I \subset r\mathbb{T}$, then

$$\mathrm{dist}(r, |\mathcal{Z}_f|) \geq \frac{\delta(1-t)}{M}.$$

We will show that under appropriate choice of small parameters δ and η,

$$\mathrm{Area}(E_s^*) \gtrsim \frac{\alpha(1-t)}{M} \left(\int_t^{t_1} M_s(r)\, dr - \frac{(1-t)M}{5} \right). \tag{3.6}$$

Then recalling (3.5), we get the assertion of the Main Lemma. If $\int_t^{t_1} M_s(r)\, dr \geq \frac{(1-t)M}{5}$, then

$$\begin{aligned} \mathrm{Area}(f^{-1}S \cap K) &\geq \mathrm{Area}(E_{n.s.}) + \mathrm{Area}(E_s^*) \\ &\gtrsim \frac{\alpha(1-t)}{M} \left((t_1 - t)M - \frac{(1-t)M}{5} \right) \gtrsim \alpha(1-t)^2. \end{aligned}$$

If $\int_t^{t_1} M_s(r)\, dr < \frac{(1-t)M}{5}$, then we simply discard the short S-arcs:

$$\begin{aligned} \mathrm{Area}(f^{-1}S \cap K) &\geq \mathrm{Area}(E_{n.s.}) \\ &\overset{(3.5)}{\geq} \frac{\alpha\eta(1-t)}{M} \left((t_1 - t)M - \frac{(1-t)M}{5} \right) \gtrsim \alpha(1-t)^2. \end{aligned}$$

Now we start proving (3.6). Let

$$\mathcal{E} = \left\{ r \in [t, t_1] \colon \mathrm{dist}(r, |\mathcal{Z}_f|) < \tfrac{\delta(1-t)}{M} \right\}$$

be an exceptional set of radii, and let $m(r)$ be the number of *very short* T-arcs J such that

$$|J| \leq \frac{\delta(1-t)}{M}.$$

Then, as above, for $r \notin \mathcal{E}$,

$$\int_{E_s^* \cap r\mathbb{T}} |\nabla \arg f|\, |dz| \geq \alpha(M_s(r) - m(r)),$$

and

$$\begin{aligned}\iint\limits_{E_s^*} |\nabla \arg f|\, d\,\mathrm{Area} &\geq \alpha \int\limits_{[t,t_1]\setminus\mathcal{E}} (M_s(r) - m(r))\, dr \\ &\geq \alpha \left(\int\limits_t^{t_1} M_s(r)\, dr - \int\limits_t^{t_1} m(r)\, dr - M|\mathcal{E}| \right) .\end{aligned}$$

The next two claims show that the second and the third terms on the right-hand side are relatively small, provided that δ is sufficiently small.

Claim 3.7. *We have*

$$\int\limits_t^{t_1} m(r)\, dr \lesssim \delta(1-t)M .$$

P r o o f. Let G be the union of all very short T-arcs in K. Then, as above,

$$\int\limits_{G\cap r\mathbb{T}} |\nabla \arg f|\, |dz| \geq 2\pi m(r) ,$$

and

$$\iint\limits_G |\nabla \arg f|\, d\,\mathrm{Area} \geq 2\pi \int\limits_t^{t_1} m(r)\, dr .$$

On the other hand, a counterpart of (3.2) together with estimate (3.3) give us

$$\iint\limits_G |\nabla \arg f|\, d\,\mathrm{Area} \lesssim M \left(\frac{\mathrm{Area}(G)}{1-t} + \sqrt{\mathrm{Area}(G)} \right) .$$

If the second term on the right-hand side is larger than the first one, then we get

$$\mathrm{Area}(G) \gtrsim \left(\frac{1}{M} \int\limits_t^{t_1} m(r)\, dr \right)^2 . \tag{3.7}$$

If the first term is larger, then $\mathrm{Area}(G) \geq (1-t)^2$ and we again arrive at (3.7).

Since G consists of very short T-arcs, we have

$$\mathrm{Area}(G) \leq \frac{\delta(1-t)}{M} \int_t^{t_1} m(r)\, dr .$$

Hence,

$$\frac{1}{M}\int\limits_t^{t_1} m(r)\,dr \lesssim \delta(1-t)\,,$$

proving the claim. ■

Claim 3.9. *We have*

$$|\mathcal{E}| \lesssim \delta(1-t)\,.$$

P r o o f. Since $\#\mathcal{Z}_f \lesssim M$, this follows from definition of $\mathcal{E}$. ■

Using these claims, we choose δ so small that

$$\int\limits_t^{t_1} m(r)\,dr + M|\mathcal{E}| \le \frac{(1-t)M}{10}\,.$$

Then

$$\iint\limits_{E_s^*} |\nabla \arg f|\, d\,\mathrm{Area} \ge \alpha \left(\int\limits_t^{t_1} M_s(r)\,dr - \frac{(1-t)M}{10}\right)\,. \tag{3.8}$$

We are ready to make the final step: to estimate from above the integral on the left-hand side of (3.8). As above (cf. (3.2)),

$$\iint\limits_{E_s^*} |\nabla \arg f|\, d\,\mathrm{Area} \lesssim M\left(\frac{\mathrm{Area}(E_s^*)}{1-t} + \sup_{\zeta\in\mathcal{Z}_f}\iint\limits_{E_s^*}\frac{d\,\mathrm{Area}(z)}{|z-\zeta|}\right)\,. \tag{3.9}$$

The next claim bounds the double integral on the right-hand side:

Claim 3.12. *Let $F\subset K_{[t,1]}$ be a closed set, $t\ge\frac{1}{2}$. Suppose that there exists $s\in(0,\frac{1-t}{2})$ such that, for each $r\in(t,1)$, the set $F(r)=F\cap r\mathbb{T}$ is a union of disjoint arcs I of length*

$$|I|\le\beta s.$$

Further, assume that each arc I is contained in a bigger arc J, the arcs J are pairwise disjoint,

$$|J|\ge s\,,$$

and the total number of arcs is $\lesssim\frac{1-t}{s}$. Given $\xi\in[t,1]$, denote

$$F_\xi = F\setminus K_{[\xi-s,\,\xi+s]}\,.$$

Then

$$\iint\limits_{F_\xi}\frac{d\,\mathrm{Area}(z)}{|z-\xi|}\lesssim\beta(1-t)\,.$$

Note that under the assumptions of this claim, $\text{Area}(F_\xi) \lesssim \beta(1-t)^2$, and estimate (3.3) gives us only

$$\iint\limits_{F_\xi} \frac{d\,\text{Area}(z)}{|z-\xi|} \lesssim \sqrt{\beta}(1-t)\,.$$

P r o o f o f C l a i m 3.12. Fix $r \in (t,1)$, $|r-\xi| \ge s$, and consider the integral

$$\int\limits_{F(r)} \frac{|dz|}{|z-\xi|}\,.$$

First, estimate the contribution of the components I of $F(r)$ that intersect the arc $\{re^{i\theta}\colon |\theta| \le |r-\xi|\}$. For each arc I,

$$\int\limits_{I} \frac{|dz|}{|z-\xi|} \le \frac{|I|}{|r-\xi|} \le \frac{\beta s}{|r-\xi|}\,.$$

The number of such arcs I is $\lesssim \frac{|r-\xi|}{s}$. Therefore, the total contribution of these arcs is $\lesssim \beta(1-t)$.

Now consider the arcs I that do not intersect the arc $\{re^{i\theta}\colon |\theta| \le |r-\xi|\}$. It suffices to consider only the arcs I lying in the upper semi-circle. For these arcs, $|re^{i\theta}-\xi| \gtrsim \theta$. We enumerate the arcs I counter-clockwise by index j, $1 \le j \lesssim \frac{1-t}{s}$. Then the contribution of the j-th arc is

$$\int\limits_{I_j} \frac{|dz|}{|z-\xi|} \lesssim \int\limits_{|r-\xi|+(j-1)s}^{|r-\xi|+(j-1)s+\beta s} \frac{d\theta}{\theta} \le \frac{\beta}{\frac{|r-\xi|}{s}+j-1}\,.$$

Summing over j, we get the bound

$$\beta\left(\log\frac{1}{|r-\xi|} + \text{Const}\right).$$

Integrating this bound by r from t to 1, we see that the contribution of these arcs is $\lesssim \beta(1-t)$ as well. ■

At last, we are able to get estimate (3.6) for $\text{Area}(E_s^*)$, and thus to finish the proof of Lemma 2.1. Without loss of generality, we assume that $t \ge \frac{1}{2}$. We apply the claim to the integral on the right-hand side of (3.9) with

$$\beta = \frac{\alpha\eta}{\delta}\,, \qquad s = \frac{\delta(1-t)}{M}$$

(recall that the parameter δ alredy has been fixed, but η has not been chosen yet). We obtain

$$\iint_{E_s^*} |\nabla \arg f| \, d\,\text{Area} \lesssim M \left(\frac{\text{Area}(E_s^*)}{1-t} + \frac{\alpha\eta(1-t)}{\delta} \right) . \tag{3.10}$$

Juxtaposing estimates (3.8) and (3.10), we get

$$\text{Area}(E_s^*) \gtrsim \frac{\alpha(1-t)}{M} \left(\int_t^{t_1} M_s(r)\, dr - \frac{(1-t)M}{10} \right) - \frac{\alpha\eta(1-t)^2}{\delta} .$$

It remains to choose η so small that the right-hand side is

$$\gtrsim \frac{\alpha(1-t)}{M} \left(\int_t^{t_1} M_s(r)\, dr - \frac{(1-t)M}{5} \right) .$$

This gives us (3.6) and completes the proof of the Main Lemma. ■

4. Functions of order zero

Here we prove Theorem 1.2. Without loss of generality, $f(0) \neq 0$ and f is not a polynomial. Then, up to a constant factor that is irrelevant here, we have

$$f(z) = \prod_{\zeta \in \mathcal{Z}_f} \left(1 - \frac{z}{\zeta} \right) .$$

Fix an $\varepsilon > 0$. Then take a very small $\delta > 0$ to be chosen later. Choose r_δ to be the radius at which the ratio

$$\frac{n(r,f)}{r^\delta}$$

attains its maximum. Note that $r_\delta \to +\infty$ as $\delta \to 0+$. Let $M = n(r_\delta, f)$. Let U be a huge constant (depending only on ε) to be chosen later. We claim that the disk $R\mathbb{D}$ with $R = U^2 r_\delta$ satisfies the equidistribution property of the theorem if δ is small enough.

Indeed, consider the annulus $K := \{z \colon Ur_\delta < |z| < R\}$. For every $r \in (Ur_\delta, R)$ the set $f^{-1}S \cap r\mathbb{T}$ contains at least M disjoint traversing S-arcs and, thereby,

$$\int_{f^{-1}S \cap r\mathbb{T}} |\nabla \arg f(z)| \, |dz| \geq M\alpha .$$

Therefore,

$$\iint_E |z|\,|\nabla \arg f(z)|\, d\,\mathrm{Area}(z) \ge M\alpha \int_{R/U}^{R} r\,dr = M\alpha \frac{R^2}{2}(1-U^{-2})\,, \tag{4.1}$$

where $E = f^{-1}S \cap K$.

Now, we estimate the double integral on the left-hand side from above. Write $f = f_1 \cdot f_2 \cdot f_3$, where

$$f_1 = \prod_{\zeta\in\mathcal{Z}_f\,,|\zeta|\le r_\delta} \left(1-\frac{z}{\zeta}\right),$$
$$f_2 = \prod_{\zeta\in\mathcal{Z}_f\,,r_\delta<|\zeta|\le U^3 r_\delta} \left(1-\frac{z}{\zeta}\right),$$
$$f_3 = \prod_{\zeta\in\mathcal{Z}_f\,,|\zeta|> U^3 r_\delta} \left(1-\frac{z}{\zeta}\right).$$

For $z \in K$ we have

$$|\nabla \arg f_1(z)| \le \sum_{\zeta\in\mathcal{Z}_f\,,|\zeta|\le r_\delta} \frac{1}{|z-\zeta|} \le \frac{U}{U-1}\frac{M}{|z|}.$$

Also

$$\begin{aligned}
|\nabla \arg f_3(z)| \;&\le\; \sum_{\zeta\in\mathcal{Z}_f\,,|\zeta|>U^3 r_\delta} \frac{1}{|z-\zeta|} \le \frac{U}{U-1} \sum_{\zeta\in\mathcal{Z}_f\,,|\zeta|>U^3 r_\delta} \frac{1}{|\zeta|}\\
&=\; \frac{U}{U-1}\sum_{j\ge 3}\; \sum_{\zeta\in\mathcal{Z}_f, U^j r_\delta<|\zeta|\le U^{j+1} r_\delta} \frac{1}{|\zeta|}\\
&\le\; \frac{U}{U-1}\sum_{j\ge 3} \frac{1}{U^j r_\delta}(U^{(j+1)\delta}-1)M\\
&=\; \frac{U}{U-1}\frac{1}{U^2 r_\delta}\sum_{j\ge 1}\frac{1}{U^j}(U^{(j+3)\delta}-1)M \le \frac{U}{U-1}\sigma(U,\delta)\frac{M}{|z|}\,,
\end{aligned}$$

where

$$\sigma(U,\delta) = \sum_{j\ge 1}\frac{1}{U^j}(U^{(j+3)\delta}-1) \to 0 \qquad \text{as } \delta\to 0 \quad \text{for any fixed } U>1\,.$$

Therefore,

$$|\nabla \arg f_1(z)| + |\nabla \arg f_3(z)| \le (1+\gamma(U,\delta))\frac{M}{|z|}\,,$$

where $\gamma(U,\delta)$ can be made arbitrarily small if U is large enough and δ is small enough. Note that the number of zeroes in f_2 does not exceed $(U^{3\delta}-1)M$. Hence,

$$\begin{aligned}\iint\limits_E |z|\,|\nabla \arg f_2(z)|\,d\,\mathrm{Area}(z) &\leq (U^{3\delta}-1)MR\iint\limits_{|z|\leq R}\frac{d\,\mathrm{Area}(z)}{|z-\zeta|}\\ &\leq (U^{3\delta}-1)M\cdot 2\pi R^2\,.\end{aligned}$$

Thus

$$\iint\limits_E |z|\,|\nabla \arg f(z)|\,d\,\mathrm{Area}(z) \leq (1+\gamma(U,\delta))M\,\mathrm{Area}(E) + (U^{3\delta}-1)M\cdot 2\pi R^2\,. \tag{4.2}$$

The rest is clear. We choose U so large such that $U^{-2} < \frac{\varepsilon}{4}$. Then we choose δ so small that

$$\gamma(U,\delta) < \frac{\varepsilon}{4}\,, \qquad \text{and} \qquad U^{3\delta}-1 < \frac{\varepsilon}{4}\,.$$

Juxtaposing (4.1) and (4.2), cancelling M, and taking into account the choice of U and δ, we get the result. ∎

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