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A probabilistic approach to q-polynomial coefficients, Euler and Stirling numbers. II

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The aim of this paper is to indicate stochastic processes which are connected with Stirling numbers of the first and the second kind and Euler numbers in a natural way. A probabilistic approach allows us to give very simple proofs of some identities for these coefficients.

To my teacher Professor Iossif Vladimirovich Ostrovskii on the occasion of his 70-th birthday

4. Stirling numbers of the second kind

We recall that *Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}$ may be defined for $n \in \mathbf{N_0}$ and integer k as numbers which equal 1 if $n = k = 0$, and 0, if $k < 0$ or $k > n$, and satisfy the following recurrence identity (see [1, Sect. 6.1])

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} . \tag{4.1}$$

Let $\Omega_n := \{\omega = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) : \varepsilon_j = 0 \text{ or } 1, j = 1, 2, \dots, n\}$ be a set of all sequences of the length n with elements 0 and 1, $\beta = \{\beta_j\}_{j=0}^{\infty}$ be a sequence of positive numbers. We define a weight $w_{n,\beta}$ on Ω_n inductively. For $n = 1$, we set

$$w_{1,\beta}((1)) = \beta_0 , \qquad w_{1,\beta}((0)) = 1 . \tag{4.2}$$

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This paper is a continuation of the paper [2]. The terminology and all meanings of the paper [2] are kept here.

Let $m > 1$. We define the weight of a chain of the length m to be

$$w_{m,\beta}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}, 1)) = w_{m-1,\beta}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})) \cdot \beta_j \,, \tag{4.3}$$

where $j = \#\{l : 1 \le l \le m-1, \varepsilon_l = 0\}$, and

$$w_{m,\beta}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}, 0)) = w_{m-1,\beta}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})) \,. \tag{4.4}$$

In other words, the weight of a chain $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}, \varepsilon_m)$ of the length m equals the product of the weight of the chain $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})$ and that of the element ε_m, which is equal to 1 if $\varepsilon_m = 0$ and to β_j if $\varepsilon_m = 1$ and $\#\{k : 1 \le k \le m-1, \varepsilon_k = 0\} = j$.

For every set $A \subset \Omega_n$, we define the weight $W_{n,\beta}(A)$ of A to be

$$W_{n,\beta}(A) := \sum_{\omega \in A} w_{n,\beta}(\omega) \,. \tag{4.5}$$

It is evident from (4.5) that the additive property of the weight is valid:

$$W_{n,\beta}(A \cup B) = W_{n,\beta}(A) + W_{n,\beta}(B) \,, \quad \text{if} \quad A \cap B = \emptyset \,. \tag{4.6}$$

For the sake of brevity we write often w_n and W_n instead of $w_{n,\beta}$ and $W_{n,\beta}$. For $n \ge 1$ and $0 \le k \le n$ we denote

$$\xi_{nk} := \xi_{nk}(\beta) := W_n(\mathbf{0}\tbinom{n}{k}) \,, \tag{4.7}$$

where $\mathbf{0}\binom{n}{k} = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \Omega_n : \#\{l : 1 \le l \le n, \varepsilon_l = 0\} = k\}$. We denote also $\xi_{00}(\beta) = 1$, $\xi_{nk}(\beta) = 0$ if $k < 0$ or $k > n$. We see that ξ_{nk} is a polynomial in the variables β_i, $0 \le i \le k$, considering β_j as independent variables.

Definition 4.1. *Polynomials ξ_{nk} are said to be Stirling polynomials of the second kind generated by the sequence β.*

The following theorem gives a recurrence relation for polynomials ξ_{nk}.

Theorem 4.1. *If $n \in \mathbf{N}$ and $0 \le k \le n$, then*

$$\xi_{nk} = \xi_{n-1,k-1} + \xi_{n-1,k}\beta_k \,. \tag{4.8}$$

P r o o f. For $j = 0, 1$, we denote $A^j := \mathbf{0}\binom{n}{k} \cap \{(\varepsilon_1, \ldots, \varepsilon_n) \in \Omega_n : \varepsilon_n = j\}$. Evidently, $\mathbf{0}\binom{n}{k} = A^0 \cup A^1$ and $A^0 \cap A^1 = \emptyset$. Therefore

$$\xi_{nk} = W_n(\mathbf{0}\tbinom{n}{k}) = W_n(A^0) + W_n(A^1) \,. \tag{4.9}$$

We evaluate $W_n(A^0)$. Obviously, $\omega = (\varepsilon_1, \dots, \varepsilon_{n-1}, 0) \in A^0$ if and only if $\omega' = (\varepsilon_1, \dots, \varepsilon_{n-1}) \in \mathbf{0}\binom{n-1}{k-1}$. In this case $w_{n,\beta}(\omega) = w_{n-1,\beta}(\omega') \cdot 1$ by (4.4). It follows from this that

$$W_n(A^0) = \sum_{\omega \in A^0} w_n(\omega) = \sum_{\omega' \in \mathbf{0}\binom{n-1}{k-1}} w_{n-1}(\omega') = W_{n-1}(\mathbf{0}\tbinom{n-1}{k-1}) = \xi_{n-1,k-1}\,. \quad (4.10)$$

Analogously, $\omega = (\varepsilon_1, \dots, \varepsilon_{n-1}, 1) \in A^1$ if and only if $\omega' = (\varepsilon_1, \dots, \varepsilon_{n-1}) \in \mathbf{0}\binom{n-1}{k}$. In this case $w_{n,\beta}(\omega) = w_{n-1,\beta}(\omega') \cdot \beta_k$ by (4.3). Therefore

$$W_n(A^1) = \sum_{\omega \in A^1} w_n(\omega) = \sum_{\omega' \in \mathbf{0}\binom{n-1}{k}} w_{n-1}(\omega')\beta_k = W_{n-1}(\mathbf{0}\tbinom{n-1}{k})\beta_k = \xi_{n-1,k}\beta_k\,. \quad (4.11)$$

Inserting (4.10) and (4.11) into (4.9), we obtain (4.8). ∎

For every positive integer l and a sequence $\beta := \{\beta_j\}_{j=0}^{\infty}$, we denote $\beta^{(l)} := \{\beta_{l+j}\}_{j=0}^{\infty}$. The $W_n^{(l)}$ will denote the weight on Ω_n generated by the sequence $\beta^{(l)}$.

Definition 4.2. *Polynomials*

$$\xi_{nk}^{(l)} := \xi_{nk}(\beta^{(l)})\,, \quad (n = 1, 2, \dots\,, \quad 0 \le k \le n)\,; \quad \xi_{00}^{(l)} := 1\,,$$

in the variables $\beta_l, \beta_{l+1}, \dots$ *are said to be associated of the rank* l *with polynomials* $\xi_{nk}(\beta)$.

The following theorem gives a relation that includes ξ_{nk} and $\xi_{nk}^{(1)}$.

Theorem 4.2. *For all* $n \ge 1$ *and* $0 \le k \le n$ *the following recurrence relation holds:*

$$\xi_{nk} = \xi_{n-1,k-1}^{(1)} + \beta_0 \xi_{n-1,k}\,. \quad (4.12)$$

P r o o f. For $j = 0, 1$ we denote $B^j := \mathbf{0}\binom{n}{k} \cap \{(\varepsilon_1, \dots, \varepsilon_n) \in \Omega_n : \varepsilon_1 = j\}$. Evidently, $\mathbf{0}\binom{n}{k} = B^0 \cup B^1$ and $B^0 \cap B^1 = \emptyset$. Therefore

$$\xi_{nk} = W_n(\mathbf{0}\tbinom{n}{k}) = W_n(B^0) + W_n(B^1)\,. \quad (4.13)$$

We evaluate $W_n(B^0)$. Obviously, $\omega = (0, \varepsilon_2, \dots, \varepsilon_n) \in B^0$ if and only if $\omega' = (\varepsilon_2, \dots, \varepsilon_n) \in \mathbf{0}\binom{n-1}{k-1}$. In this case

$$w_{n,\beta}(\omega) = w_{n-1,\beta^{(1)}}(\omega')$$

by (4.2)–(4.4). (For example, if $\omega = (0,1,0,0,1,0,1)$ $(n = 7, \omega' = (1,0,0,1,0,1))$, then $w_{7,\beta}(\omega) = 1 \cdot \beta_1 \cdot 1 \cdot 1 \cdot \beta_3 \cdot 1 \cdot \beta_4$, $w_{6,\beta^{(1)}}(\omega') = \beta_{1+0} \cdot 1 \cdot 1 \cdot \beta_{1+2} \cdot 1 \cdot \beta_{1+3}$.) Therefore,

$$\begin{aligned} W_n(B^0) &= W_{n,\beta}(B^0) = \sum_{\omega \in B^0} w_{n,\beta}(\omega) = \sum_{\omega' \in \mathbf{0}\binom{n-1}{k-1}} w_{n-1,\beta^{(1)}}(\omega') \\ &= W_{n-1,\beta^{(1)}}(\mathbf{0}\tbinom{n-1}{k-1}) = W^{(1)}_{n-1}(\mathbf{0}\tbinom{n-1}{k-1}) = \xi^{(1)}_{n-1,k-1} \,. \qquad (4.14) \end{aligned}$$

Analogously, we evaluate $W_n(B^1)$. We have: $\omega = (1, \varepsilon_2, \ldots, \varepsilon_n) \in B^1$ if and only if $\omega' = (\varepsilon_2, \ldots, \varepsilon_n) \in \mathbf{0}\binom{n-1}{k}$. In this case

$$w_{n,\beta}(\omega) = \beta_0 \cdot w_{n-1,\beta}(\omega')$$

by (4.2)–(4.4). (For example, if $\omega = (1,1,0,0,1,0,1)$ $(n = 7, \omega' = (1,0,0,1,0,1))$, then $w_{7,\beta}(\omega) = \beta_0 \cdot \beta_0 \cdot 1 \cdot 1 \cdot \beta_2 \cdot 1 \cdot \beta_3$, $w_{6,\beta}(\omega') = \beta_0 \cdot 1 \cdot 1 \cdot \beta_2 \cdot 1 \cdot \beta_3$.) Therefore

$$\begin{aligned} W_n(B^1) &= W_{n,\beta}(B^1) = \sum_{\omega \in B^1} w_n(\omega) = \beta_0 \sum_{\omega' \in \mathbf{0}\binom{n-1}{k}} w_{n-1,\beta}(\omega') \\ &= \beta_0 W_{n-1\beta}(\mathbf{0}\tbinom{n-1}{k}) = \beta_0 \xi_{n-1,k} \,. \qquad (4.15) \end{aligned}$$

Inserting (4.14) and (4.15) into (4.13), we obtain (4.12). ■

Let us consider a particular case. We put

$$\beta_j := j \quad \text{for all} \quad j \geq 0 \,. \qquad (4.16)$$

(Therefore, $w_n(\omega) = 0$ for every chain $\omega = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ such that $\varepsilon_1 = 1$.) Then we get numbers $\tilde{\xi}_{nk} := \xi_{nk}(\{j\}_{j=0}^{\infty})$ satisfying the following recurrence relation (see (4.8))

$$\tilde{\xi}_{nk} = \tilde{\xi}_{n-1,k-1} + \tilde{\xi}_{n-1,k} \cdot k \qquad (4.17)$$

and conditions $\tilde{\xi}_{00} = 1$, $\tilde{\xi}_{nk} = 0$ if $k < 0$ or $k > n$. The theorem below follows directly from the definition of Stirling numbers of the second kind $\left\{ n \atop k \right\}$ (see (4.1)).

Theorem 4.3. *Let $n \in \mathbf{N}$, $0 \leq k \leq n$. Then*

$$\left\{ n \atop k \right\} = W_n(\mathbf{0}\tbinom{n}{k}) \,, \qquad (4.18)$$

where W_n denotes the weight on Ω_n generated by the sequence $\beta = \{j\}_{j=0}^{\infty}$ with the help of (4.2)–(4.5).

In the following theorem we give the proof of the known fact (see, for example, [1, formula (6.20)]), based on Theorem 4.3.

Theorem 4.4. *If* $n \in \mathbf{N}$ *and* $0 \le m \le n$, *then*

$$\left\{ {n \atop m} \right\} = \sum_{l=m}^{n} \left\{ {l-1 \atop m-1} \right\} m^{n-l} . \tag{4.19}$$

P r o o f. Let $F_l := \mathbf{0}\binom{n}{m} \cap \{(\varepsilon_1, \ldots, \varepsilon_n) : \varepsilon_l = 0, \varepsilon_{l+1} = \varepsilon_{l+2} = \ldots = \varepsilon_n = 1\}$ for every $l = m, m+1, \ldots, n$. Here, l gives the place of the last 0 in the chains $\omega \in F_l$. It is evident that these sets form a partition of $\mathbf{0}\binom{n}{m}$. We calculate $W_n(F_l)$. Obviously, $\omega \in F_l$ if and only if ω has the form $\omega = (\omega', 0, 1_{(n-l)})$, where $\omega' \in \mathbf{0}\binom{l-1}{m-1}$. (We recall that $j_{(k)}$ denotes the sequence $\underbrace{j, j, \ldots, j}_{k}$.) In this case

$$w_{n,\beta}(\omega) = w_{l-1,\beta}(\omega') \cdot 1 \cdot \beta_m^{n-l}$$

by (4.3), (4.4). Therefore, by (4.18) and (4.16),

$$W_n(F_l) = W_{l-1}(\mathbf{0}\tbinom{l-1}{m-1})\beta_m^{n-l} = \left\{ {l-1 \atop m-1} \right\} m^{n-l} . \tag{4.20}$$

Inserting (4.18) and (4.20) into $W_n(\mathbf{0}\binom{n}{m}) = \sum_{l=m}^{n} W_n(F_l)$, we obtain (4.19). ■

The following theorem gives relations between Stirling numbers of the second kind and associated ones with them.

Theorem 4.5. 1) *If* $n, m \in \mathbf{N}$ *and* $1 \le m \le n$, *then*

$$\left\{ {n \atop m} \right\} = \sum_{j=1}^{m} j \left\{ {n-j-1 \atop m-j} \right\}^{(j)} . \tag{4.21}$$

2) *If* $n, \nu \in \mathbf{N}$, $1 \le \nu \le n-1$, $0 \le m \le n$, *then*

$$\left\{ {n \atop m} \right\} = \sum_{k=0}^{\nu} \left\{ {\nu \atop k} \right\} \left\{ {n-\nu \atop m-k} \right\}^{(k)} . \tag{4.22}$$

P r o o f. 1) For $j = 0, 1, 2, \ldots, m$, we consider the sets

$$G_j := \mathbf{0}\tbinom{n}{m} \cap \{(\varepsilon_1, \ldots, \varepsilon_n) \in \Omega_n : \varepsilon_1 = \ldots = \varepsilon_j = 0, \varepsilon_{j+1} = 1\}.$$

(Here, $j+1$ gives the place of the first 1 in the chains $\omega \in G_j$.) Evidently, G_j form a partition of $\mathbf{0}\binom{n}{m}$. We evaluate $W_n(G_j)$. Obviously, $\omega \in G_j$ if and only if ω has the form $\omega = (0_{(j)}, 1, \omega')$, where $\omega' \in \mathbf{0}\binom{n-j-1}{m-j}$. In this case (compare the proof of Theorem 4.2)

$$w_{n,\beta}(\omega) = \underbrace{1 \cdot \ldots \cdot 1}_{j} \cdot \beta_j \cdot w_{n-j-1,\beta^{(j)}}(\omega') .$$

(For example, if $\omega = (0,0,1,0,1,0,1)$ ($n=7$, $j=2$, $\omega' = (0,1,0,1)$), then $w_{7,\beta}(\omega) = 1 \cdot 1 \cdot \beta_2 \cdot 1 \cdot \beta_3 \cdot 1 \cdot \beta_4$, $w_{4,\beta^{(2)}}(\omega') = 1 \cdot \beta_{2+1} \cdot 1 \cdot \beta_{2+2}$.) Hence, by the definition of the weight W_n, we have

$$\begin{aligned} W_n(G_j) &= \sum_{\omega \in G_j} w_{n,\beta}(\omega) = \beta_j \sum_{\omega' \in \mathbf{0}\binom{n-j-1}{m-j}} w_{n-j-1,\beta^{(j)}}(\omega') \\ &= \beta_j W^{(j)}_{n-j-1}(\mathbf{0}\tbinom{n-j-1}{m-j}) = j \left\{ \begin{matrix} n-j-1 \\ m-j \end{matrix} \right\}^{(j)} . \end{aligned} \tag{4.23}$$

We recall that $W^{(j)}$ is a weight generated by the sequence $\{j+i\}_{i=0}^{\infty}$ and $\left\{ \begin{matrix} a \\ b \end{matrix} \right\}^{(j)} := W^{(j)}_n(\mathbf{0}\binom{a}{b})$ are numbers associated with Stirling ones of the rank j. Inserting (4.18) and (4.23) into $W_n(\mathbf{0}\binom{n}{m}) = \sum_{j=0}^{m} W_n(G_j)$, we get (4.21).

2) For $k = 0,1,2,\ldots,\nu$, we consider the sets

$$\begin{aligned} R_k := \{\omega = (\varepsilon_1, \ldots, \varepsilon_\nu, \varepsilon_{\nu+1}, \ldots, \varepsilon_n) \in \Omega_n : \omega' &= (\varepsilon_1, \ldots, \varepsilon_\nu) \in \mathbf{0}\tbinom{\nu}{k} , \\ \omega'' &= (\varepsilon_{\nu+1}, \ldots, \varepsilon_n) \in \mathbf{0}\tbinom{n-\nu}{m-k} \} . \end{aligned}$$

Evidently, the sets R_k form a partition of $\mathbf{0}\binom{n}{m}$. We evaluate $W_n(R_k)$. For every $\omega = (\omega', \omega'') \in R_k$ ($\omega' \in \mathbf{0}\binom{\nu}{k}$, $\omega'' \in \mathbf{0}\binom{n-\nu}{m-k})\}$) we have

$$w_{n,\beta}(\omega) = w_{\nu,\beta}(\omega') w_{n-\nu,\beta^{(k)}}(\omega'') .$$

(For example, if $n=7$, $\nu=3$, $m=3$, $k=1$, $\omega' = (1,0,1)$, $\omega'' = (0,1,1,0)$, $\omega = (\omega', \omega'')$, then $w_{3,\beta}(\omega') = \beta_0 \cdot 1 \cdot \beta_1$, $w_{4,\beta^{(1)}}(\omega'') = 1 \cdot \beta_{1+1} \cdot \beta_{1+1} \cdot 1$, $w_{7,\beta}(\omega) = \beta_0 \cdot 1 \cdot \beta_1 \cdot 1 \cdot \beta_2 \cdot \beta_2 \cdot 1$.) By the definition of the weight W_n, we have

$$\begin{aligned} W_n(R_k) &= \sum_{\omega' \in \mathbf{0}\binom{\nu}{k}} \sum_{\omega'' \in \mathbf{0}\binom{n-\nu}{m-k}} w_{\nu,\beta}(\omega') w_{n-\nu,\beta^{(k)}}(\omega'') \\ &= \left(\sum_{\omega' \in \mathbf{0}\binom{\nu}{k}} w_{\nu,\beta}(\omega') \right) \cdot \left(\sum_{\omega'' \in \mathbf{0}\binom{n-\nu}{m-k}} w_{n-\nu,\beta^{(k)}}(\omega'') \right) \\ &= W_\nu(\mathbf{0}\tbinom{\nu}{k}) W^{(k)}_{n-\nu}(\mathbf{0}\tbinom{n-\nu}{m-k}) = \left\{ \begin{matrix} \nu \\ k \end{matrix} \right\} \left\{ \begin{matrix} n-\nu \\ m-k \end{matrix} \right\}^{(k)} . \end{aligned} \tag{4.24}$$

Inserting (4.18) and (4.24) into $W_n(\mathbf{0}\binom{n}{m}) = \sum_{k=0}^{\nu} W_n(R_k)$, we obtain (4.22). ■

R e m a r k 4.1. It is easy to generalize Theorems 4.4 and 4.5 to the case of an arbitrary sequence $\{\beta_j\}$.

5. Stirling numbers of the first kind

We recall that *Stirling numbers of the first kind* $\begin{bmatrix} n \\ k \end{bmatrix}$ may be defined for $n \in \mathbf{N_0}$ and integer k as numbers which equal 1 if $n = k = 0$, and 0, if $k < 0$ or $k > n$, and satisfy the following recurrence identity (see [1, Sect. 6.1])

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} . \tag{5.1}$$

As in Section 4, let $\Omega_n = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) : \varepsilon_j = 0, 1; 1 \le j \le n\}$, $\gamma = \{\gamma_j\}_{j=1}^{\infty}$ be a sequence of positive numbers. We introduce a weight $w_{n,\gamma}$ on the set Ω_n inductively. We put for $n = 1$,

$$w_{1,\gamma}((1)) = \gamma_1 , \qquad w_{1,\gamma}((0)) = 1 , \tag{5.2}$$

and for $m > 1$,

$$\begin{aligned} w_{m,\gamma}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}, 1)) &= w_{m-1,\gamma}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})) \cdot \gamma_m , \\ w_{m,\gamma}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}, 0)) &= w_{m-1,\gamma}((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})) . \end{aligned} \tag{5.3}$$

We define $W_{n,\gamma}(A)$ for all $A \subset \Omega_n$ as in (4.5). For the sake of brevity we write w_n and W_n instead of $w_{n,\gamma}$ and $W_{n,\gamma}$, respectively.

For every sequence $\gamma = \{\gamma_j\}_{j=1}^{\infty}$, for all $n \in \mathbf{N}$ and integer k, $0 \le k \le n$, we define polynomials η_{nk} in variables γ_j as follows:

$$\eta_{nk} := \eta_{nk}(\gamma) := W_n(\mathbf{0}\tbinom{n}{k}) . \tag{5.4}$$

For every sequence γ we define also $\eta_{00}(\gamma) = 1$, $\eta_{nk}(\gamma) = 0$ if $k < 0$ or $k > n$. For each nonnegative integer n and $0 \le k \le n$, η_{nk} is a polynomial in the variables γ_i, $1 \le i \le n$.

Definition 5.1. *We say that polynomials η_{nk} are Stirling polynomials of the first kind generated by the sequence γ.*

Polynomials $\eta_{nk}(\gamma)$ satisfy the following recurrence relation.

Theorem 5.1. *Let $n \in \mathbf{N}$, $0 \le k \le n$, and $\eta_{nk}(\gamma)$ be polynomials in $\gamma_1, \ldots, \gamma_n$ defined by* (5.4)*. Then*

$$\eta_{nk}(\gamma) = \eta_{n-1,k-1}(\gamma) + \eta_{n-1,k}(\gamma) \cdot \gamma_n . \tag{5.5}$$

This is an analogue of Theorem 4.1 and the proof is the same. ■

As in the previous section, we write $\gamma^{(l)} := \{\gamma_{l+j}\}_{j=1}^{\infty}$ for every $l \in \mathbf{N}$.

Definition 5.2. *Polynomials* $\eta_{nk}^{(l)} := \eta_{nk}(\gamma^{(l)})$ *in the variables* $\gamma_{l+1}, \gamma_{l+2}, \ldots$ *are said to be associated ones of the rank* l *with polynomials* $\eta_{nk}(\gamma)$.

We consider now a particular case:

$$\gamma_j := j-1 \quad \text{for all} \quad j \geq 1\,. \tag{5.6}$$

(Therefore, $w_n(\omega) = 0$ for every chain $\omega = (\,\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\,)$ such that $\varepsilon_1 = 1$.) We get a set of numbers $\tilde{\eta}_{nk} := \eta_{nk}(\{j-1\}_{j=1}^{\infty})$, which satisfy the recurrence relation

$$\tilde{\eta}_{nk} = \tilde{\eta}_{n-1,k-1} + \tilde{\eta}_{n-1,k} \cdot (n-1)$$

and conditions $\tilde{\eta}_{00} = 1$, $\tilde{\eta}_{nk} = 0$ if $k < 0$ or $k > n$. These numbers are called as *Stirling numbers of the first kind* and are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}$ (see (5.1)).

As a result, we can derive the following statement.

Theorem 5.2. *Let* $n \in \mathbf{N}$, $0 \leq k \leq n$. *Then*

$$\begin{bmatrix} n \\ k \end{bmatrix} = W_n(\mathbf{0}\tbinom{n}{k})\,, \tag{5.7}$$

where W_n *is the weight on* Ω_n *generated by the sequence* $\gamma = \{j-1\}_{j=1}^{\infty}$ *with the help of* (5.2), (5.3), (4.5).

Using (5.7), we give very simple proof of the following known fact (see, for example, [1, formula (6.21)]).

Theorem 5.3. *If* $n \in \mathbf{N}$ *and* m *is an integer such that* $0 \leq m \leq n$, *then*

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{l=m}^{n} \begin{bmatrix} l-1 \\ m-1 \end{bmatrix} l(l+1)(l+2)\ldots(n-1)\,. \tag{5.8}$$

P r o o f. The proof is very similar to that of Theorem 4.4. We consider the sets F_l, $l = m, m+1, \ldots, n$, $k = 0, 1, \ldots, m$, introduced in the proof of the first part of Theorem 4.4. We find from (5.2) and (5.3) that

$$W_n(F_l) = W_{l-1}(\mathbf{0}\tbinom{l-1}{m-1})\gamma_{l+1}\gamma_{l+2}\cdot\ldots\cdot\gamma_n = \begin{bmatrix} l-1 \\ m-1 \end{bmatrix} l(l+1)(l+2)\ldots(n-1)\,.$$

Repeating the reasoning from the proof of Theorem 4.4, we obtain (5.8). ■

The following theorem is an analogue of Theorem 4.5.

Theorem 5.4. 1) *If* $n, m \in \mathbf{N}$, $1 \le m \le n$, *then*

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{j=1}^{m} j \begin{bmatrix} n-j-1 \\ m-j \end{bmatrix}^{(j)} . \tag{5.9}$$

2) *If* $n, \nu, m \in \mathbf{N}$, $1 \le \nu, m \le n$, *then*

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{k=0}^{\nu} \begin{bmatrix} \nu \\ k \end{bmatrix} \begin{bmatrix} n-\nu \\ m-k \end{bmatrix}^{(\nu)} . \tag{5.10}$$

P r o o f. The proof is similar to that of Theorem 4.5. We consider the sets G_j $(1, 2, \ldots, m)$, R_k, $k = 0, 1, \ldots, \nu$, introduced there. In our case the weights of these sets are equal to

$$W_{n+m+1}(G_j) = \gamma_{j+1} \cdot \begin{bmatrix} n-j-1 \\ m-j \end{bmatrix}^{(j)} = j \cdot \begin{bmatrix} n-j-1 \\ m-j \end{bmatrix}^{(j)} ,$$

$$W_{n+m+1}(R_k) = W_\nu(\mathbf{0}\tbinom{\nu}{k}) W^{(\nu)}_{n-\nu}(\mathbf{0}\tbinom{n-\nu}{m-k}) = \begin{bmatrix} \nu \\ k \end{bmatrix} \begin{bmatrix} n-\nu \\ m-k \end{bmatrix}^{(\nu)} .$$

The theorem is now immediate. ■

R e m a r k 5.1. It is easy to generalize Theorems 5.3 and 5.4 to the case of an arbitrary sequence $\{\gamma_j\}$.

6. Euler numbers

Euler numbers $\left\langle {n \atop k} \right\rangle$ $(n \in \mathbf{N_0}, k \in \mathbf{Z})$ may be defined as numbers which equal 1 if $n = k = 0$, and 0, if $k < 0$ or $k > n$, and satisfy the following recurrence identity (see [1, Sect. 6.1])

$$\left\langle {n \atop k} \right\rangle = (n-k) \left\langle {n-1 \atop k-1} \right\rangle + (k+1) \left\langle {n-1 \atop k} \right\rangle . \tag{6.1}$$

As before, let Ω_n be the set of all sequences of the length n with elements 0 and 1, $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=0}^{\infty}$ be two sequences of positive numbers. Let us introduce a weight on Ω_n by induction on n. For $n = 1$, we set

$$w_1((1)) = \beta_0 , \qquad w_1((0)) = \alpha_1 . \tag{6.2}$$

Let $m > 1$. We define the weight of a chain of the length m as follows:

$$\begin{aligned} w_m((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}, 1)) &= w_{m-1}((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1})) \cdot \beta_k\,, \\ w_m((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}, 0)) &= w_{m-1}((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1})) \cdot \alpha_{m-k}\,, \end{aligned} \tag{6.3}$$

where $k = \#\{j : 1 \le j \le m-1, \varepsilon_j = 0\}$. (We do not indicate the dependence w_n on α and β.) In other words, the weight of a chain of the length m equals the product of the weight of the chain consisting of the first $m-1$ terms of a given one and of the weight of the m^{th} term which is equal to α_{m-k}, if this term is 0, and β_k, if it is 1 and if k terms are 0 among the first $m-1$ ones of a given chain. Evidently, definition (6.2) is consistent with definition (6.3), that is (6.2) follows from (6.3) if we take $m = 1$ and if we assume that the first term at the right-hand side of both equalities (6.3) equals 1. As before, we define the weight $W_n(A)$ of a set $A \subset \Omega_n$ by the formula (4.5).

For every $n \in \mathbf{N}$ and integer k such that $0 \le k \le n$, we define

$$\zeta_{nk} := \zeta_{nk}(\alpha, \beta) := W_n(\mathbf{0}\tbinom{n}{k})\,. \tag{6.4}$$

By definition we put $\zeta_{00}(\alpha, \beta) = 1$ and $\zeta_{nk}(\alpha, \beta) = 0$ whenever $k < 0$ or $k > n$. It is evident that ζ_{nk} are polynomials in the variables α_i, β_j (if we consider α_i, β_j as independent variables).

Definition 6.1. *Polynomials $\zeta_{nk}(\alpha, \beta)$ are said to be Euler polynomials, generated by sequences α and β.*

The following theorem gives a recurrence relation for the polynomials ζ_{nk}.

Theorem 6.1. *Let $n \in \mathbf{N}$, $0 \le k \le n$, and $\zeta_{nk}(\alpha, \beta)$ be polynomials defined by* (6.4). *Then*

$$\zeta_{nk}(\alpha, \beta) = \zeta_{n-1,k-1}(\alpha, \beta)\alpha_{n-k+1} + \zeta_{n-1,k}(\alpha, \beta)\beta_k\,. \tag{6.5}$$

P r o o f. The proof is analogous to that of Theorems 4.1 and 5.1. We only note that if A_0 and A_1 are defined as in the proof of Theorem 4.1, then

$$W_n(A_0) = W_{n-1}(\mathbf{0}\tbinom{n-1}{k-1})\alpha_{n-(k-1)}\,, \quad W_n(A_1) = W_{n-1}(\mathbf{0}\tbinom{n-1}{k})\beta_k\,.$$

■

Definition 6.2. *For every $\nu \in \mathbf{N}$ and integer μ such that $0 \le \mu \le \nu$, we define the polynomials*

$$\zeta_{nk}^{(\nu,\mu)}(\alpha,\beta) := \zeta_{nk}(\alpha^{(\nu-\mu)}, \beta^{(\mu)})$$

and call them polynomials associated with the polynomials $\zeta_{nk}(\alpha,\beta)$ of rank (ν,μ). They are polynomials in variables $\alpha_{\nu-\mu+1}, \alpha_{\nu-\mu+2}, \ldots, \beta_\mu, \beta_{\mu+1}, \ldots$. (We recall that if $\delta = \{\delta_j\}_{j=j_0}^\infty$ is a sequence and l is a nonnegative integer, then we denote $\delta^{(l)} := \{\delta_{l+j}\}_{j=j_0}^\infty$.)

We consider a particular case. Let

$$\alpha_l = l-1 \text{ for all} l \ge 1\,, \qquad \beta_k = k+1 \text{ for all } k \ge 0\,.$$

We obtain a set of numbers $\tilde\zeta_{nk} := \zeta_{nk}(\{l-1\}_{l=1}^\infty, \{k+1\}_{k=0}^\infty)$, $n \in \mathbf{N}_0$, $0 \le k \le n$, such that

$$\tilde\zeta_{nk} = \tilde\zeta_{n-1,k-1} \cdot (n-k) + \tilde\zeta_{n-1,k} \cdot (k+1)\,, \tag{6.6}$$

and $\tilde\zeta_{00} = 1$, $\tilde\zeta_{nk} = 0$ whenever $k < 0$ or $k > n$. These numbers are called *Euler numbers* and are denoted by $\left\langle {n \atop k} \right\rangle$ (see (6.1)). By (1.3) and (6.6), the following theorem holds.

Theorem 6.2. *Let $n \in \mathbf{N}$ and $0 \le k \le n$. Then*

$$\left\langle {n \atop k} \right\rangle = W_n(\mathbf{0}(\tbinom{n}{k}))\,, \tag{6.7}$$

where the weight W_n on Ω_n is generated by the sequences $\alpha = \{l-1\}_{l=1}^\infty$, $\beta = \{k+1\}_{k=0}^\infty$ by means of (6.2), (6.3), *and* (4.5).

The following theorem is an analogue of Theorems 4.4 and 5.3.

Theorem 6.3. 1) *If $n, m \in \mathbf{N}_0$, $0 \le m \le n$, then*

$$\left\langle {n \atop m} \right\rangle = \sum_{l=m+1}^{n} \left\langle {l-1 \atop m-1} \right\rangle (l-m)(m+1)^{n-l}\,. \tag{6.8}$$

2) *If $n, m \in \mathbf{N}_0$, then*

$$\left\langle {n+m+1 \atop m} \right\rangle = \sum_{k=0}^{m} \left\langle {n+k \atop k} \right\rangle (k+1)(n+1)^{m-k}\,. \tag{6.9}$$

P r o o f. For all $l = m, m+1, \ldots, n$ and $k = 0, 1, \ldots, m$ we introduce the sets F_l and H_k in the same way as in the proof of Theorem 4.4. By Theorem 6.2 we have

$$W_n(F_l) = W_{l-1}(\mathbf{0}\tbinom{l-1}{m-1})\alpha_{l-(m-1)}\beta_m^{n-l} = \left\langle {l-1 \atop m-1} \right\rangle (l-m)(m+1)^{n-l},$$

$$W_{n+m+1}(H_k) = W_{n+k}(\mathbf{0}\tbinom{n+k}{k})\beta_k\alpha_{n+2}^{m-k} = \left\langle {n+k \atop k} \right\rangle (k+1)(n+1)^{m-k}.$$

The theorem is now immediate. ■

Theorem 6.4. 1) *If $n, m \in \mathbf{N}$, $1 \le m \le n$, then*

$$\left\langle {n \atop m} \right\rangle = \sum_{j=1}^{n-m} j \left\langle {n-j-1 \atop m-j} \right\rangle^{(j+1,1)}. \tag{6.10}$$

2) *If $n, m, \nu \in \mathbf{N}_0$, $0 \le m \le n$, $1 \le \nu \le n-1$, then*

$$\left\langle {n \atop m} \right\rangle = \sum_{k=0}^{\nu} \left\langle {\nu \atop k} \right\rangle \left\langle {n-\nu \atop m-k} \right\rangle^{(\nu,k)}. \tag{6.11}$$

P r o o f. 1) Just as in the proof of Theorem 4.5, we consider the sets G_j, $j = 0, 1, \ldots, m$. We have in our case

$$W_n(G_j) = \beta_0^j \alpha_{j+1} W_{n-j-1}^{(j+1,1)}(\mathbf{0}\tbinom{n-j-1}{m-1}) = j \cdot \left\langle {n-j-1 \atop m-1} \right\rangle^{(j+1,1)}. \tag{6.12}$$

2) As in the proof of the second proposition of Theorem 4.5, we consider the sets R_k, $k = 0, 1, 2, \ldots, \nu$. In our case we have

$$W_n(\mathbf{0}\tbinom{n}{m}) = \sum_{k=0}^{\nu} W_n(R_k) = \sum_{k=0}^{\nu} W_\nu(\mathbf{0}\tbinom{\nu}{k}) W_{n-\nu}^{(\nu,k)}(\mathbf{0}\tbinom{n-\nu}{m-k})$$

$$= \sum_{k=0}^{\nu} \left\langle {\nu \atop k} \right\rangle \left\langle {n-\nu \atop m-k} \right\rangle^{(\nu,k)}.$$

■

R e m a r k 6.2. It is easy to generalize Theorems 6.3 and 6.4 to the case of arbitrary sequences $\{\alpha_j\}$ and $\{\beta_j\}$.

References

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[2] *A. Il'inskii*, A probabilistic approach to q-polynomial coefficients, Euler and Stirling numbers. I. — *Mat. fiz., analiz, geom.* (2004), v. 11, № 4, c. 434–448.