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Short Notes

D'Alembert–Liouville–Ostrogradskii formula and related results

I. Roitberg

Chair of Mathematical Analysis, Chernigov State Pedagogical University
53 Getmana Polubotka, Chernigov, 14013, Ukraine

E-mail:i_roitberg@yahoo.com

A. Sakhnovich

Branch of Hydroacoustics, Marine Institute of Hydrophysics
National Academy of Sciences of Ukraine
3 Preobrazhenskaya, Odessa, 65026, Ukraine

E-mail:al_sakhnov@yahoo.com

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Results, that generalize previous important results of the d'Alembert–Liouville–Ostrogradskii formula type by F.S. Rofe-Beketov, are obtained. The $2p \times 2p$ fundamental solution of the first order system is recovered by its $2p \times p$ block Y_0. Applications to the asymptotics of the continuous analogs of polynomial kernels and to the pseudo-Hermitian quantum mechanics are treated. Similar to the F.S. Rofe-Beketov results the invertibility of the $p \times p$ blocks of Y_0 on the interval is not required.

1. Introduction

Suppose y_0 satisfies Sturm–Liouville equation $-y'' + qy = \lambda y$ $(y' = \frac{dy}{dx})$. Then the well-known D'Alembert–Liouville–Ostrogradskii formula

$$y(x) = y_0(x) \int_0^x y_0(\xi)^{-2} d\xi$$

gives another solution of the Sturm–Liouville equation. (See [2] for this formula and its matrix generalization.) Further developments and interesting applications

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to the spectral theory have been obtained in [4–6] and references therein. Two separate cases have been treated by F.S. Rofe-Beketov in [4–6]: canonical system

$$w'(x,\lambda) = i\lambda J H(x) w(x,\lambda) \quad (\lambda = \overline{\lambda}, \quad H = H^*, \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}) \tag{1.1}$$

with $m \times m$ Hamiltonian H, and non-self-adjoint system

$$w'(x,\lambda) = G(x,\lambda) w(x,\lambda) \tag{1.2}$$

with 2×2 matrix function G. Here $m = 2p$ and I_p is $p \times p$ identity matrix. It is essential to mention that the invertibility of given y_0 (or $p \times p$ block of the fundamental solution w) on the interval that is required in the initial D'Alembert–Liouville–Ostrogradskii formula (1) (or subsection 4.3 [2]) is not required in the formulas in [4–6] anymore.

In this note we shall consider a slightly more general first order system that includes systems (1.1) and (1.2) as well as a class of pseudo-Hermitian systems of m equations, in particular. Applications to the asymptotics of the continuous analogs of polynomial kernels and to the pseudo-Hermitian quantum mechanics will be treated. Similar to the F.S. Rofe-Beketov results the invertibility of the $p \times p$ block of the given $2p \times p$ solution on the interval is not required.

We denote by $\mathbb{R}$ the real axis and by $\overline{\lambda}$ the complex conjugate to λ scalar (or the matrix with the complex conjugate entries).

The authors are grateful to F.S. Rofe-Beketov for introduction to this topic and very fruitful discussion.

2. Main theorem

Theorem 2.1. *Suppose $m \times p$ ($m = 2p$) matrix functions Y_0 and $\widetilde{Y}_0$ satisfy systems*

$$Y_0'(x,\lambda) = G(x,\lambda) Y_0(x,\lambda), \quad \widetilde{Y}_0'(x,\lambda) = \widetilde{G}(x,\lambda) \widetilde{Y}_0(x,\lambda), \tag{2.1}$$

where G and $\widetilde{G}$ are $m \times m$ locally summable matrix functions. Fix also an absolutely continuous $m \times m$ matrix function $D(x)$ and suppose additionally that

$$\det\, Y_0^* Y_0 \neq 0, \quad \det\, \widetilde{Y}_0^* D \widetilde{Y}_0 \neq 0, \quad \widetilde{Y}_0^* Y_0 \equiv 0. \tag{2.2}$$

Then the matrix function

$$Y(x,\lambda) = Y_0(x,\lambda) g_0(x,\lambda) + D(x) \widetilde{Y}_0(x,\lambda) \widetilde{g}_0(x,\lambda), \tag{2.3}$$

where $p \times p$ matrix functions g_0 and $\widetilde{g}_0$ are given by the equations

$$\widetilde{g}_0' + (\widetilde{Y}_0^* D \widetilde{Y}_0)^{-1} \widetilde{Y}_0^* (D' + D\widetilde{G} - GD) \widetilde{Y}_0 \widetilde{g}_0 = 0, \tag{2.4}$$

$$g_0' + (Y_0^* Y_0)^{-1} Y_0^* \Big((D' + D\widetilde{G} - GD)\widetilde{Y}_0 \widetilde{g}_0 + D\widetilde{Y}_0 \widetilde{g}_0' \Big) = 0, \tag{2.5}$$

satisfies the first system in (2.1) also:

$$Y'(x,\lambda) = G(x,\lambda)Y(x,\lambda). \tag{2.6}$$

P r o o f. The proof is straightforward. Indeed, by (2.1) and (2.3) we have

$$Y' = GY_0 g_0 + Y_0 g_0' + (D\widetilde{Y}_0 \widetilde{g}_0)' = GY + (D' + D\widetilde{G} - GD)\widetilde{Y}_0 \widetilde{g}_0 + D\widetilde{Y}_0 \widetilde{g}_0' + Y_0 g_0'.$$

Therefore (2.6) is equivalent to the equality

$$(D' + D\widetilde{G} - GD)\widetilde{Y}_0 \widetilde{g}_0 + D\widetilde{Y}_0 \widetilde{g}_0' + Y_0 g_0' = 0. \tag{2.7}$$

In view of the first and second relations in (2.2) we have $\det Z \neq 0$, where

$$Z(x,\lambda) := \begin{bmatrix} \widetilde{Y}_0(x,\lambda)^* \\ Y_0(x,\lambda)^* \end{bmatrix}. \tag{2.8}$$

So we multiply both sides of (2.7) by Z and taking into account the third relation in (2.2) we see that (2.7) is equivalent to the equations (2.4) and (2.5). Thus (2.6) holds. ■

3. Examples

3.1. In the Subsections **3.1**, **3.2** we shall consider a particular case

$$D = I_m, \quad \widetilde{G}(x,\lambda) = -G(x,\lambda)^*, \quad \widetilde{Y}_0(0)^* Y_0(0) = 0. \tag{3.1}$$

Corollary 3.1. *Suppose G and $\widetilde{G}$ are locally summable in the interval $\Gamma \subseteq \mathbb{R}$ ($0 \in \Gamma$). Let equalities (2.1) and (3.1) hold and assume that*

$$\det\, Y_0(0)^* Y_0(0) \neq 0, \quad \det\, \widetilde{Y}_0(0)^* \widetilde{Y}_0(0) \neq 0. \tag{3.2}$$

Then relations (2.2) hold and matrix function

$$Y(x,\lambda) = Y_0(x,\lambda) g_0(x,\lambda) + \widetilde{Y}_0(x,\lambda) \widetilde{g}_0(x,\lambda), \tag{3.3}$$

where

$$\widetilde{g}_0(x,\lambda) = \big(\widetilde{Y}_0(x,\lambda)^* \widetilde{Y}_0(x,\lambda)\big)^{-1}, \quad g_0(x,\lambda) = \int_0^x \big(Y_0(\xi,\lambda)^* Y_0(\xi,\lambda)\big)^{-1}$$

$$\times Y_0(\xi,\lambda)^* \big(G(\xi,\lambda) - \widetilde{G}(\xi,\lambda)\big) \widetilde{Y}_0(\xi,\lambda) \big(\widetilde{Y}_0(\xi,\lambda)^* \widetilde{Y}_0(\xi,\lambda)\big)^{-1} d\xi, \tag{3.4}$$

satisfies (2.6).

P r o o f. The first two relations in (2.2) are immediate from (2.1) and (3.2). From (2.1) and the second relation in (3.1) it follows that $(\widetilde{Y}_0^* Y_0)' \equiv 0$. So in view of the third relation in (3.1) we get the third relation in (2.2). Therefore the conditions of Theorem 2.1 are fulfilled. As $D = I_m$ and $-G = \widetilde{G}^*$ one easily checks that $\widetilde{g}_0$ given by (3.4) satisfies (2.4). Using this and $Y_0^* \widetilde{Y}_0 \equiv 0$, we derive that g_0 given by (3.4) satisfies (2.5). Hence Y constructed in this corollary satisfies conditions of Theorem 2.1, i.e., equality (2.6) is proved. ■

Corollary 3.1 includes the case of system (1.1). In this case we assume

$$Y_0(0)^* Y_0(0) > 0, \quad Y_0(0)^* J Y_0(0) = 0. \tag{3.5}$$

According to the second relation in (3.1) and (3.5) we can put here $\widetilde{Y}_0 = JY_0$. For the canonical system with $H \geq 0$ and $\Im\lambda < 0$ the asymptotics of the forms $Z(x,\lambda)JZ(x,\lambda)^*$ (analogs of the polynomial kernels) have been studied in [7]. The asymptotics of Z in the almost periodic case is based on formula (3.3) (see Theorem 6.2 [6]). From Theorem 6.2 [6] follows

Corollary 3.2. *Suppose Hamiltonian H and $m \times p$ solution Y_0 of system (1.1) are uniformly almost periodic on $\mathbb{R}$ and (3.5) holds. Then for Z of the form (2.8), $\lambda = \overline{\lambda}$ and $|x| \to \infty$ we have*

$$Z(x,\lambda)JZ(x,\lambda)^*$$

$$= T(x,\lambda) \begin{bmatrix} Y_0(x,\lambda)^* Y_0(x,\lambda) & 0 \\ 0 & \big(Y_0(x,\lambda)^* Y_0(x,\lambda)\big)^{-1} \end{bmatrix} T(x,\lambda), \tag{3.6}$$

where $T(x,\lambda) = \begin{bmatrix} (K(\lambda) + o(1))x & I_p \\ I_p & 0 \end{bmatrix}$,

$$K(\lambda) = K(\lambda)^* = \lim_{|x| \to \infty} x^{-1} g_0(x,\lambda).$$

3.2. Non-self-adjoint PT-symmetric and pseudo-Hermitian systems are actively studied last years following the important paper [1] (see further references in [3, 8]). Operator $\mathcal{H}$ is called pseudo-Hermitian if $\mathcal{H}^* = \eta\mathcal{H}\eta^{-1}$, where operator η is a Hermitian invertible linear operator [3]. Putting $\eta = CP$, where $C = C^*$ is $m \times m$ matrix and $(Pf)(x) = f(-x)$, we see that system

$$w'(x) = i\lambda H(x) w(x,\lambda) \quad (x \in \mathbb{R}, \quad \lambda = \overline{\lambda}) \tag{3.7}$$

is pseudo-Hermitian if the locally summable $m \times m$ matrix function H satisfies equality

$$H(x)^* = -CH(-x)C^{-1}. \tag{3.8}$$

Corollary 3.3. *Suppose $m \times p$ matrix function Y_0 satisfies pseudo-Hermitian system (3.7), (3.8) and relations*

$$Y_0(0)^* Y_0(0) > 0, \quad Y_0(0)^* C Y_0(0) = 0 \tag{3.9}$$

hold. Put also

$$\widetilde{Y}_0(x, \lambda) = C Y_0(-x, \lambda). \tag{3.10}$$

Then the conditions of Corollary 3.1 are satisfied.

3.3. If system (1.2) with 2×2 matrix function G is given, we put in Theorem 2.1 $p = 1$,

$$\widetilde{G} = jJ\overline{G}Jj, \quad \widetilde{Y}_0 = jJ\overline{Y}_0, \tag{3.11}$$

$$j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D(x) = \begin{bmatrix} 1 & 0 \\ 0 & \varphi(x) \end{bmatrix}. \tag{3.12}$$

Equality $\widetilde{Y}_0^* Y_0 \equiv 0$ is now fulfilled automatically, and Theorem 6.6 [6] follows. System (1.1) includes the case of the self-adjoint Sturm–Liouville system and system (1.2) includes non-self-adjoint Sturm–Liouville scalar equation. D'Alembert–Liouville–Ostrogradskii formula for the Sturm–Liouville case may prove useful to judge on the existence of the subordinate solutions in the framework of the Gilbert–Pearson theory.

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