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On a retarded PDE system for a von Kármán plate with thermal effects in the flow of gas

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We prove existence of a compact global attractor of finite fractal dimension and existence of a finite set of asymptotically determining functionals for a retarded PDE system for a von Kármán plate with thermal effects in the flow of gas. Moreover, we show that asymptotical dynamics of the entire system is determined by the dynamics of the single component u, which describes displacement of the plate.

1. Introduction

Nonlinear oscillations of a clamped plate in the presence of thermal effects can be described by the following equations:

$$P_\alpha u_{tt} + (\epsilon_1 - \epsilon_2\Delta)u_t + \Delta^2 u - [u, v+\eta] + \Delta\theta = p(x,t), \quad x \in \Omega, \tag{1}$$

$$\theta_t - \Delta\theta - \Delta u_t = 0, \tag{2}$$

$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \theta|_{\partial\Omega} = 0, \tag{3}$$

where $v = v(u)$ is Airy's stress function defined by

$$\Delta^2 v = -[u,u], \qquad v|_{\partial\Omega} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0, \tag{4}$$

Ω is a smooth bounded domain in $\mathbb{R}^2$, n is the outward unit normal vector to $\partial\Omega$, Δ is the Laplace operator, $P_\alpha = (1-\alpha\Delta)$. The von Kármán brackets are defined by $[u,v] = \partial^2_{x_1}u\cdot\partial^2_{x_2}v + \partial^2_{x_2}u\cdot\partial^2_{x_1}v - 2\partial^2_{x_1x_2}u\cdot\partial^2_{x_1x_2}v$. The function $u = u(x,t)$

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describes transverse displacement of the plate, the function $\theta = \theta(x,t)$ denotes the temperature; $\eta(x) \in H^4(\Omega)$ is a given function determined by mechanical loads. The parameter $\alpha > 0$ accounts for rotational inertia. The parameters $\epsilon_1 \geq 0$, $\epsilon_2 > 0$ account for mechanical dissipation.

If the plate is located in a supersonic gas flow ($U > 1$) that moves in the direction of the x_1-axis with the velocity $U > 1$, then aerodynamic pressure on the plate can be described by (see, e.g., [1], and also [2, 3])

$$p(x,t) = p_0 - \nu\left(u_t + U\partial_{x_1}u + q(u,x,t)\right), \tag{5}$$

where

$$q(u,x,t) = \frac{1}{2\pi}\int_{-t^*}^{0} ds \int_{0}^{2\pi} d\theta [M_\theta^2 \overline{u}](x_1 - (U + \sin\theta)s, x_2 - s\cos\theta, t - s). \tag{6}$$

Here $\overline{u}$ is the extension of $u(x,t)$ that vanishes outside Ω, $M_\theta = \sin\theta \cdot \partial_{x_1} + \cos\theta \cdot \partial_{x_2}$.

Thus, we have a nonlinear PDE system with the time delay $t^* = l/(U-1)$, where l is a size of Ω in the direction of the x_1-axis. Therefore initial conditions must be chosen in the form

$$u(0) = u_0, \quad u_t(0) = u_1, \quad \theta(0) = \theta_0, \quad u|_{(-t^*,0)} = \varphi_0. \tag{7}$$

It was shown in [2] that the problem (1)–(7) also describes the plate in a subsonic gas flow ($0 < U < 1$). In this case the time delay is defined by $t^* = \inf\{t : (x_1 - (U + \sin\theta)s, x_2 - s\cos\theta) \notin \Omega$ for all $(x_1, x_2) \in \Omega,\ \theta \in [0, 2\pi],\ s > t\}$.

Similar problems were studied in [3–5]. An isothermal variant of the problem (1)–(7) was considered in [3]. Rotational inertia of elements of a plate was taken into account, i.e., α, $\epsilon_2 > 0$. Existence of a compact global attractor and finiteness of number of essential modes were proved for that problem.

Nonlinear oscillations of a plate in the Berger approach without thermal effects were studied in [4, 5]. The transversal load on the plate was described by the same term as in [3], but rotational inertia was neglected there, i.e., $\alpha = \epsilon_2 = 0$. Existence of a compact global attractor, it's finite dimensionality, and finiteness of number of essential modes were proved for this problem. Note, that we can use the same method as in [4] to prove finite dimensionality of the attractor of the problem considered in [3].

Another approach to the problem of aeroelastisity is considering of a coupled PDEs systems, in which the first equation describes displacement of a plate and the second describes a flow of gas moving over the plate. The detailed survey of results and unsolved problems, concerning such systems, is given in [6]. In particular, a stabilization result for the entire system (plate + gas) in the case of subsonic flow ($0 < U < 1$) and structural dissipation (i.e., the damping term has the form $(\epsilon_1 - \epsilon_2\Delta)u_t$) is stated there. It is claimed that every trajectory of the entire system tends to the set of fixed points of the problem. The thermoelastic variant of the abovementioned problem without mechanical dissipation is considered in [7], where stabilization result of the same type is obtained.

In the present paper we use ideas and methods from [3] to prove existence of a compact global attractor for the problem (1)–(7) and results from [8] to prove it's finite dimensionality in the case $\alpha,\ \epsilon_2 > 0$. It is also proved that there exist a finite set of determining functionals for the problem (1)–(7). The important result is that the dynamics of the entire system is asymptotically determined by the dynamics of the single component $u(t)$. Whether these results are true in the case $\epsilon_1 = \epsilon_2 = 0$ (i.e., without mechanical dissipation), is an open question. The main difficulty in this case is to prove dissipativity of the system.

2. Existence and uniqueness of a solution

We will denote the norm in the Sobolev space $H^s(\Omega)$ by $||\cdot||_s$ and the norm and the scalar product in $L^2(\Omega)$ by $||\cdot||$ and $(\cdot,\cdot)$, respectively. In this paper we will use the following functional spaces: $\mathcal{H} = H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ with the norm $||(u_0,u_1,\theta_0)||^2_{\mathcal{H}} = ||u_0||_2^2 + ||u_1||_1^2 + ||\theta_0||^2$; $\mathcal{F} = \mathcal{H} \times L^2(-t^*,0;H_0^2(\Omega))$ with the norm $||(u_0,u_1,\theta_0,\phi_0(\tau))||^2_{\mathcal{F}} = ||(u_0,u_1,\theta_0)||^2_{\mathcal{H}} + \int_{-t^*}^0 ||\phi_0(\tau)||_2^2 d\tau$; $\mathcal{W}_T = \{(u,\theta) : u(x,t) \in L^\infty(-t^*,T;H_0^2(\Omega)),\ u_t(x,t) \in L^\infty(0,T;H_0^1(\Omega)), \theta(x,t) \in L^\infty(0,T;L^2(\Omega))\}$. We also define the equivalent scalar product in $H_0^1(\Omega)$: $(u,v)_{1,\alpha} = (u,v) + \alpha(\nabla u, \nabla v)$. We will assume that initial data $(u_0,u_1,\theta_0,\varphi_0) \in \mathcal{F}$.

Definition 1. *Function $(u(t),\theta(t))$ is sad to be a weak solution to the problem (1)–(7) if $(u(t),\theta(t)) \in \mathcal{W}_T$, it satisfies (1)–(6) in the sense of distributions and initial conditions (7) hold.*

To prove existence and uniqueness of a weak solution we need the following estimate for the retarded term $q(u,x,t)$.

Lemma 1. *Let $u(t) \in L^2(-t^*,T;H_0^2(\Omega))$. Then*

$$||q(u,t)||_s^2 \le Ct^* \int_{t-t^*}^{t} ||u(\tau)||^2_{2+s}, \quad s = 0,-1, \quad t \in [0,T]. \tag{8}$$

It was shown in [2] that using change of variables we can reduce the retarded term (6) to the form used in [3], where the estimate (8) was proved.

Theorem 1 (existence and uniqueness of a weak solution). *Let* $W_0 = (u_0, u_1, \theta_0, \varphi_0) \in \mathcal{F}$. *Then for every interval* $(0,T)$ *there exists a unique weak solution* $(u(t), \theta(t))$ *to the problem (1)–(7) with the initial conditions* W_0. *Moreover,* $y(t) = (u(t), u_t(t), \theta(t)) \in C(0,T;\mathcal{H})$, $\theta(t) \in L^2(0,T;H_0^1(\Omega))$, *and the energy equality holds:*

$$E(y(t)) = E(y(0)) + \int_0^t \{-(\epsilon_1+\nu)||u_t(\tau)||^2 - \epsilon_2||\nabla u_t(\tau)||^2 - ||\nabla\theta(\tau)||^2$$
$$-U\nu(\partial_{x_1}u(\tau), u_t(\tau)) - \nu(q(u,x,\tau), u_t(\tau))\}\,d\tau, \tag{9}$$

where

$$E(y(t)) = E_0(y(t)) - \frac{1}{2}([u(t),u(t)],\eta) - (p_0, u(t)),$$
$$E_0(y(t)) = \frac{1}{2}\left(||u_t(t)||^2 + \alpha||\nabla u_t(t)||^2 + ||\Delta u(t)||^2 + \frac{1}{2}||\Delta v(u(t))||^2 + ||\theta(t)||^2\right).$$

If $W^j(t) = (u^j(t), u_t^j(t), \theta^j(t), u^j(t+\tau))$, $\tau \in [-t^*, 0]$, $j = 1,2$, *are two solutions to (1)–(7) with the initial data* $W_0^j = (u_0^j, u_1^j, \theta_0^j, \varphi_0^j) \in \mathcal{F}$ *such that* $||W_0^j||_{\mathcal{F}} \le R$, *then*

$$||W^1(t) - W^2(t)||_{\mathcal{F}} \le C(R,T)||W_0^1 - W_0^2||_{\mathcal{F}}, \quad t \le T. \tag{10}$$

P r o o f. The proof uses Galerkin's approximations. As it is standard, we give only a sketch of the proof. Let $\{e_k\}$ be the orthonormal in $H_0^1(\Omega)$ basis of eigenvectors of the problem

$$\Delta^2 u = \lambda P_\alpha u, \quad u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \tag{11}$$

i.e., eigenvectors of the positive self-adjoint operator A defined by $(Au, v)_{1,\alpha} = (\Delta u, \Delta v)$ with the domain $H^3(\Omega)\bigcap H_0^2(\Omega)$, and let $\{\lambda_k\}$ be the sequence of eigenvalues of A. Let $\{\bar{e}_k\}$ and $\{\bar{\lambda}_k\}$ be eigenvectors and eigenvalues of $-\Delta$ with the domain $H_0^1(\Omega)\bigcap H^2(\Omega)$, respectively. We suppose that $\{\bar{e}_k\}$ are orthonormal in $L^2(\Omega)$. Assume that approximate solutions of the problem (1)–(7) have the form

$$u^m(x,t) = \sum_{k=1}^m g_k(t)e_k(x), \quad \theta^m(x,t) = \sum_{k=1}^m \bar{g}_k(t)\bar{e}_k(x)$$

and satisfy

$$(P_\alpha u_{tt}^m(t), e_k) + ((\epsilon_1 - \epsilon_2\Delta)u_t^m(t), e_k) + (\Delta u^m(t), \Delta e_k) + (\Delta\theta^m(t), e_k)$$
$$= (p_0, e_k) + ([u^m(t), v(u^m(t)) + \eta], e_k)$$
$$-\nu\left\{(u_t^m(t) + U\partial_{x_1}u^m(t) + q(u^m, x, t), e_k)\right\}, \quad k = 1, \dots, m,$$
$$(\theta_t^m(t), \bar{e}_k) - (\Delta\theta^m(t), \bar{e}_k) - (\Delta u_t^m(t), \bar{e}_k) = 0, \quad k = 1, \dots, m,$$

with the initial conditions

$$u^m(0) = P_m u_0,\ u_t^m(0) = P_m u_1,\ \theta_m(0) = \overline{P}_m \theta_0,\ u_m(\tau) = P_m \varphi_0(\tau),\ \tau \in [-t^*, 0],$$

where P_m and $\overline{P}_m$ are projectors on $Lin\{e_k\}_{k=1}^m$ and $Lin\{\overline{e}_k\}_{k=1}^m$, respectively.

Similarly as in [9] we can rewrite the system for g_k, $\bar{g}_k$ in integral form and obtain existence of an approximate solution on some interval $(0, T')$. Similarly as in [3], using Lemma 1 and the estimates for the von Kármán brackets given by Lemma 1.1 [10], we can establish the energy equality (9) for approximate solutions and the a priori bound

$$E_0(y^m(t)) \le c_1 \left(1 + E_0(y^m(0)) + \int_{-t^*}^{0} ||u^m(\tau)||^2 d\tau \right) e^{c_2 t} \le C_{T'}, \quad t \in [0, T']. \tag{12}$$

This bound and Theorem 2.3.2 form [9] enable us to get continuation of a local solution on every interval $(0, T)$. Thus, we obtain a global approximate solution to (1)–(7). Estimate (12) implies $*$-weak compactness of $\{y^m(t)\}$ in $L^\infty(0, T; \mathcal{H})$. This enables us to prove existence of a solution to (1)–(7). Using the standard techniques from [11, Ch. 3], we prove that this solution to (1)–(7) is strong continuous in $\mathcal{H}$ and satisfies the energy equality (9). We prove uniqueness of a solution using Gronwall's lemma. In the same way we obtain (10). The proof of Theorem 1 is now complete.

3. Existence of a global attractor

Due to Theorem 1 we can define a strongly continuous semigroup S_t on $\mathcal{F}$ by the formula

$$S_t(u_0, u_1, \theta_0, \varphi_0(s)) = (u(t), u_t(t), \theta(t), u(t+s)), \quad s \in (-t^*, 0),\ t > 0,$$

where $(u(t), \theta(t))$ is a weak solution to (1)–(7). For the dynamical system $(S_t, \mathcal{F})$ we have the following result.

Theorem 2 (existence of a global attractor). *If $\epsilon_2 > 0$, then the dynamical system $(S_t, \mathcal{F})$ possesses a compact global attractor $\mathcal{A}$, i.e., there exists a compact set $\mathcal{A}$ such that $S_t \mathcal{A} = \mathcal{A}\ \forall t > 0$ and*

$$\lim_{t \to +\infty} \sup_{W \in B} dist_{\mathcal{F}}(S_t W, \mathcal{A}) = 0$$

for every bounded set $B \subset \mathcal{F}$.

The proof of the theorem follows the well-known scheme (see, e.g., Theorem 5.1 from [12, Ch. 1]): it is sufficient to prove that the semigroup S_t is dissipative and asymptotically compact.

Lemma 2 (dissipativity). *The dynamical system $(S_t, \mathcal{F})$ is dissipative, i.e., there exists $R > 0$ such that for every bounded set $B \subset \mathcal{F}$ there exists $t_0(B)$ such that $||S_t W||_{\mathcal{F}} \le R$ for all $t > t_0(B)$ and all $W \in B$.*

P r o o f. Similarly as in [3], we use the functional $V(y) = E(y) + \mu\Phi(y)$ defined for $y \in \mathcal{H}$, where $\mu > 0$ and

$$\Phi(y) = (P_\alpha u_0, u_1) + 1/2((\epsilon_1 + \nu - \epsilon_2\Delta)u_0, u_0), \quad y = (u_0, u_1, \theta_0). \tag{13}$$

It is easy to see that for some $d > 0$ and $\mu > 0$ small enough there exist c_1, $c_2 > 0$ such that

$$c_1(1 + E_0(y)) \le V_d(y) \le c_2(1 + E_0(y)),$$

where $V_d(y) = V(y) + d$ [10, Lemma 3.2]. After a simple calculation we obtain that

$$\frac{d}{dt}V_d(y(t)) \le -(\epsilon_1 + \nu)||u_t(t)||^2 - \epsilon_2||\nabla u_t(t)||^2 - ||\nabla\theta(t)||^2 - \mu(||\Delta u(t)||^2$$
$$+||\Delta v(u(t))||^2) + (\delta+\mu)||u_t(t)||_1^2 + \mu||\nabla\theta(t)||^2 + C_1||\nabla u(t)||^2 + C_2||q(u,x,t)||_1^2 + C_3$$

for every $\delta > 0$ and $0 < \mu < \mu_0$. Similarly as in [3], we can choose δ and μ small enough to obtain the estimate

$$E_0(y(t)) \le C_1 + C_2\left\{1 + E_0(y(t^*)) + \int_0^{t^*} E_0(y(\tau))d\tau\right\}e^{-\delta t}, \quad t \ge t^*,$$

for some $\delta > 0$. This implies the lemma.

Now we introduce the set $K_A^\sigma = (v_0, v_1, \zeta_0, \phi(s)) \subset \mathcal{F}$ such that

$$||v_0||_{2+\sigma}^2 + ||v_1||_{1+\sigma}^2 + ||\zeta_0||_{2\sigma}^2 + \operatorname*{ess\,sup}_{s\in(-t^*,0)}(||\phi(s)||_{2+\sigma}^2 + ||\phi_s(s)||_{1+\sigma}^2) \le A,$$

where $0 < \sigma < 1/2$, $A > 0$. Evidently, K_A^σ is compact in $\mathcal{F}$.

Lemma 3 (asymptotical compactness). *There exists $A > 0$ such that for every bounded set $B \subset \mathcal{F}$*

$$\lim_{t\to+\infty}\sup_{W\in B} dist_{\mathcal{F}}(S_t W, K_A^\sigma) = 0, \quad 0 < \sigma < 1/2.$$

P r o o f. Let T_t be the evolution operator corresponding to the linear PDEs system

$$P_\alpha u_{tt} + (\epsilon_1 - \epsilon_2\Delta)u_t + \Delta^2 u + \Delta\theta = 0, \quad x \in \Omega, \tag{14}$$

$$\theta_t - \Delta\theta - \Delta u_t = 0, \tag{15}$$

$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \theta|_{\partial\Omega} = 0, \tag{16}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad \theta(0) = \theta_0. \tag{17}$$

Similarly as in [10] we prove that T_t is a semigroup of contractions on $H_0^{2+\sigma}(\Omega) \times H_0^{1+\sigma}(\Omega) \times H_0^{2\sigma}(\Omega) = \mathcal{H}^\sigma$ for $0 < \sigma < 1/2$, and $||T_t||_{\mathcal{H}^\sigma} \le Me^{-\gamma t}$. Then a solution to (1)–(7) can be represented by means of Duhamel's principle:

$$(u(t), u_t(t), \theta(t)) = T_t(u_0, u_1, \theta_0) + \int_0^t T_{t-\tau}(0, P_\alpha^{-1}(M(u(\tau))), 0)d\tau,$$

where $M(u(\tau)) = [u, v(u) + \eta] + p_0 - \nu\left(u_t + U\partial_{x_1}u + q(u,x,t)\right)$. Without loss of generality we can assume that initial data lie in the absorbing ball. Then $||M(u(\tau))||_{-1+\sigma} \le C_R$ for $0 < \sigma < 1/2$. Since $||T_t||_{\mathcal{H}^\sigma} \to 0$, $t \to +\infty$, the statement of the lemma holds. Lemmas 2 and 3 implies Theorem 2.

4. Finite dimensionality of the attractor

The following theorem takes place.

Theorem 3 (finite dimensionality of the attractor). *If the conditions of Theorem 2 hold, the attractor $\mathcal{A}$ of the dynamical system $(S_t, \mathcal{F})$ is of finite fractal dimension.*

First we prove a lemma that will be used also in the proof of existence of finite number of determining functionals.

Lemma 4 (stabilizability inequality). *Let $(u_1(t), \theta_1(t)), (u_2(t), \theta_2(t))$ be two solutions to (1)–(7) such that $||(u_j(0), \partial_t u_j(0), \theta_j(0), u(\tau))||_{\mathcal{F}}^2 \le R^2$, $\tau \in [-t^*, 0]$. Let $w = u_1 - u_2, \zeta = \theta_1 - \theta_2$. Then*

$$||(w, w_t, \zeta)(t)||_{\mathcal{H}}^2 \le C(R)e^{-\xi(t-t_0)}\left(||(w, w_t, \zeta)(t_0)||_{\mathcal{H}}^2 + \int_{t_0-t^*}^{t_0} ||\nabla w(\tau)||^2 d\tau\right)$$

$$+C(R)\int_{t_0}^{t} e^{-\xi(t-\tau)}||w(\tau)||_{2-\mu}^2, \quad 0 < t_0 < t, \tag{18}$$

for some $\xi > 0$ and arbitrary $0 < \mu < 1$.

P r o o f. We denote $N(t) = ||w_t(t)||^2_{1,\alpha} + ||\Delta w(t)||^2 + ||\zeta(t)||^2$ and $\Phi(t) = \Phi(w(t), w_t(t), \zeta(t))$, where Φ is defined by (13). It is easy to see that there exists $\beta > 0$ small enough such that

$$c_1 N(t) \leq (N + \beta\Phi)(t) \leq c_2 N(t) \quad \forall t > 0 \tag{19}$$

for some positive c_1, c_2. After a simple calculation we obtain

$$\begin{aligned} \frac{d}{dt}(N + \beta\Phi)(t) &= -(\epsilon_1 + \nu)||w_t(t)||^2 - \epsilon_2||\nabla w_t(t)||^2 - ||\nabla\zeta(t)||^2 \\ &+ \beta||w_t(t)||^2_{1,\alpha} - \beta||\Delta w(t)||^2 + ([u_1, v(u_1)] - [u_2, v(u_2)], w_t)(t) + ([w, \eta], w_t)(t) \\ &- \nu U(\partial_{x_1} w, w_t)(t) - \nu(q(u, x, t), w_t(t)) + \beta([u_1, v(u_1)] - [u_2, v(u_2)], w)(t) \\ &+ \beta([w, \eta], w)(t) + \beta(\nabla\zeta, \nabla w)(t) - \nu U\beta(\partial_{x_1} w, w)(t) - \beta\nu(q(u, x, t), w(t)). \end{aligned} \tag{20}$$

Due to Lemma 1.1 from [10] the following estimates are valid:

$$\begin{aligned} &|([u_1, v(u_1)] - [u_2, v(u_2)], w_t)(t) + ([w, \eta], w_t)(t)| \\ &\qquad \leq C(R, \epsilon)||w(t)||^2_{2-\mu} + \epsilon||w_t(t)||^2_1; \end{aligned} \tag{21}$$

$$\begin{aligned} &|([u_1, v(u_1)] - [u_2, v(u_2)], w)(t) + ([w, \eta], w)(t)| \\ &\qquad \leq C(R)(||w(t)||^2_{2-\mu} + ||\nabla w(t)||^2) \end{aligned} \tag{22}$$

for arbitrary $0 < \mu < 1$ and $\epsilon > 0$. Lemma 1 gives that

$$|(q(u, x, t), w_t(t))| \leq C_\epsilon \int_{t-t^*}^{t} ||\nabla w(\tau)||^2 d\tau + \epsilon||w_t(t)||^2_1; \tag{23}$$

$$|(q(u, x, t), w(t))| \leq C \left(\int_{t-t^*}^{t} ||\nabla w(\tau)||^2 d\tau + ||w_t(t)||^2_1 \right) \tag{24}$$

for every $\epsilon > 0$. Using Schwartz inequality for the other terms in (20) and choosing ϵ and β small enough, we obtain that for some $\xi > 0$

$$\begin{aligned} &\frac{d}{dt}(N + \beta\Phi)(t) + \xi(N + \beta\Phi)(t) \\ &\leq C(R) \left(||w(t)||^2_{2-\mu} + \int_{t-t^*}^{t} ||\nabla w(\tau)||^2 d\tau \right), \quad 0 < \mu < 1. \end{aligned}$$

Multiplying this inequality by $e^{\xi t}$, integrating from t_0 to t, and changing order of variables in the integral term, we obtain

$$(N+\beta\Phi)(t) \le e^{-\xi(t-t_0)}(N+\beta\Phi)(t_0)$$
$$+C(R)\left(e^{-\xi(t-t_0)}\int\limits_{t_0-t^*}^{t_0}||\nabla w(\tau)||^2 d\tau + \int\limits_{t_0}^{t} e^{-\xi(t-\tau)}||w(\tau)||^2_{2-\mu}d\tau\right).$$

Applying (19) to the last estimate, we finish the proof of the lemma.

The proof of finite dimensionality of the attractor of the problem (1)–(7) is based on the idea from [8]. For convenience we list the definition and the theorem we need bellow.

Definition 2. *Let X be a separable Hilbert space. A seminorm $n(x)$ is said to be compact if $n(x_m) \to 0$ for every sequence $\{x_m\}_{m=1}^{\infty} \subset X$ such that $x_m \to 0$ weakly in X.*

Theorem 4. *Let X be a separable Hilbert space and A be a bounded closed set in X. Assume that there exists mapping $V : A \to X$ such that*
(i) $A \subseteq VA$;
(ii) V is Lipschitz on A, i.e., there exists $L > 0$ such that

$$||Va_1 - Va_2|| \le L||a_1 - a_2|| \quad \text{for all} \quad a_1, a_2 \in A.$$

(iii) there exist compact seminorms $n_1(x)$ and $n_2(x)$ on X such that

$$||Va_1 - Va_2|| \le \eta||a_1 - a_2|| + K[n_1(a_1 - a_2) + n_2(Va_1 - Va_2)], \qquad (25)$$

for all a_1, $a_2 \in A$, where $0 < \eta < 1$ and $K > 0$ are constants. Then A is a compact set in X of the finite fractal dimension.

P r o o f o f T h e o r e m 3. It is convenient to use "pieces" of trajectories to prove finite dimensionality of an attractor. Similarly as in [8], we define the space $\mathcal{X}_T = \mathcal{F} \times L^2(0,T;\mathcal{H})$ with the following norm:

$$||U||^2_{\mathcal{X}} = ||(u_0, u_1, \theta_0, \phi_0(\tau))||^2_{\mathcal{F}} + \int\limits_0^T ||(v_0(t), v_1(t), \xi(t))||^2_{\mathcal{H}}dt,$$

where $U = (u_0, u_1, \theta_0, \phi_0(\tau), v_0(t), v_1(t), \xi(t))$, $\tau \in [-t^*, 0]$, $t \in [0,T]$. The constant $T > 0$ will be determined later. We introduce the following seminorm on the space $\mathcal{X}_T$:

$$n_T(U) = \int\limits_0^T ||v_0(t)||^2_{2-\mu}dt + \int\limits_{-t^*}^0 ||\phi_0(t)||^2_{2-\mu}dt.$$

The Dubinsky theorem give us compactness of the seminorm n_T on $\mathcal{X}_T$.

Define the set $\mathcal{A}_T \subset \mathcal{X}_T$ as the follows: $\mathcal{A}_T = \{(u_0, u_1, \theta_0, \phi_0(\tau), u(t), u_t(t), \theta(t)),\ \tau \in [-t^*, 0],\ t \in [0, T])\}$, where $(u_0, u_1, \theta_0, \phi(\tau)) \in \mathcal{A}$ and $(u(t), \theta(t))$ is the week solution to (1)–(7) with the initial data $(u_0, u_1, \theta_0, \phi(\tau)) \in \mathcal{A}$. The operator $V_T : \mathcal{A}_T \to \mathcal{X}_T$ is defined by the formula

$$V_T: \ (u_0, u_1, \theta_0, \phi(\tau), u(t), u_t(t), \theta(t))$$
$$\mapsto (S_T(u_0, u_1, \theta_0, \phi(\tau)), u(T+t), u_t(T+t), \theta(T+t)), \quad \tau \in [-t^*, 0],\ t \in [0, T].$$

Now we will prove that the operator V_T and the set $\mathcal{A}_T$ satisfy conditions of Theorem 4. In the rest of the proof C is a generic constant depending on the radius of dissipativity and the parameter ξ from (18).

Condition (i) easily follows from the fact $\mathcal{A}$ is the attractor. Estimate (10) implies the operator S_t, and therefore V_T, are Lipschitz.

Let us prove (iii). Our starting point is inequality (18). Let $U_1, U_2 \in \mathcal{A}_T$. We denote $U_1 - U_2 = (w_0, w_1, \zeta_0, \varphi_0(\tau), w(t), w_t(t), \zeta(t))$. Replacing t_0 with T in (18), we get

$$||(w, w_t, \zeta)(t)||^2_{\mathcal{H}} - C \left(\int\limits_T^t e^{-\xi(t-\tau)} ||w(\tau)||^2_{2-\mu} d\tau + \int\limits_{T-t^*}^{T} ||\nabla w(\tau)||^2 d\tau \right)$$
$$\leq ||(w, w_t, \zeta)(T)||^2_{\mathcal{H}}.$$

Setting in (18) $t_0 = 0$ and $t = T$ and using the previous inequality, we get

$$||(w, w_t, \zeta)(t)||^2_{\mathcal{H}} \leq Ce^{-\xi T} \left(||(w, w_t, \zeta)(0)||^2_{\mathcal{H}} + \int\limits_{-t^*}^{0} ||\nabla w(\tau)||^2 d\tau \right)$$
$$+C \left(\int\limits_0^T e^{-\xi(T-\tau)} ||w(\tau)||^2_{2-\mu} d\tau + \int\limits_T^t e^{-\xi(t-\tau)} ||w(\tau)||^2_{2-\mu} d\tau + \int\limits_{T-t^*}^{T} ||\nabla w(\tau)||^2 d\tau \right)$$

for $t > T$. Integrating this inequality with respect to t from $T - t^*$ to $2T$, we obtain

$$\int\limits_{T-t^*}^{2T} ||(w, w_t, \zeta)(t)||^2_{\mathcal{H}} dt \leq C(T + t^*)e^{-\xi T} \left(||(w, w_t, \zeta)(0)||^2_{\mathcal{H}} + \int\limits_{-t^*}^{0} ||\nabla w(\tau)||^2 d\tau \right)$$

$$+C(T+t^*)\left(\int\limits_0^T e^{-\xi(T-\tau)}||w(\tau)||^2_{2-\mu}d\tau+\int\limits_{T-t^*}^{T}||\nabla w(\tau)||^2 d\tau\right)$$

$$+\frac{C}{\xi}e^{2\xi T}\int\limits_{T-t^*}^{2T}||w(\tau)||^2_{2-\mu}d\tau. \tag{26}$$

Thus, from (18) and (26) we obtain that

$$||(w,w_t,\zeta)(T)||^2_{\mathcal{H}}+\int\limits_{T-t^*}^{2T}||(w,w_t,\zeta)(t)||^2_{\mathcal{H}}dt$$

$$\leq C(T+t^*+1)e^{-\xi T}\left(||(w,w_t,\zeta)(0)||^2_{\mathcal{H}}+\int\limits_{-t^*}^{0}||\nabla w(\tau)||^2 d\tau\right)$$

$$+C(T)\left(n_T(U_1-U_2)+n_T(V_TU_1-V_TU_2)\right). \tag{27}$$

Choosing T large enough we get that V_T satisfies condition (iii) of Theorem 4.

Thus, for some $T>0$ the set $\mathcal{A}_T$ is of finite fractal dimension. Let the operator $P:\mathcal{X}_T\to\mathcal{F}$ is defined by the formula

$$P((u_0,u_1,\theta_0,\phi_0(\tau),v_0(t),v_1(t),\xi(t))=(u_0,u_1,\theta_0,\phi_0(\tau)),\ t\in[0,T],\ \tau\in[-t^*,0].$$

Sines P is Lipschitz continuous,

$$\dim_f^{\mathcal{F}}\mathcal{A}=\dim_f^{\mathcal{F}}P\mathcal{A}_T\leq\dim_f^{\mathcal{X}_T}\mathcal{A}_T<\infty.$$

Here $\dim_f^{\mathcal{F}}$ denotes fractal dimension of a set in a space $\mathcal{F}$. Theorem 3 is proved.

5. Determining functionals

In the previous section we prove finite dimensionality of the attractor for the problem (1)–(7), but study of its structure in most cases seems not to be possible. In view of this fact it is important to find minimal (or close to minimal) set of natural parameters of the problem that uniquely determine long-time behaviour of the system. General approach to the problem of existence of a finite set of determining parameters is discussed in [13] (see also [12]). In this paper we use the method based on the notion of completeness defect.

Definition 3. *Let V and H be reflexive Banach spaces such that V is densely and continuously embedded in H. The completeness defect of the set $\mathcal{L}$ of linear functionals on V with respect to H is said to be a quantity*

$$\epsilon_{\mathcal{L}}(V,H)=\sup\left\{||w||_H\ :w\in V,\ l(w)=0,\ l\in\mathcal{L},\ ||w||_V\leq 1\right\}.$$

Lemma 4 enables us to prove that asymptotical behaviour of a solution is determined by the component u.

Theorem 5 (existence of a finite set of determining functionals). *Let $\mathcal{L} = \{l_j\}_{j=1}^N$ be a set of functionals on $H_0^2(\Omega)$ with completeness defect $\epsilon_{\mathcal{L}}(H_0^2(\Omega), H_0^{2-\mu}(\Omega)) = \epsilon(\mathcal{L}) < \xi/C(R)$, were ξ and $C(R)$ are constants form Lemma 4 and $\mu > 0$. Then $\mathcal{L}$ is a set of asymptotically determining functionals for the problem (1)–(7) in the following sense: for every two solutions $(u^1, u_t^1, \theta^1)(t)$, $(u^2, u_t^2, \theta^2)(t)$ to the problem (1)–(7) the relation*

$$\lim_{t\to+\infty} \int\limits_{t-t^*}^{t} (l_k(u^1(s)) - l_k(u^2(s)))ds = 0, \quad k = 1, \ldots, N,$$

implies that $||(u^1, u_t^1, \theta^1)(t) - (u^2, u_t^2, \theta^2)(t)||_{\mathcal{H}} \to 0$ when $t \to +\infty$.

P r o o f. Without loss of generality we can consider data from the absorbing ball only. Our starting point is estimate (18). As

$$||u||_{2-\mu} \le C_{\mathcal{L}} \cdot \max_{j=1,\ldots,N} |l_j(u)| + \epsilon(\mathcal{L})||u||_2, \quad 0 < \mu < 1$$

(for the proof see, e.g., [12, Ch. 5]), (18) yields that

$$||(w, w_t, \zeta)(t)||_{\mathcal{H}}^2 \le C(R)e^{-\xi(t-t_0)} \left(||(w, w_t, \zeta)(t_0)||_{\mathcal{H}}^2 + \int\limits_{t_0-t^*}^{t_0} ||\nabla w(\tau)||^2 d\tau \right)$$
$$+C(R) \left(\epsilon(\mathcal{L}) \int\limits_{t_0}^{t} e^{-\xi(t-\tau)} ||(w, w_t, \zeta)(\tau)||_{\mathcal{H}}^2 d\tau + C_{\mathcal{L}} \int\limits_{t_0}^{t} e^{-\xi(t-\tau)} N_{\mathcal{L}}(w(\tau)) d\tau \right), \tag{28}$$

where $w = u^1 - u^2$, $\zeta = \theta^2 - \theta^2$, $N_{\mathcal{L}}(w(\tau)) = max_{j=1,\ldots,N}|l_j(w(\tau))|$. Denote $\Psi(t) = e^{\xi t}||(w, w_t, \zeta)(t)||$, $\delta = C(R)\epsilon(\mathcal{L})$. Then, applying Gronowall's lemma to (28), we get that

$$\int\limits_{t_0}^{t} \Psi(\tau)d\tau \le e^{\delta(t-t_0)}C(R) \left(\Psi(t_0) + e^{\xi t_0} \int\limits_{t_0-t^*}^{t_0} ||\nabla w(\tau)||^2 d\tau \right)$$
$$+Ce^{\delta t} \int\limits_{t_0}^{t} e^{-\delta\tau} d\tau \int\limits_{t_0}^{\tau} e^{\xi s} N_{\mathcal{L}}(w(s)) ds.$$

Here and to the end of the proof C is a generic constant independent on t. Integrating by parts and taking into account (28), we obtain

$$||(w,w_t,\zeta)(t)||^2_{\mathcal{H}} \leq 2C(R)e^{-(\xi-\delta)(t-t_0)}\left(||(w,w_t,\zeta)(t_0)||^2_{\mathcal{H}} + \int\limits_{t_0-t^*}^{t_0} ||\nabla w(\tau)||^2 d\tau\right)$$
$$+C\int\limits_{t_0}^{t} e^{-(\xi-\delta)(t-\tau)} N_{\mathcal{L}}(w(\tau))d\tau.$$

If $\delta = C(R)\epsilon(\mathcal{L}) < \xi$, then $\omega = \xi - \delta > 0$. Evidently, for every $0 < a < t - t_0$

$$||(w,w_t,\zeta)(t)||^2_{\mathcal{H}} \leq 2C(R)e^{-\omega(t-t_0)}\left(||(w,w_t,\zeta)(t_0)||^2_{\mathcal{H}} + \int\limits_{t_0-t^*}^{t_0} ||\nabla w(\tau)||^2 d\tau\right)$$
$$+C\int\limits_{t-a}^{t} e^{-\omega(t-\tau)} N_{\mathcal{L}}(w(\tau))d\tau + C\int\limits_{t_0}^{t-a} e^{-\omega(t-\tau)} N_{\mathcal{L}}(w(\tau))d\tau.$$

Using dissipativity of $(S_t, \mathcal{F})$, we obtain that

$$||(w,w_t,\zeta)(t)||^2_{\mathcal{H}} \leq C_1(R)e^{-\omega(t-t_0)} + C\int\limits_{t-a}^{t} e^{-\omega(t-\tau)} N_{\mathcal{L}}(w(\tau))d\tau + C(R,\mathcal{L})e^{-\omega a}$$

for $t \geq t_0$ and $0 < a < t - t_0$. There R is the radius of absorbing ball. Fixing a and letting t to infinity, we get that

$$\limsup_{t\to+\infty} ||(w,w_t,\zeta)(t)||^2_{\mathcal{H}} \leq C(R,\mathcal{L})e^{-\omega a}$$

for every $a > 0$. This implies the statement of the theorem.

E x a m p l e. Let $\{e_k\}_{k=1}^{\infty}$ be eigenvectors of the operator A defined in Sect. 2. It is easy to see that for the set $\mathcal{L}_N = \{e_k\}_{k=1}^{N}$, where e_k are considered as functionals on $H_0^2(\Omega)$, completeness defect $\epsilon_{\mathcal{L}_N}(H_0^2(\Omega), H_0^{2-\mu}(\Omega)) = \lambda_N^{-\mu/2}$. As $\lambda_N \to +\infty$, the completeness defect of $\mathcal{L}_N$ tends to 0 while $N \to +\infty$ and can be made as small as we need.

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