Математическая физика, анализ, геометрия
1994, т. 1, № 3/4, с. 416–423

УДК 519.46

Simplicity of A. van Daele algebra for finite-dimensional C^*–Hopf algebras

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Received November 1, 1993

We have proved that a A. van Daele *-algebra related with a C^*-Hopf finite algebras is always simple.

1. The quantum group theory [1] is now developing intensively. It generates also interest in the Hopf algebras [2] and some related problems.

Let A and B be two Hopf *-algebras which form a dual pair (see definition 2 below). A. van Daele [3] introduced construction of a *-algebra AB for such A and B. In the present paper this algebra is studied under the assumption that A and B are both finitely dimensional Hopf C^*-algebras. We prove that AB is semisimple. However, if A is either a group algebra over C with the usual structure of the Hopf *-algebra, or an 8-dimensional Kac algebra [4], then the *-algebra AB is simple. Thus, it is reasonable to ask question about simplicity of the AB algebra in general.

This question was solved in the positive in section 4. Section 3 contains the proof of semisimplicity of AB, and section 2 presents auxiliary information.

All necessary preliminary information and results can be found in [1]-[4].

2. Let us recall the definition of a Hopf *-algebra.

Definition 1. Let A be a *-algebra over C with identity. Let Δ be a *-homomorphism of A into $A \otimes A$ such that $(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$. Let $\varepsilon : A \to C$ be a *-homomorphism such that $(\varepsilon \otimes I)\Delta = (I \otimes \varepsilon)\Delta = I$. Finally assume that $S : A \to A$ is a linear, antimultiplicative map such that $S(S(a)^*)^* = a$ for all $a \in A$ and such that $d(S \otimes I)\Delta(a) = \varepsilon(a)1$ for all $a \in A$, where $d : A \otimes A \to A$ is the multiplication map defined by $d(a \otimes b) = ab$. Then A is called a Hopf *-algebra and Δ, ε, S are called the comultiplication, the counit and the antipode of A, respectively.

Definition 2. Let A, B be two Hopf *-algebras. A bilinear map $\langle \cdot,\cdot \rangle : A \times B \to C$ is called a pairing if we have

$$\langle \Delta(a), b_1 \otimes b_2 \rangle = \langle a, b_1 b_2 \rangle,$$

$$\langle a_1 \otimes a_2, \Delta(b) \rangle = \langle a_1 a_2, b \rangle,$$

$$\langle a^*,b\rangle = \langle a,S(b)^*\rangle^-,$$
$$\langle a,1\rangle = \varepsilon(a),$$
$$\langle 1,b\rangle = \varepsilon(b),$$
$$\langle Sa,b\rangle = \langle a,Sb\rangle.$$

for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. If this pairing is non-degenerate, then (A,B) is called a dual pair of Hopf *-algebras.

We will indicate the construction of the *-algebra AB.

In [3] A. van Daele has defined the linear map $R : B \otimes A \to A \otimes B$

$$R(b \otimes a) = \sum_{(a),(b)} \langle a_{(2)},b_{(1)}\rangle\, a_{(1)} \otimes b_{(2)},$$

where

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)},$$
$$\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}.$$

Now we denote by J the involution on A and B and by σ the flip on $A \otimes B$.

In $A \otimes B$ a product can be defined as follows:

$$xy = (d \otimes d)(I \otimes R \otimes I)(x \otimes y),\ \ (x, y \in A \otimes B) \tag{1}$$

and an involution in $A \otimes B$ as follows:

$$x^* = R(J \otimes J)\sigma(x),\ \ (x \in A \otimes B). \tag{2}$$

It turns out that (1), (2) impose on $A \otimes B$ the structure of the *-algebra. For a detailed proof we refer to [3]. We will only show that $x^{**} = x$ for any $x \in A \otimes B$.

Lemma 1. $\left(R(J \otimes J)\sigma\right)^2 = I$ *on* $A \otimes B$.

Proof: Let $a \in A$ and $b \in B$. Then,

$$R(J \otimes J)\sigma(a \otimes b) = R(b^* \otimes a^*) = \sum_{(a),(b)} \langle a^*_{(2)},b^*_{(1)}\rangle\, a^*_{(1)} \otimes b^*_{(2)} =$$
$$= \sum_{(a),(b)} \langle S^{-1}a_{(2)},b_{(1)}\rangle^-\, a^*_{(1)} \otimes b^*_{(2)}.$$

If we apply $R(J \otimes J)\sigma$ once more, we get

$$\left(R(J \otimes J)\sigma\right)^2(a \otimes b) = \sum_{(a),(b)} \langle S^{-1}a_{(2)},b_{(1)}\rangle\, R(b_{(2)} \otimes a_{(1)}) =$$
$$= \sum_{(a),(b)} \langle S^{-1}a_{(3)},b_{(1)}\rangle \langle a_{(2)},b_{(2)}\rangle\, a_{(1)} \otimes b_{(3)} =$$
$$= \sum_{(a),(b)} \langle (1 \otimes S^{-1})a_{(2)} \otimes a_{(3)},b_{(2)} \otimes b_{(1)}\rangle\, a_{(1)} \otimes b_{(3)} =$$
$$= \sum_{(a),(b)} \langle (S \otimes 1)\Delta(a_{(2)}),\sigma(S^{-1} \otimes S^{-1})\Delta(b_{(1)})\rangle\, a_{(1)} \otimes b_{(2)} =$$

$$= \sum_{(a),(b)} \langle (S \otimes 1)\Delta(a_{(2)}), \Delta(S^{-1}b_{(1)}) \rangle a_{(1)} \otimes b_{(2)} =$$

$$= \sum_{(a),(b)} \langle d(S \otimes 1)\Delta(a_{(2)}), (S^{-1}b_{(1)}) \rangle a_{(1)} \otimes b_{(2)} =$$

$$= \sum_{(a),(b)} \varepsilon(a_{(2)})\varepsilon(S^{-1}b_{(1)})a_{(1)} \otimes b_{(2)} = (1 \otimes \varepsilon)\Delta(a) \otimes (\varepsilon \otimes 1)\Delta(b) = a \otimes b.$$

Q.E.D.

It may be proved that A and B are subalgebras of $A \otimes B$.

Proposition 1. *The maps $a \to a \otimes 1$ and $b \to 1 \otimes b$ are *-homomorphisms.*

P r o o f: At first we remark that, if $a \in A$, then

$$R(1 \otimes a) = \sum_{(a)} \langle a_{(2)}, 1 \rangle a_{(1)} \otimes 1 = (1 \otimes \varepsilon)\Delta(a) \otimes 1 = a \otimes 1,$$

$$(a \otimes 1)^* = R(1 \otimes a^*) = a^* \otimes 1.$$

So, for $a, c \in A$ we obtain

$$(c \otimes 1)(a \otimes 1) = (d \otimes d)(I \otimes R \otimes I)(c \otimes 1 \otimes a \otimes 1) =$$

$$= (d \otimes d)(c \otimes a \otimes 1 \otimes 1) = (ca \otimes 1).$$

Q.E.D.

Proposition 2. *For all $a \in A$ and $b \in B$ we have*

$$(a \otimes b) = (a \otimes 1)(1 \otimes b),$$

$$R(b \otimes a) = (1 \otimes b)(a \otimes 1).$$

P r o o f: These formulae follow from (1). Q.E.D.

The algebra $(A \otimes B)$ with the structure of (1) and (2) is denoted by AB.

3. Theorem 1. *Let A and B be the dual pair of Hopf C^*-algebras, and let μ_A and μ_B be the two Haar measures for A and B, respectively; then $\mu_A \otimes \mu_b$ is a faithful central state of AB. Moreover, the *-algebra AB is semisimple.*

P r o o f: It is enough to show that $x \neq 0 \Rightarrow xx^* \neq 0$ for any $x \in AB$ and that $\mu_A \otimes \mu_B$ is central. First of all, it is to be noted

$$(a_i \otimes b_i)(a_j \otimes b_j)^* = (a_i \otimes 1)(1 \otimes b_i)(1 \otimes b_j^*)(a_j^* \otimes 1) =$$

$$= (a_i \otimes 1)(1 \otimes b_i b_j^*)(a_j^* \otimes 1) = \sum_{(a_j^*)(b_i b_j^*)} \langle (a_j^*)_{(2)}, (b_i b_j^*)_{(1)} \rangle a_i (a_j^*)_{(1)} \otimes (b_i b_j^*)_{(2)}.$$

Applying the formulae

$$(I \otimes \mu_A)(\Delta a) = (\mu_A \otimes I)(\Delta a) = \mu_A(a)1,$$

$$(\varepsilon \otimes I)(\Delta a) = a, \qquad a \in A,$$

we get

$$\mu_A \otimes \mu_B\left((a_i \otimes b_i)(a_j \otimes b_j)^*\right) =$$

$$= \mu_A \Big(\sum_{(a_j^*)(b_i b_j^*)} \langle (a_j^*)_{(2)}, \mu_B((b_i b_j^*)_{(2)})(b_i b_j^*)_{(1)} \rangle \, a_i (a_j^*)_{(1)} \Big) =$$

$$= \mu_A \Big(\sum_{(a_j^*)} \langle (a_j^*)_{(2)}, \mu_B((b_i b_j^*) 1 \rangle \, a_i (a_j^*)_{(1)} \Big) =$$

$$= \mu_A \Big(\mu_B(b_i b_j^*) \sum_{(a_j^*)} \varepsilon((a_j^*)_{(2)}) a_i (a_j^*)_{(1)} \Big) = \mu_A(a_i a_j^*)\, \mu_B(b_i b_j^*).$$

Therefore,

$$\mu_A \otimes \mu_B \left(\Big(\sum_i a_i \otimes b_i \Big) \Big(\sum_i a_i \otimes b_i \Big)^* \right) = \sum_{i,j} \mu_A(a_i a_j^*)\ \mu_B(b_i b_j^*). \tag{3}$$

Let $\mu = \mu_A \otimes \mu_B$.

It follows from (3) that $\mu(xx^*) = \mu\left(\iota(x)\iota(x)^*\right)$, $x \in A \otimes B$, where $\iota : A \otimes B \to AB$ is the identity map and $A \otimes B$ the C^*-algebra with the ordinary multiplication.

If $\mu\left(\iota(x)\iota(x)^*\right) = 0$, then $\mu(xx^*) = 0$ and $x = 0$. Therefore $\iota(x) = 0$ and μ is faithful. Thus, $x \neq 0 \Rightarrow xx^* \neq 0$ for any $x \in AB$.

Since μ_A and μ_B are central states of A and B, respectively (see [4]; [1,appendix]), $\mu_A \otimes \mu_B$ is a central state of AB. Q.E.D.

E x a m p l e. Let G be a finite group and let A be the group algebra over C with the usual structure of the Hopf *-algebra:

$$\Delta(g) = g \otimes g, \qquad S(g) = g^{-1}, \qquad \varepsilon(g) = 1 \quad , \; g^* = g^{-1}.$$

Let B be the algebra of complex functions on G with the usual structure of the Hopf *-algebra :

$$(\Delta f)(g \otimes g') = f(gg'); \; (Sf)(g) = f(g^{-1}), \; \varepsilon(f) = f(e), \; f^*(g) = \overline{f(g)}.$$

It easy to verify that $\langle g,f \rangle = f(g)$ defines the pairing between these two algebras.

It is obvious that $\delta_h(h_1 h_2) = \sum_p \delta_{hp}(h_1)\, \delta_{p^{-1}}(h_2)$ and

$$\Delta(\delta_h(.)) = \sum_p \delta_{hp}(.) \otimes \delta_{p^{-1}}(.).$$

Then,

$$R\left(\delta_h(.) \otimes g\right) = \sum_p \langle g, \delta_{hp}(.) \rangle\, g \otimes \delta_{p^{-1}}(.) = \sum_p \delta_{hp}(g)\, g \otimes \delta_{p^{-1}}(.) = g \otimes \delta_{g^{-1}h}(.).$$

It turns out that $e \otimes \delta_e(.)$ is the minimal projector of AB. Indeed, for any $g \otimes f \in AB$, we have

$$(e \otimes \delta_e(.))(g \otimes f)(e \otimes \delta_e(.)) = (e \otimes \delta_e(.))(g \otimes 1)(e \otimes f)(e \otimes \delta_e(.)) =$$

$$= (g \otimes \delta_{g^{-1}}(.))(e \otimes f(e.)\,\delta_e(.)) = g \otimes f(.)\,\delta_{g^{-1}}(.)\,\delta_e(.).$$

Since,

$$g \otimes \delta_{g^{-1}}(.)\,\delta_e(.) = \begin{cases} e \otimes \delta_e(.), & g = e \\ 0, & g \neq e \end{cases},$$

then $e \otimes \delta_e(.)$ is the minimal projector. Next,

$$(g^{-1} \otimes 1)(e \otimes \delta_h(.))(g \otimes 1) = (g^{-1} \otimes 1)(g \otimes \delta_{g^{-1}h}(.)) = e \otimes \delta_{g^{-1}h}(.).$$

It follows from the expression above that $e \otimes \delta_h(.)$ and $e \otimes \delta_e(.)$ are equivalent.

Thus, it implies that

$$(e \otimes \delta_h(.))(AB)(e \otimes \delta_h(.)) = c(e \otimes \delta_h(.)), \qquad c \in C,$$

for all $h \in G$. Therefore,

$$(e \otimes \delta_h)(g \otimes 1) = (e \otimes \delta_h(.))(g \otimes 1)(e \otimes \delta_{g^{-1}h}(.)) \neq 0.$$

Now it is easy to see that the AB algebra is *-isomorphic to a simple matrix algebra $|G| \times |G|$.

Thus, the AB algebra is simple. Q.E.D.

4. Theorem 2. *If A and B are a dual pair of finite-dimensional Hopf C^*-algebras, then AB is simple.*

P r o o f: Since $\varepsilon : A \to C$ is the one-dimensional representation of the algebra A, then there exists the one-dimensional central idempotent $\tau \in A$, such that

$$\tau a = \tau\varepsilon(a), \text{ for all } a \in A.$$

Let $A = A^{(\tau)} + \Sigma_k A^{(k)}$, $B = B^{(\tau)} + \Sigma_s B^{(s)}$, where $A^{(\tau)},\ldots,B^{(s)}$ are simple algebras.

Let $\pi^{(k)}(a) = e^{(k)}ae^{(k)}$ be the irreducible representation of the A algebra, where $a \in A$, $e^{(k)} = \Sigma_i e_{ii}^{(k)}$, $\{ e_{ij}^{(k)} \}_{i,j}$ are matrix units of $A^{(k)}$ algebras.

We denote by $\pi^{(k)}(.)_{pq}$ the matrix elements of $\pi^{(k)}$.

Then $\pi^{(l)}(e_{ij}^{(k)})_{pq} = \delta_{lk}\delta_{ip}\delta_{jg}$.

Let us define elements $\varphi_{pq}^{(k)} \in B$ by

$$\varphi_{pq}^{(k)}(a) = \pi^{(k)}(a)_{pq}, \quad a \in A.$$

Note that $\varphi_{11}^{(\tau)}(a) = \varepsilon(a)$, $a \in A$.

Lemma 2. *We have*

$$\Delta\varphi_{pq}^{(k)} = \sum_m \varphi_{pm}^{(k)} \otimes \varphi_{mq}^{(k)}, \qquad \Delta\varphi_{11}^{(\tau)} = \varphi_{11}^{(\tau)} \otimes \varphi_{11}^{(\tau)},$$

$$\varepsilon\left(\varphi_{pq}^{(k)}\right) = \delta_{pq}, \qquad \varphi_{pq}^{(k)*} = S\varphi_{qp}^{(k)}.$$

P r o o f. For $a_1, a_2 \in A$ we have

$$\Delta\varphi_{pq}^{(k)}(a_1 \otimes a_2) = \varphi_{pq}^{(k)}(a_1 a_2) = \pi^{(k)}(a_1 a_2)_{pq} =$$
$$= \left(\pi^{(k)}(a_1)\,\pi^{(k)}(a_2)\right)_{pq} = \sum_m \pi^{(k)}(a_1)_{pm}\,\pi^{(k)}(a_2)_{mq} =$$
$$= \sum_m \varphi_{pm}^{(k)}(a_1)\,\varphi_{mq}^{(k)}(a_2) = \sum_m (\varphi_{pm}^{(k)} \otimes \varphi_{mq}^{(k)})(a_1 \otimes a_2).$$

Since $\dim \pi^{(\tau)} = 1$, $\Delta\varphi_{11}^{(\tau)} = \varphi_{11}^{(\tau)} \otimes \varphi_{11}^{(\tau)}$, then

$$\varepsilon\left(\varphi_{pq}^{(k)}\right) = \varphi_{pq}^{(k)}(1) = \pi^{(k)}(1)_{(pq)} = \delta_{pq}.$$

$$\varphi_{pq}^{(k)*}(a) = \overline{\pi(S(a)^*)}_{pq} = \pi(S(a))_{qp} = S\varphi_{qp}^{(k)}(a).\ \text{Q.E.D.}$$

The equality

$$\langle e_{ij}^{(k)}, \varphi_{pq}^{(l)} \rangle = \varphi_{pq}^{(l)}(e_{ij}^{(k)})$$

defines the pairing between A and B. Consider now any algebra $A^{(k)}$, for example $A^{(1)}$.

Remark 1. $\mu_B((\varphi_{1q}^{(1)})^*(\varphi_{1r}^{(1)})) = 0$ for any q, r, $q \neq r$.

Indeed, by virtue of Lemma 2, all $\{\varphi_{pq}^{(1)}\}_{p,q}$ are the matrix elements of some representation $\varphi^{(1)}$ of the group algebra B [1,section 4]. Since $\varphi_{pq}^{(1)}(a) = \pi^{(1)}(a)_{pq}$ and π is irreducible, then $\varphi^{(1)}$ is also irreducible. Hence ([1,proposition 4.8],[5]), the matrix elements of $\varphi(1)$ are orthogonal, i.e. $\mu_B((\varphi_{1q}^{(1)})^*(\varphi_{1r}^{(1)})) = 0$, $q \neq r$.

Lemma 3. *$\tau \otimes 1$ is the minimal projection of AB.*

Proof. Let

$$a = \lambda_\tau \tau + \sum_{i,j,k} \lambda_{ij}^{(k)} e_{ij}^{(k)},$$
$$b = \beta_\tau \varphi_{11}^{(\tau)} + \sum_{i,j,s} \beta_{ij}^{(s)} \varphi_{ij}^{(s)},$$

Then, using $\tau a = \tau\varepsilon(a)$ and definition 1, we have

$$(\tau \otimes 1)(a \otimes b)(\tau \otimes 1) = (\tau \otimes 1)(a \otimes 1)(1 \otimes b)(\tau \otimes 1) =$$
$$= (\lambda_\tau \tau \otimes 1)(1 \otimes b)(\tau \otimes 1) = \lambda_\tau (\tau \otimes 1) \sum_{(b)(\tau)} \langle \tau_{(2)}, b_{(1)} \rangle \tau_{(1)} \otimes b_{(2)} =$$
$$= \lambda_\tau \sum_{(b)} \langle \tau, b_{(1)} \rangle \tau \otimes b_{(2)} = \lambda_\tau \beta\, \varphi_{11}^{(\tau)}(\tau) \otimes \varphi_{11}^{(\tau)}, \qquad \beta = \text{const}.$$

Since the only term $\langle \tau, \varphi_{11}^{(\tau)} \rangle = \varphi_{11}^{(\tau)}(\tau)$ in the latter sum is not 0, the equality holds true. Thus,

$$(\tau \otimes 1)(AB)(\tau \otimes 1) = \alpha\, \tau \otimes \varphi_{11}^{(\tau)} = \alpha\, \tau \otimes 1.$$

From the expression above it follows that $(\tau \otimes 1)$ is the minimal projection. Q. E.D.

Lemma 4. *Let P_1 and P_2 be two equivalent minimal projections and u be the corresponding partial isometry. If $tr\, u = 0$, $tr\, P_i = 1$, $i = 1, 2$, then P_1 and P_2 are orthogonal.*

P r o o f. From the conditions of the lemma it follows that $P_1 u P_2 = u$. Consider the polar expansion of $P_1\, P_2$

$$P_1\, P_2 = v \mid P_1\, P_2 \mid = \lambda^{1/2} v\, P_2,$$

where λ is such that $P_2\, P_1\, P_2 = \lambda P_2$ and v is the partial isometry such that $P_1\, v\, P_2 = v$.

But then $v^*\, u = P_2$, and hence

$$v = v\, P_2 = vv^* u.$$

Since $vv^* = P_1$, then $v = P_1 u = u$.

Therefore, $tr\, P_1\, P_2 = \lambda^{1/2} tr\, u = 0$ and P_1 and P_2 are orthogonal. Q.E.D.

We continue to prove Theorem 2.

Now we consider the basis $\{ z_{ij}^{(r)} \}_{r,i,j}$ of the (group) algebra A introduced in ([4], sect. 6) with the following properties:

$$\Delta z_{ij}^{(r)} = \sum_l z_{il}^{(r)} \otimes z_{lj}^{(r)},$$

$$\varepsilon(z_{ij}^{(r)}) = \delta_{ij},$$

$$S z_{ij}^{(r)} = z_{ji}^{(r)*}.$$

The reader can compare the relation with lemma 2. Let $e_{11}^{(1)} = \Sigma_{i,j,r} \nu_{ji}^{(r)} z_{ji}^{(r)}$. Then,

$$(\tau \otimes 1)(1 \otimes b)(e_{11}^{(1)} \otimes 1) = (\tau \otimes 1) \sum_{(b),i,j,l,r} \langle\, \nu_{ij}^{(r)} z_{lj}^{(r)}, b_1 \,\rangle\, z_{il}^{(r)} \otimes b_2 =$$

$$= \sum_{(b),i,j,r} \langle\, \nu_{ij}^{(r)} z_{ij}^{(r)}, b_1 \,\rangle\, \tau \otimes b_2 = \sum_{(b)} \langle\, e_{11}^{(1)}, b_1 \,\rangle\, \tau \otimes b_2.$$

If $b = \varphi_{1t}^{(1)}$, $t = 1,\dots,n$, $n = \dim A^{(1)}$, then we obtain

$$(\tau \otimes 1)(1 \otimes \varphi_{1t}^{(1)})(e_{11}^{(1)} \otimes 1) = \tau \otimes \varphi_{1t}^{(1)}, \quad t = 1,\dots,n.$$

Thus, there exist n minimal subprojections of the projection $e_{11}^{(1)} \otimes 1$ which we denote by P_t, $t = 1,\dots,n$. Observe that $\tau \otimes 1$ and P_t $(t = 1,\dots,n)$ are equivalent. Hence, all P_t are mutually equivalent. We will prove that the projections P_t, $t = 1,\dots,n$ are mutually orthogonal.

Since $\tau \otimes \varphi_{1r}^{(1)}$ intertwines the operators $\tau \otimes 1$ and P_r, $\tau \otimes \varphi_{1q}^{(1)}$ and intertwines the operators $\tau \otimes 1$ and P_q, $(\tau \otimes \varphi_{1q}^{(1)})^* (\tau \otimes \varphi_{1r}^{(1)})$ intertwines P_q and P_r.

To prove orthogonality of projectors P_t, $t = 1,\dots,n$, we use the arguments of the proof of lemma 4.

Since P_q and P_r are equivalent, then there exists partial isometry u such that $P_r u P_q = u$. Then,

$$(\tau \otimes \varphi_{1q}^{(1)})^* (\tau \otimes \varphi_{1r}^{(1)}) = P_q (\tau \otimes \varphi_{1q}^{(1)})^* (\tau \otimes \varphi_{1r}^{(1)}) P_r = \lambda^{1/2} u. \qquad (4)$$

Applying $\mu_A \otimes \mu_B$ to both sides of (4), we get

$$\mu_A \otimes \mu_B (\tau \otimes \varphi_{1q}^{(1)})^* (\tau \otimes \varphi_{1r}^{(1)}) = \mu_A (\tau) \mu_B ((\varphi_{1q}^{(1)})^* (\varphi_{1r}^{(1)})) = \lambda^{1/2} \mu_A \otimes \mu_B (u).$$

By virtue of remark 1, $\mu_B((\varphi_{1q}^{(1)})^* (\varphi_{1r}^{(1)})) = 0$, $q \neq r$. Therefore, $\mu_A \otimes \mu_B(u) = 0$. By virtue of Lemma 4, the projections P_q and P_r are orthogonal.

To complete the proof of Theorem 2, it suffices to observe that $\Sigma_t P_t = e_{11}^{(1)} \otimes 1$.

In fact , let $\Sigma_t P_t = P$. Since P_t, $t = 1,\ldots,n$ are mutually orthogonal and equivalent to the minimal projection, then $\dim P = \alpha n$, where α is a normalizing multiple. Note that

$$P \subseteq e_{11}^{(1)} \otimes 1.$$

Since $\mu_A \otimes \mu_B (e_{11}^{(1)} \otimes 1) = \alpha n$, then $\dim (e_{11}^{(1)} \otimes 1) = \alpha n$.

Therefore, $P = e_{11}^{(1)} \otimes 1$.

Thus, the $*$-algebra AB is simple and $\dim AB = m$, where $m = 1 + \Sigma_k n_k^2$, $n_k = \dim A^{(k)}$. Q.E.D.

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