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Star products on conic Poisson manifolds of constant rank

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We use the method of B.V. Fedosov to construct a star-product on a conic manifold equipped with a Poisson bracket of constant rank.

1. Description of the problem and notations

A cone (or conic manifold) is a smooth paracompact manifold X with a free action of the multiplicative group R_+ *. We denote $\mathcal{O}^s(X)$ (or $\mathcal{O}^s$) the space of C^∞ homogeneous functions f of degree s, i.e., such that $f(\lambda x) = \lambda^s f(x)$ (we will also denote $\mathcal{O}^s$ the corresponding sheaf on X). We denote $\hat{\mathcal{O}}^s(X)$ or $\hat{\mathcal{O}}^s$ the space of symbols of degree s, i.e., of formal series:

$$a = \sum a_{s-k} \text{ with } a_{s-k} \in \mathcal{O}^{s-k},\ k \text{ integer, } k \geq 0. \tag{1.1}$$

We denote $\hat{\mathcal{O}}$ the algebra $\hat{\mathcal{O}} = \bigcup \hat{\mathcal{O}}^k$ ($k \in Z$). The algebra $\mathcal{P}$ of formal differential operators acts on $\hat{\mathcal{O}}$; an operator $P \in \mathcal{P}$ of degree m is a formal series:

$$P = \sum_{k \leq m} P_k, \tag{1.2}$$

where each P_k is a linear differential operator, homogeneous of degree k with respect to homothetics of X. Similarily we have a set $\mathcal{P}_2$ of formal bilinear differential operators: a bilinear operator of degree $\leq m$ is a formal series:

$$L(a, b) = \sum_{k \leq m} L_k(a, b). \tag{1.3}$$

* I.e., X is isomorphic with $Y \times R$, where $Y = X/R$, is the basis, and homotheties are given by: $t(y, r) = (y, tr)$; the choice of an isomorphism corresponds to the choice of a function $r > 0$ homogeneous of degree 1.

where L_k is a bilinear differential operator, homogeneous of degree k, i.e., it is locally a finite sum of the form $L_k(a, b) = \sum p_{\alpha\beta}(x) \partial^{\alpha} a \, \partial^{\beta} b$, and it is homogeneous of degree k with respect to homotheties. Such an operator L defines a composition law (product) on $\hat{\mathcal{O}}$.

A Poisson bracket on X is an antisymmetric bilinear differential operator of order 1: f, $g \to \{ f, g \}$, satisfying the Jacobi identity i.e.,

$$\{ f, f \} = 0 \ \ (antisymmetry),$$

$$\{ fg, h \} = f \{ g, h \} + \{ f, h \} g \ \ (order\ \ 1),$$

$$\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0 \ \ (Jacobi\ \ identity). \qquad (1.4)$$

It is homogeneous of degree-1 if $\{ f, g \}$ is homogeneous of degree $deg \{ f, g \} = deg (f) + deg (g) - 1$ when f and g are homogeneous.

A star-product on X is a law $L \in \mathcal{P}_2$ which is associative, i.e., $L(a, L(b, c)) = L(L(a, b), c)$, and unitary, i.e., $L(1, a) = L(a, 1) = a$. Such an L defines an associative algebra structure on $\hat{\mathcal{O}}$ for which the unit is 1. The dominant term of L is necessarily $L_0(a, b) = ab$ (slightly more generally if L is just associative, its dominant term L_m is necessarily of order 0 as a differential operator, i.e., of the form $L_m(a, b) = fab$ for some $f \in \mathcal{O}^m$. If f is invertible, L has a unit u with dominant term f^{-1}, whose total formal series one constructs elementary by induction on the degree; then $u^{-1}Lu$ is an equivalent associative and unitary law).

If L is associative and unitary the relations on terms of degree-1 and -2 imposed by associativity imply that the bilinear operator describing the dominant term of commutators [a, b]:

$$\{ a, b \} = L_{-1}(a, b) - L_{-1}(b, a) \qquad (1.5)$$

is a Poisson bracket, homogeneous of degree-1

A natural problem is then to construct and classify, up to equivalence, star-products associated to a given Poisson bracket (two laws L and L' are conjugate if there exists an invertible $P \in \mathcal{P}$ such that $L' = PLP^{-1}$). This was done by M. D. De Wilde and P. Lecomte [DL1, 2] in the semi-classical case, where $X = Y \times R_+$, with Y a symplectic manifold, $\{ \ \}_X = h \{ \ \}_Y$, and the "Planck constant" h is a positive homogeneous function of degree-1 (the inverse of the canonical variable of R_+), cf. also [OMY1, 2]. In [BG] (cf. also [B1, 2]) we studied the case where X is a symplectic cone. Here we will show the following result, using the "elementary" method of Fedosov [F2].

Theorem 1. *If X is a conic manifold equipped with a homogeneous Poisson bracket of degree-1, of constant rank, there exists an associated star-product.*

Remark 1. The relative position of the leaves of the foliation of the Poisson bracket with respect to the infinitesimal generator of homotheties (radial vector) does not enter in this analysis — in contrast to what usually happens for P. D. E's.

For Poisson brackets of nonconstant rank it is in general not known if there exists an associated star-product.

2. The local model — filtered Weyl algebras

Let X be a cone, and $E \to X$ a symplectic vector bundle over X. On each fiber E_x there is a Poisson bracket $c_x = \sum c_{ij}\, \partial_i\, \partial_j$, where the ∂_i are the fiber derivations, in some vector basis of the fiber, and the coefficients c_{ij} are constant along the fibers, i.e., only depend (smoothly) on the basis point x. Locally one may choose coordinates ξ_i, linear along the fibers, and homogeneous of degree 1/2 with respect to homotheties, so that the c_{ij} are constant (the degree must be 1/2 if { } is of degree-1).

Let W be the associated fiber bundle of "homogeneous filtered Weyl algebras": its sections of degree m are symbols:

$$f = \sum f_k\,(x, \xi)$$

with f_k homogeneous of degree $m - k$. The composition law is given by

$$f * g = \exp \frac{1}{2}\, c_x\,(\partial_\xi\,.\,\partial_\eta)\,f(x, \xi) g(x, \eta)\,\big|_{\eta = \xi} =$$

$$= \sum \frac{1}{k!}\left(c\,(\partial_\xi\,.\,\partial_\eta)^k\, f(x, \xi)\, g(x, \eta)\right)\Big|_{\eta = \xi} \tag{2.1}$$

which is well defined as a symbol (formal series of homogeneous functions) because the Poisson bracket c is homogeneous of degree-1, so $\frac{1}{k!}\, c\,(\partial_\xi\,.\,\partial_\eta)^k\, f(x, \xi)\, g(x, \eta)$ is of degree $\leq deg\,(f) + deg\,(g) - k \to -\infty$. ($W$ should be thought of as a sheaf on conic open sets of E).

We will also use the algebra $\hat{W}$ of jets of infinite order of sections of W along the zero section { $\xi = 0$ } of E (it is a sheaf on conic open sets of X). Its sections of degree k can be written, locally with coordinates ξ_j as above, as formal power series:

$$f = \sum f_\alpha(x)\, \xi^\alpha, \tag{2.2}$$

where the f_α are symbols on X of degree $\leq k - |\,\alpha\,| - 1/2\; deg\,(\xi)$. The star-product (2.1) is equally well defined on $\hat{W}$.

In the general case we will construct below the desired star-product algebra as the subalgebra of such a Weyl algebra killed by suitable derivations, described together as the coefficients of a connection as in [F2]. We recall the structure of derivations of $\hat{W}$ (or W):

Lemma 1. *Let D be a derivation of degree k of $\hat{W}$ (i.e., $D\hat{W}^{m} \subset \hat{W}^{m+k}$). Locally, if we choose local x-coordinates on X and ξ-coordinates in the fibers as above, so that the $\{\xi_i, \xi_j\}$ are constant, D can be written:*

$$Df = \sum a_j(x)\, \partial f / \partial x_j + [b, f] \tag{2.3}$$

with a_j symbols of degree $\le k + \deg x_j$, and b of degree $\le k + 1$.

P r o o f. The center of W consists of functions $f = f(x)$ constant along fibers E_x. If f is central, then so is Df since $[Df, g] = D[f, g] - [f, Dg]$. Let $D_0 = \sum a_j(x)\, \partial f / \partial x_j$ be the vector field on X such that $Df = D_0 f$ for central f, and denote again D_0 the extension to W defined by our choice of coordinates. Then the derivation $D - D_0$ kills central functions and this implies that it is an interior derivation. This last assertion is proved as follows: we choose dual coordinates ξ_i^* so that $[\xi_i^*, \xi_j] = \delta_{ij}$; more generally we have $\partial f / \partial \xi_i = [\xi_i^*, f]$. Let us set $\beta_i = (D - D_0)\, \xi_i^*$. The equalities

$$(D - D_0)[\xi_i^*, \xi_j^*] = [(D - D_0)\, \xi_i^*\ \xi_j^*] + [\xi_i^*, (D - D_0)\xi_j^*] = 0. \tag{2.4}$$

i.e., $\partial \beta_i / \partial \xi_j = \partial \beta_j / \partial \xi_i$ mean that $\beta = \sum \beta_i\, d\xi_i$ is a closed form, so it has primitive b (along the fibers), and we have $(D - D_0) f = [b, f]$ (in fact a there is a canonical global primitive: $b = \int_0^1 \xi \cdot \partial_\xi \llcorner \beta(t\xi)\, dt$).

R e m a r k 2. The formulas above equally apply to a conic vector bundle E equipped with a vector Poisson bracket $\{\ \} = \sum c_{ij}\, \partial_i\, \partial_j$ which is not symplectic, i.e., the coefficients $c_{ij}(x)$ are again smooth functions of the base x alone but the matrix (c_{ij}) is no longer invertible. Formula (2.1) still defines a Weyl algebra W or a formal algebra of jets $\hat{W}$. However, if the rank of the Poisson bracket (i.e., of the matrix (c_{ij})) is not maximal, in particular if it is not constant, the structure of derivations is more complicated and Lemma 1 does not hold.

3. Connection associated to a good coordinate system

Let X be a cone with a Poisson bracket homogeneous of degree-1, of constant rank. Then there is an associated foliation F, generated by all hamiltonians vector fields h_f (these generate a subvector bundle — of constant rank — of TX, and the Frobenius integrability condition follows from the Jacobi identity). The tangent bundle TF of the foliation F is a conic symplectic vector-bundle with basis X, as above, and there is an associated Weyl algebra $\hat{W}_{TF}$.

As we said above, we will construct a star-algebra associated to $\{\ \}_X$ as a subalgebra of sections of $\hat{W}_{TF}$ killed by a suitable connection. We first show how this can be done locally, in a "good" system of local coordinates.

Lemma 2. *Near any given point $x_0 \in X$, there exist local coordinates $x_1,\dots, x_n$ (of homogeneous degree 1/2) such that the matrix $\{ x_i, x_j \}_{1 \le i,j \le k}$ is constant, invertible ($k = rank\ of \{\ \}$).*

P r o o f. We first choose functions x'_j (homogeneous of degree $1/2$) so that the hamiltonian fields $h_{x'_1},\dots, h_{x'_k}$ form a basis of TF at x_0, and $h_{x'_1},\dots, h_{x'_{k-1}}$ are linearily independant from the radial vector field ρ, infinitesimal generator of honothetics (let us notice that if ρ is tangent to a leaf F at some point, it is tangent to F everywhere because F and its homothetics meet along the ray through that point, and two leaves with a comman point are equal).

We may now modify these coordinates recursively imposing $x_1 = x'_1$, and for $1 \le i,j \le k$: $\{ x_i, x_j \} = constant = \{ x'_i, x'_j \}(x_0)$, $x_j = x'_j$ on a conic initial manifold transvesal to the h_{x_i}, $i < j$ (this ensures homogeneity, and the initial transversality hypothesis, that $h_{x'_1},\dots, h_{x'_{k-1}}$ are linearily independant of ρ, ensures the existence of such initial manifolds). The remaining coordinates $x_{k+1},\dots, x_n$ may be chosen arbitrarily. We will call such a coordinate system a "good coordinate system".

To such a good coordinate system we associate a first canonical "integrable" F-connection ∇ with coefficients in $W_{TF} \otimes \Omega_{TF}$. The starting point is the following: the canonical tangent form of F, with coefficients in TF can be written $\tau = \sum dx_i\, \partial / \partial\xi_i$ (in the TF-coordinates as above). To this corresponds

$$\delta = \sum_{i=1}^{k} \xi_i^* \, d\xi_i, \text{ with } \xi_i^* \text{ the dual basis of } \xi_i \text{ for } \{\ \}. \tag{3.1}$$

τ is invariant by all changes of coordinates preserving leaves, so δ is invariant by all changes of coordinates preserving $\{\ \}$, i.e., preserving leaves with their symplectic structure.

With good coordinates x_i as above we have a first connection:

$$\nabla = D - \delta \tag{3.2}$$

with δ as above and

$$D = \sum_1^k dx_i\,(\partial / \partial x_i)^F, \tag{3.3}$$

where we denote $(\partial / \partial x_i)^F$ the vector field tangent to F such that $(\partial / \partial x_i)^F(x_j) = \delta_{ij}$, if $1 \le i, j \le k$, with δ_{ij}, the Kronecker symbol (the vector field $h_{x_i^*}$). D is the canonical extension of the exterior derivative d^F defined by the choice (x_i, ξ_i) of coordinates in TF as above. Obviously, we have

$$D^2 = 0, \;\; [D, \delta] = 0, \quad \delta^2 = 1 \otimes \omega\,, \tag{3.4}$$

where ω is the F-symplectic form associated to $\{\ \}$.

Thus the curvature form of ∇ central, and $Ad\nabla$ is integrable. The sections of W killed by $Ad\,\nabla$ are the symbols $f(x, \xi)$ such that $\nabla f - [\delta, f] = 0$, i.e., such that f only depends on $x_1 + \xi_1, \dots, x_k + \xi_k$. Obviously, these sections form a star-algebra on X, isomorphic to the standard star-algebra equipped with the Moyal-Weyl product (in our "good" system of coordinates).

4. Global construction

The connection D above is not invariant by changes of coordinates; if $x' = (x_i')$ is another good system of coordinates, $\xi' = (\xi_i')$ the corresponding coordinates of the fibers of TF, we have $\xi' = A\xi$ with $A = dx' / dx$, so $D\xi' = dAA^{-1}\xi'$. The linear operator $dA.A^{-1}$ is infinitesimally symplectic, of the form $\xi \to [\lambda, \xi]$, with $\lambda = \sum \lambda_{ijk}\,\xi_i\,\xi_j\,dx_k$ a second order section of W "without constant term" (the λ_{ijk} are uniquely determined by this and the symmetry condition $\lambda_{ijk} = \lambda_{jik}$).

In the new set of coordinates, with ∇' the new connection, we get

$$\nabla = \nabla' + \lambda$$

with $\lambda = \sum \lambda_{ijk}\,\xi_i\,\xi_j\,dx_k$ the Weyl symbol of order 2 as above.

We will call Weyl connection a derivation $\nabla\colon \hat{W} \to \hat{W} \otimes \Omega_F$ which locally, in good coordinates as above, can be written

$$\nabla = D - \delta + \lambda \tag{4.1}$$

with $\lambda = \sum \lambda_j\,(x, \xi)\,dx_j = \sum \lambda_{i\,\alpha}(x)\,\xi^\alpha\,dx_i$, $|\,\alpha\,| \ge 2$ a differential form whose coefficients are symbols of degree ≤ 1 ($Ad\,\lambda$ of degree ≤ 0), vanishing of order ≥ 2 for $\xi = 0$, i.e., the coefficients λ_j are of degree $\le 1/2$ (recall that the dx_i are homogeneous of degree $1/2$), $\lambda_{i,\alpha}$ of degree $\leq -(1+\alpha)/2$, $|\alpha| \geq 2$). Such connections exist locally as we saw above (with the coefficients of λ polynomials of order 2), and also globally because they can obviously be patched together by means of a partition of unity (so far this just corresponds to the construction of a symplectic connection on F). A Weyl connection extends naturally as an antiderivation of $\hat{W} \otimes \Omega_F$.

We now introduce a new of weight (valuatior) w on $\hat{W}$ (or W) and $\hat{W} \otimes \Omega_F$, which measures the vanishing order along the zero-section $\xi = 0$:

$$w(x) = -1, \ (w(f(x)) = -2 deg f), \ w(\xi) = 0, \ w(dx) = 0.$$

Obviously, we have

$$w(f * g) \leq w(f) + w(g), \ w([f, g]) \leq w(f) + w(g)$$

(note that the graded algebra corresponding to the weight w is not commutative).

We have further

$$w(Ad\delta) = w(\partial / \partial\xi_i) = 0, \quad w(D) \geq 1, \quad w(\lambda) \geq 1$$

so the leading term of ∇ is $-\delta$ which essentially the same as the fiber exterior derivative d_ξ, exchanging the dx_i, $d\xi_i$.

In the next lines we denote $\tilde{\alpha}$ the differential form deduced from α by exchange of dx_i, $d\xi_i$.

Let $R = \nabla^2 = 1 \otimes \omega + r$ be the curvature. We have $\nabla(r) = 0$ because ∇ kills both R and ω. Hence $d_\xi \tilde{r} = 0 +$ terms of higher weight.

Let us now suppose $w(r) \geq k$ (this is always true for some integer $k \geq 1$). Let α be the 1-form such that

$$\tilde{\alpha} = \int_0^1 \xi . \partial_\xi \llcorner \tilde{r}(t\xi) \, dt$$

(note that α is globally and canonically defined). We have

$$w(\alpha) = w(R) \geq 1 \ \ (deg\, \alpha = deg\, R \leq 0), \ \text{et} \ w(\tilde{R} - d_\xi \tilde{\alpha}) > w(R),$$

so that the curvature of the modified connection $\nabla + \alpha$ is $R_\alpha = (\nabla + \alpha)^2 = R + + \nabla(\alpha) + \alpha^2 = 1 \otimes \omega +$ terms of weight $> w(r) = k$.

Thus by successive approximations we construct ∇ (i.e., the "Taylor series" of λ) globally so that $\nabla^2 = 1 \otimes \omega$, i.e., ∇ has central curvature and $Ad\,\nabla$ is "integrable".

5. End of the construction

Let finally A be the sub-algebra of W consisting of those f such that $\nabla f = 0$. Then the associated graded algebra $Gr\, A$ is the algebra of functions $f(x, \xi)$ constant on the leaves of $gr\, \nabla$, i.e., such that

$$df + \sum - \partial f / \partial \xi_i + \{ \lambda_i , f \} = 0$$

in any "good" local system of coordinates (x, ξ) as above. Along the zero-section $\{ \xi = 0 \}$ these leaves are tangent to the manifolds ($x + \xi = constant$), so they are transverse to the zero section $\{ \xi = 0 \}$. It follows that the restriction $f \in A \to f_{|X}$ $(f(x, \xi) \to f(x, 0))$ is one to one, and

$$\sigma([f, g])_{|X} = \{ \sigma f, \sigma g \}_{\xi\, |X} = \{ f_{|X} , g_{|X} \}_X .$$

We have thus constructed a star-algebra as announced.

R e m a r k 3. The star-algebra thus constructed is "minimally non-commutative" in the sense that its star-product can be expressed in terms of derivations tangent to the leaves of F, and it can be embedded in a Weyl algebra of rank k ($k =$ rank of $\{ \}$). Its center is maximally large: $Z(gr\, A) = gr\, Z(A)$. With this restriction one can classify star-products associated to a given $\{ \}$ along the same as Fedosov [F2]. Otherwise classification seems an ill-posed problem — e.g., classifying star-products associated with the zero Poisson bracket amounts to classifying all star-products associated to all $\{ \}$ of higher homogeneity degree-1.

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*-Произведения на конических пуассоновых многообразиях постоянного ранга

Луи Буте де Монвель

Мы используем метод Б. В. Федосова для построения *-произведения на коническом многообразии, оснащенном скобками Пуассона постоянного ранга.

*-Добуток на конічних пуассонових многовидах сталого рангу

Луї Буте де Монвель

Ми використовуємо метод Б. В. Федосова для побудови *-добутку на конічному многовиді, оснащеному дужками Пуассона сталого рангу.