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A relation between two results about entire functions of exponential type

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A theorem about polynomials stated at the beginning of this paper is first extended to entire functions of small exponential type and the result then used to deduce the multiplier theorem of Beurling and Malliavin. A precise characterization of the order of magnitude of a non-zero periodic function's Fourier coefficients, compatible with that function's vanishing on some interval, is also given.

The sum $\sum_{-\infty}^{\infty} \log^+|P(n)|/(1+n^2)$ may be looked on as a discrete analogue of the integral $\int_{-\infty}^{\infty} (\log^+|P(t)|/(1+t^2))\,dt$. It is well known that the size of a *polynomial* $P(z)$ is controlled, in the whole complex plane, by the latter expression; it is thus not too surprising that the former one can also, with suitable precautions, be used for the same purpose. One has indeed the

Theorem. *There are numerical constants* $\eta_0 > 0$ *and k such that, for any polynomial* $P(z)$ *with* $\sum_{-\infty}^{\infty} \log^+|P(n)|/(1+n^2) = \eta \le \eta_0$, *the relation* $|P(z)| \le C_\eta e^{k\eta|z|}$ *holds for all complex z, with* C_η *depending on* η *but not on P.*

This result can be found on p. 520 of [1]. In it, the restriction to *small* values of $\eta > 0$ is really necessary, as is shown by examples. Such a theorem for *even* polynomials $P(z)$ with $P(0) = 1$ was already published in 1966 (see [2]), and the main work is in the proof for that case. From there, the passage to general polynomials is rather easy. Treatment of the special case is straightforward in principle, but made intricate by various technical difficulties.

The establishment of the result was originally motivated by a desire to deduce from it a new proof of the multiplier theorem due to Beurling and Malliavin ([3]), but up to now that project has not been realized. The purpose of the present article is to show how the deduction can be carried out. In doing so we will obtain some auxiliary propositions of independent interest.

I thank Henrik Pedersen for having called my attention to some mistakes and obscurities in a preliminary version of this paper.

1. Extension of result cited above to entire functions of small exponential type

The carrying out of such an extension is proposed as problem 24 in [1] (p. 518); there the reader is asked to imitate the proof of the result for polynomials. One can also, however, arrive at the extension directly.

Lemma. *Let* $f(z) = \prod_{1}^{\infty} (1 - z^2/\lambda_k^2)$, *with the* $\lambda_k > 0$, *be of exponential type* α. *Then, for all sufficiently large integers* N, $f_N(z) = \prod_{\lambda_k \le N} (1 - z^2/\lambda_k^2)$ *satisfies*

$$\sum_{1}^{\infty} (\log^+|f_N(n)|)/n^2 \le \sum_{1}^{\infty} (\log^+|f(n)|)/n^2 + C\alpha + o(1).$$

Here, C *is a numerical constant and the* $o(1)$ *term tends to zero as* $N \to \infty$.

P r o o f: Let $\nu(t)$ denote the number of λ_k (counting multiplicities) in $(0, t]$ and put $\nu_N(t) = \nu(t) - \nu(N)$ for $t \ge N$, with $\nu_N(t) = 0$ for $0 \le t \le N$. The last function is in fact zero on an interval $[0, N+\varepsilon)$, where $\varepsilon > 0$.

We have

$$\log|f_N(x)| = \log|f(x)| - \int_N^{\infty} \log|1 - x^2/t^2|\, d\nu_N(t).$$

Integration by parts converts the right side to

$$\log|f(x)| + \int_N^{\infty} \frac{2x^2}{t^2 - x^2} \cdot \frac{\nu_N(t)}{t}\, dt$$

because $\nu(t)$ is $O(t)$ for large t. Here, the *integral* — call it $g_N(x)$ — is positive and increasing for $0 \le x \le N$, so $\sum_{1}^{N-1} g_N(n)/n^2 \le 4\int_0^N (g_N(x)/x^2)\, dx$, and

$$\sum_{1}^{N-1} \frac{\log^+|f_N(n)|}{n^2} \le \sum_{1}^{N-1} \frac{\log^+|f(n)|}{n^2} + 4\int_0^N \frac{g_N(x)}{x^2}\,dx. \tag{1}$$

Since $f(z)$ is of exponential type α, a standard application of Jensen's formula gives $\nu(t)/t \le e\alpha + o(1)$ for $t \to \infty$ (see [1], problem 1, p. 5). Thence,

$$\int_0^N \frac{g_N(x)}{x^2}\,dx = \int_0^N\int_N^\infty \frac{2}{t^2 - x^2}\cdot\frac{\nu_N(t)}{t}\,dt dx =$$

$$= \int_N^\infty \frac{\nu_N(t)}{t}\log\left|\frac{t+N}{t-N}\right|\frac{dt}{t} \le \frac{\pi^2}{4}(e\alpha + o(1)), \tag{2}$$

with the $o(1)$ term tending to zero as $N \to \infty$. (We've used

$\int_1^\infty \log((\tau+1)/(\tau-1))d\tau/\tau = \pi^2/4$.)

To estimate $\sum_N^\infty (\log^+|f_N(n)|)/n^2$, note that $\log|f_N(x)| = \int_0^N \log|x^2/t^2 - 1|\,d\nu(t)$ is increasing for $x \ge N$, so, in $[N, \infty)$, $\log|f_N(x)|$ is ≥ 0 precisely on an interval of the form $[x_0, \infty)$, where $x_0 \ge N$. Let M be the first integer $\ge x_0$. Then, since $\log|f_N(x)|$ increases on $[N, \infty)$,

$$\sum_N^\infty (\log^+|f_N(n)|)/n^2 = \sum_M^\infty (\log|f_N(n)|)/n^2 \le$$

$$\le 4\int_M^\infty (\log|f_N(x)|/x^2)dx = 4\int_M^\infty\int_0^N \log\left(\frac{x^2}{t^2} - 1\right)d\nu(t)\,\frac{dx}{x^2}.$$

The last expression, after doing its inner integral by parts, becomes

$$4\nu(N)\int_M^\infty \log\left(\frac{x^2}{N^2} - 1\right)\frac{dx}{x^2} + 4\int_M^\infty\int_0^N \frac{2}{x^2 - t^2}\cdot\frac{\nu(t)}{t}\,dt dx.$$

The first term is $4\,\frac{\nu(N)}{N}\int\limits_{M/N}^{\infty}\log(\xi^2-1)\,\frac{d\xi}{\xi^2}\le$

$$\le\frac{4\nu(N)}{N}\int\limits_{\sqrt{2}}^{\infty}\log(\xi^2-1)\,\frac{d\xi}{\xi^2}\le(4e\alpha+o(1))\log(3+2\sqrt{2})$$

with $o(1)$ tending to zero as $N\to\infty$. The second is

$$4\int\limits_0^N\frac{\nu(t)}{t}\log\left|\frac{t+M}{t-M}\right|\frac{dt}{t}\le 4\int\limits_0^M\frac{\nu(t)}{t}\log\left|\frac{t+M}{t-M}\right|\frac{dt}{t}=$$

$$=4\int\limits_0^1\frac{\nu(M\tau)}{M\tau}\log\left|\frac{\tau+1}{\tau-1}\right|\frac{d\tau}{\tau}\le\pi^2(e\alpha+o(1)),$$

where $o(1)$ again tends to zero as $N\to\infty$ (because $M\ge N$ and $\int\limits_0^1\log((1+\tau)/(1-\tau))\,d\tau/\tau=\pi^2/4$). Thus,

$$\sum_N^{\infty}(\log^+|f_N(n)|)/n^2\le(\pi^2+4\log(3+2\sqrt{2}))(e\alpha+o(1)).$$

Combination of this last with (2) and (1) yields the lemma.

Theorem. *Let $f(z)$ be entire, of exponential type α, and suppose that*

$$\sum_{-\infty}^{\infty}\frac{\log^+|f(n)|}{1+n^2}=\eta.$$

Provided that α and η are both less than a certain numerical constant c_0, we have, for all z,

$$|f(z)|\le C_{\alpha,\eta}e^{\kappa(\alpha+\eta)|z|}$$

with a numerical constant κ, and $C_{\alpha,\eta}$ depending on α and η, but not on f.

P r o o f: Suppose to begin with that $f(z)$ is even and has only real zeros, and that $f(0)=1$. Then the condition in the hypothesis makes $\sum_{1}^{\infty}(\log^{+}|f(n)|)/n^{2}\le 2\eta$, so by the lemma, $\sum_{1}^{\infty}(\log^{+}|f_N(n)|)/n^{2}\le C\alpha+2\eta+\varepsilon_N$ (with $\varepsilon_N\to 0$ as $N\to\infty$) for the polynomials $f_N(z)$ considered there. Thence,

$$\sum_{-\infty}^{\infty}(\log^{+}|f_N(n)|)/(1+n^{2})\le 2C\alpha+4\eta+2\varepsilon_N,$$

so, if $2C\alpha+4\eta$ is < the number η_0 figuring in the theorem cited in the introduction we have by that result, for sufficiently large N,

$$|f_N(z)|\le Ke^{(2C\alpha+4\eta+2\varepsilon_N)k|z|}$$

with K depending only on $2C\alpha+4\eta+2\varepsilon_N$. On making $N\to\infty$, we get an estimate of the desired form for $f(z)$.

When $f(z)$ is even and one at the origin, but with *complex* zeros, we can form the even function $g(z)$ with *real* zeros having the same moduli as those of f, and $g(0)=1$. Then $g(z)$ is of exponential type and indeed $|f(z)|\le|g(i|z|)|$, whereas $|g(x)|\le|f(x)|$ on $\mathbf{R}$. The result just obtained therefore implies that $g(z)$, and *hence* $f(z)$, satisfies the inequality in question when the hypothesis holds for f.

In the *general* case, we may take $f(z)$ to be *real* on $\mathbf{R}$ since $f(z)+\overline{f(\bar z)}$ and $(f(z)-\overline{f(\bar z)})/2i$ both have that property. Then, corresponding to any $\eta>0$, one has a constant M_η such that $\sum_{-\infty}^{\infty}(\log u(n))/(1+n^2)<3\eta$ for $u(z)$ equal either to $1+z^2(f(z)+f(-z))^2/M_\eta$ or to $1+(f(z)-f(-z))^2/M_\eta$ whenever the hypothesis holds for f (for details, see [1], pp. 519-522). Both of these functions $u(z)$ are entire, of exponential type $\le 2\alpha$, even, and 1 at the origin, so, when η and α are small enough, our estimate holds *for them*. One of the same form then holds for $f(z)$. Done.

2. A weak multiplier theorem

It is better to call the result of this § a

Lemma. *Let $f(z)$ be entire, of exponential type α, with* $\int_{-\infty}^{\infty}(\log^{+}|f(x)|/(1+x^{2}))dx<\infty$. *If α is small enough, there is an entire function $\varphi(z)\not\equiv 0$ of exponential type $\le\pi$ with $|\varphi(x)|$ and $|f(x)\varphi(x)|$ bounded on* $\mathbf{R}$.

P r o o f. By Fourier analysis and duality. We first reduce our situation to one involving an entire function $g(z)$ of modulus ≥ 1 on $\mathbf{R}$, having all its zeros in $\mathcal{I}z < 0$. For that purpose, let $G(z) = 1 + f(z)\overline{f(\bar{z})}$; this function is entire, of exponential type $\leq 2\alpha$, and ≥ 1 (sic!) on $\mathbf{R}$. Also, $\int_{-\infty}^{\infty} (\log G(x) / (1 + x^2))dx < \infty$, so by a well known theorem of Akhiezer ([1], pp. 55-58; [4], pp. 125, 132; [5], p. 567) one can write $G(z) = g(z)\overline{g(\bar{z})}$ with $g(z)$ entire, of exponential type $\leq \alpha$ (the *same* in the upper and lower half planes, see [1], p. 66), having all its zeros in $\mathcal{I}z < 0$. Since $|g(x)| = \sqrt{1 + |f(x)|^2}$ on $\mathbf{R}$, the lemma will follow if we can get an entire function $\varphi(z) \not\equiv 0$ of exponential type $\leq \pi$ with $|\varphi(x)g(x)|$ bounded on $\mathbf{R}$.

We have $\int_{-\infty}^{\infty} (\log |g(x)| / (1 + x^2))\, dx < \infty$, so for $\mathcal{I}z > 0$,

$$\log |g(z)| \leq \alpha\, \mathcal{I}z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathcal{I}z \log |g(t)|}{|z - t|^2}\, dt$$

and similarly, for $x \in \mathbf{R}$,

$$\log |g(x)| \leq \alpha + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log |g(t + i)|}{(x - t)^2 + 1}\, dt$$

(see [1], pp. 38 and 47-52; [4], pp. 92,93; [5], p. 311). Substituting $x = n$ into the second formula, dividing by $1 + n^2$ and adding, one finds with the help of the first formula and the relation $\sum_{-\infty}^{\infty} 1 / ((t - n)^2 + 1)(n^2 + 1) \leq \text{const} / (t^2 + 1)$ that

$$\sum_{-\infty}^{\infty} \frac{\log |g(n)|}{1 + n^2} < \infty, \tag{3}$$

$\log |g(t)|$ being ≥ 0 on $\mathbf{R}$.

Denote the zeros of $g(z)$ by λ_k; we have $\mathcal{I}\lambda_k < 0$ for each k, and the λ_k may without loss of generality be taken as distinct. If indeed that is not so, we may split each multiple zero λ_k of $g(z)$ into simple ones very close to it (by adding different small negative imaginary quantities to λ_k without, however, altering the corresponding exponential factors in g's Hadamard factorization) and then, after multiplication by a suitable constant, the new function will be in modulus $\geq |g(x)|$ on $\mathbf{R}$ and still satisfy (3). That new function can then be used instead of $g(z)$ during the following discussion.

I say that if α (the type of $f(z)$) is *small* enough, the $e^{i\lambda_k t}$ *cannot be complete* in $L_2(-\pi, \pi)$. For each λ_k, $e^{i\lambda_k t}$ has, on $(-\pi, \pi)$, the Fourier series

$$\sum_{-\infty}^{\infty} (-1)^n \frac{\sin \pi\lambda_k}{\pi} \cdot \frac{1}{\lambda_k - n} e^{int},$$

so if the exponentials *were* complete in $L_2(-\pi, \pi)$, the functions of n equal to $(-1)^n/(\lambda_k - n)$, *and hence those equal to* $1/(\lambda_k - n)$ would, by Parseval's theorem, be complete in $\ell_2(\mathbf{Z})$.

But that cannot be true when α is small. Otherwise there would be a sequence of finite sums $s_r(n)$, each of the form $\sum_k a_k/(\lambda_k - n)$, with, in $\ell_2(\mathbf{Z})$, $s_r(n) \underset{r}{\to} \delta(n)$ (the Kronecker δ, equal to 1 for $n = 0$ and to 0 for $n \neq 0$). Each $s_r(n)$ can be expressed as $g_r(n)/g(n)$ with $g_r(z)$ an entire function of exponential type $\leq \alpha$; $g_r(z)$ is obtained by dividing $g(z)$ by the factors $\lambda_k - z$ corresponding to the denominators in the sum $s_r(n)$ and then multiplying the quotient by other linear factors, fewer in number. We thus have

$$\sum_{-\infty}^{\infty} |(g_r(n)/g(n)) - \delta(n)|^2 \underset{r}{\to} 0.$$

This implies first of all that $|g_r(n)| \leq \text{const} \cdot |g(n)|$ on $\mathbf{Z}$ and then, by dominated convergence, the definition of $\delta(n)$ and (3), that

$$\sum_{-\infty}^{\infty} \frac{\log^+ |g_r(n)/g(0)|}{1+n^2} \underset{r}{\to} 0.$$

For small $\alpha \geq 0$, this and the theorem of §1 imply that $|g_r(z)| \leq K_\gamma |g(0)| e^{\kappa\gamma|z|}$ when r is large. Here, κ is a numerical constant, γ may be taken to be any number $> \alpha$, and K_γ depends only on γ. A subsequence of the $g_r(z)$ thus tends to an entire function $h(z)$ of exponential type $\leq \kappa\gamma$ with $h(n) = \delta(n)g(0)$ on $\mathbf{Z}$. That, however, is *impossible* for $\kappa\gamma < 1/e$, since $h(z)/g(0)$, although 1 at 0, would vanish at every $n \neq 0$ in $\mathbf{Z}$ (see proof of lemma in §1). This means that for small α, the $e^{i\lambda_k t}$ are not complete in $L_2(-\pi, \pi)$, as claimed.

That being the case, we have a *non-zero* $\Psi(t) \in L_2(-\pi, \pi)$ with $\int_{-\pi}^{\pi} e^{i\lambda_k t}\Psi(t)\,dt = 0$ at each λ_k. The function $\psi(z) = \int_{-\pi}^{\pi} e^{izt}\,\Psi(t)dt$, entire, of exponential type $\le \pi$, and $\not\equiv 0$, thus vanishes at each zero of $g(z)$, so the ratio $\varphi(z) = \psi(z)/g(z)$ is entire, and of exponential type by a theorem of Lindelöf ([1], p. 22). This ratio is *bounded* on $\mathbf{R}$, for $\psi(x)$ *clearly* is, and $|g(x)| \ge 1$ there. We can thence conclude that $\varphi(z)$ is in fact of exponential type $\le \pi$ (see [4], p. 127 or [5], pp. 207-208, p. 315 and p. 605). The product $\varphi(x)g(x) = \psi(x)$ *is* bounded for real x, so, since $\psi(x) \not\equiv 0$, we are done.

R e m a r k: The following observations (prompted by a question of Peter Jones) really fall outside the scope of the present discussion, but their inclusion here is perhaps nevertheless worthwhile.

Given any $W(n) \ge 1$ *such that* $\sum_{-\infty}^{\infty} (\log W(n)/(1+n^2)) < \infty$, *there are* c_n, *not all zero, with* $\sum_{-\infty}^{\infty} |c_n| W(n) < \infty$ *(and hence in particular* $|c_n| \le \text{const}/W(n)$*), such that* $\sum_{-\infty}^{\infty} c_n e^{in\lambda}$ *vanishes on an interval* $-h \le \lambda \le h$, *where* $h > 0$.

Verification of this fact uses an idea from the proof just given. There is no loss of generality in supposing that $W(n) \to \infty$ for $n \to \pm\infty$, for otherwise one may replace $W(n)$ by $(1+n^2)W(n)$ in what follows.

When $h > 0$ is *small enough*, finite linear combinations of the $e^{i\lambda n}/W(n)$ with $-h \le \lambda \le h$ *cannot be uniformly dense* in $\mathcal{C}_0(\mathbf{Z})$. Otherwise there would be a sequence of finite sums $g_r(z)$, each of the form $\sum_{-h\le\lambda\le h} a_\lambda e^{i\lambda z}$, with $g_r(n)/W(n)$ tending *uniformly on* $\mathbf{Z}$ to $\delta(n)$ (the Kronecker δ-function) as $r \to \infty$. From this one arrives at a contradiction for small $h > 0$ by arguing just as in the above proof but using the condition on $\log W(n)$ instead of (3).

Because $W(n) \to \infty$ as $n \to \pm\infty$, the result just found gives us by duality a sequence of b_n, not all zero, with $\sum_{-\infty}^{\infty} |b_n| < \infty$ and $\sum_{-\infty}^{\infty} (e^{i\lambda n} / W(n))\, b_n = 0$ for $-h \le \lambda \le h$. Our statement thus holds with $c_n = b_n / W(n)$.

Suppose now that we *have* such a sequence $\{ c_n \}$. Let us form a complex measure μ on $\mathbf{Z}$ by putting $\mu(\{ n \}) = c_n$. On taking the convolution of μ with the function $\Delta(x) = \max(1 - 4|x|, 0)$, one obtains a non-zero $u(x)$ in $L_1(-\infty, \infty)$, *vanishing* on each interval $n + \frac{1}{4} \le x \le n + \frac{3}{4}$, $n \in \mathbf{Z}$, and with $|u(x)| \le \text{const} / W(n)$ if $|x - n| \le \frac{1}{4}$ for such n.

The Fourier transform $\int_{-\infty}^{\infty} e^{i\lambda x} u(x)dx$ vanishes *also*, for $-h \le \lambda \le h$.

No regularity is required of $W(n)$ for these results, beyond the condition that $\sum_{-\infty}^{\infty} (\log W(n) / (1 + n^2)) < \infty$.

One can give a *necessary and sufficient* condition on $W(n) \ge 1$ for the conclusion of the above statement to hold. That condition is not altogether explicit, and we limit our discussion to functions $W(n)$ tending to ∞ as $n \to \pm\infty$ in order to save time.

Given any $h > 0$ we let $W_h(z)$ denote, for each complex z, the supremum of $|\varphi(z)|$ for the entire functions $\varphi(z)$ of exponential type $\le h$, bounded on the real axis and with $|\varphi(n) / W(n)| \le 1$ for $n \in \mathbf{Z}$.

Provided that $W(n) \ge 1$ tends to ∞ as $n \to \pm\infty$, a sequence of c_n, not all zero, with $\sum_{-\infty}^{\infty} |c_n| W(n) < \infty$ and $\sum_{-\infty}^{\infty} c_n e^{in\lambda}$ vanishing on an interval of positive length exists if and only if $\sum_{-\infty}^{\infty} (\log W_h(n) / (1 + n^2)) < \infty$ for sufficiently small values of $h > 0$.

Proof of the *sufficiency* is like that of our first observation; it suffices to note that the functions $g_r(z)$ figuring therein actually satisfy (by definition!) the relation $|g_r(n)| \le \text{const} \cdot W_h(n)$ for $n \in \mathbf{Z}$.

The *necessity* follows from classical results. If a sequence $\{ c_n \}$ having the stipulated properties exists, the function $\sum_{-\infty}^{\infty} c_n e^{ina} e^{in\lambda}$ will, for suitable choice of the parameter a, vanish on an interval $-h \le \lambda \le h$, $h > 0$; finite linear combinations of the $e^{i\lambda n} / W(n)$ with λ from that interval are therefore *not* uniformly dense in $\mathcal{C}_0(\mathbf{Z})$. A result of Mergelian ([1], p. 174) thence implies (a fortiori!) that $\int_{-\infty}^{\infty} (\log W_h(x) / (1 + x^2))\, dx < \infty$. From this it is easy to deduce by an argument like the one on pp. 523-524 of [1] that $\sum_{-\infty}^{\infty} (\log W_h(n) / (1 + n^2)) < \infty$. (Cf. also the proof of (11) in §4 below and then the one for (3).)

3. A lemma about Poisson integrals

If, in the lemma of §2, one could take the type α of $f(z)$ to be *arbitrarily large*, we would *have* the theorem on the multiplier of Beurling and Malliavin. Such an extension is possible. For it, we will need a result about the Poisson integral

$$U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{I}z\, U(t)}{|z - t|^2}\, dt$$

formed, for $\mathcal{I}z > 0$, from a *positive* function $U(t) \not\equiv 0$ with $\int_{-\infty}^{\infty} (U(t) / (1 + t^2))dt < \infty$.

For each $x \in \mathbf{R}$, the ratio $U(x + iy) / y$ is, when $y > 0$, a *strictly decreasing* function of y, tending (by dominated convergence) to 0 as $y \to \infty$. Given any fixed $a > 0$, there is hence a definite $Y(x) \ge 0$ such that $U(x + iy) < ay$ for $y > Y(x)$, while $U(x + iy) \ge ay$ for $0 \le y \le Y(x)$ (it is possible that $Y(x) = 0$ if $U(t)$ vanishes at x).

The set

$$\mathcal{D}_a = \{ (x, y);\ y > Y(x) \}$$

is thus a certain domain in the upper half plane, and $\Omega_a = \{ \mathcal{I}z \ge 0 \} \sim \mathcal{D}_a$ a certain closed region lying above and on the real axis, whose interior may consist of several components. Concerning Ω_a one has, from [6], the important

Lemma. (Beurling and Malliavin, 1967). *For $a > 0$ we have*

$$\iint_{\Omega_a} \frac{dxdy}{1 + x^2 + y^2} < \infty.$$

For the reader's convenience, we sketch the proof. Fixing a consider the function $V(z) = a\mathscr{I}z - U(z)$, harmonic and > 0 in $\mathscr{D}_a$ and *zero* on its boundary $y = Y(x)$. Fix any $y_0 > 0$ for which $V(iy_0) > 0$.

For each $r > Y(0)$ there is an open arc $\sigma(r)$ of the circle $|z| = r$ lying entirely in $\mathscr{D}_a$, with endpoints on the curve $y = Y(x)$ and on opposite sides of the y-axis. The *union* of these $\sigma(r)$ is a certain domain $\mathscr{D} \subseteq \mathscr{D}_a$ (perhaps properly). Put $\Omega = \{ \mathscr{I}z \geq 0 \} \sim \mathscr{D}$; then $\Omega \supseteq \Omega_a$ and we proceed to show that $\iint_\Omega dxdy / (1 + x^2 + y^2) < \infty$, which implies the lemma.

If $R > y_0$, we denote by $\mathscr{D}(R)$ the part of $\mathscr{D}$ lying *within* the arc $\sigma(R)$, making $iy_0 \in \mathscr{D}(R)$. The harmonic function $V(z)$ is $\leq aR$ on $\sigma(R)$ and *zero* on the rest of $\partial\mathscr{D}(R)$, so we have

$$V(iy_0) \leq aR\omega_R(\sigma(R), iy_0), \tag{4}$$

where $\omega_R(\ ,\ )$ is harmonic measure for $\mathscr{D}(R)$.

Writing $r\theta(r)$ for the *length* of $\sigma(r)$, we have by a formula of Ahlfors and Carleman ([7], p. 102),

$$\omega_R(\sigma(R), iy_0) \leq \text{const.}\exp\left(-\pi \int_{y_0}^{R} \frac{dr}{r\theta(r)} \right).$$

The endpoints of each arc $\sigma(r)$ are of the form $re^{i\varphi(r)}$, $re^{i(\pi - \psi(r))}$, with $\varphi(r)$ and $\psi(r)$ both ≥ 0 and $< \pi/2$. Thence, $\theta(r) = \pi - \varphi(r) - \psi(r)$, and we can use the relation $1/\theta(r) \geq 1/\pi + (\varphi(r) + \psi(r))/\pi^2$ in estimating the integral on the right. In that way it is found that

$$\omega_R(\sigma(R), iy_0) \leq \frac{\text{const.}y_0}{R} \exp\left(-\frac{1}{\pi} \int_{y_0}^{R} \frac{(\varphi(r) + \psi(r))}{r} dr \right).$$

The integral here is just $\iint_{\Omega \cap \{y_0 < |z| < R\}} dxdy / (x^2 + y^2)$, so if $\iint_\Omega dxdy / (1 + x^2 + y^2) = \infty$, we must have $\omega_R(\sigma(R), iy_0) = o(1/R)$ for $R \to \infty$. That, substituted into (4), gives $V(iy_0) = 0$, a contradiction. We are done.

4. Construction of two majorants

Take now any fixed entire function $F(z)$ of exponential type with $|F(x)| \geq 1$ on $\mathbf{R}$ and $\int_{-\infty}^{\infty} (\log|F(x)|/(1+x^2))dx < \infty$; $\log|F(x)|$ is then continuously differentiable and indeed real analytic at the points $x \in \mathbf{R}$. There is no loss of generality in supposing that all the zeros of $F(z)$ lie in $\mathcal{I}z < 0$ ([1], p. 54; [4], p.90), and that property we henceforth *assume*.

We first show how to construct, for any $h > 0$, a *majorant* $W(x)$ of $|F(x)|$ with $|\log W(x) - \log W(x')| \leq h|x - x'|$ on $\mathbf{R}$ and also $\int_{-\infty}^{\infty} (\log W(x)/(1+x^2))\, dx < \infty$.

Start by taking the open set

$$\mathcal{O} = \{ x \in \mathbf{R};\ \log|F(\xi)| - \log|F(x)| > h(\xi - x) \text{ for some } \xi > x \}$$

outside of which $d \log|F(x)|/dx$ is everywhere $\leq h$; $\mathcal{O}$ is a finite or countable union of disjoint open intervals. All of those must have finite length, for otherwise there would be a sequence of ξ_k tending to ∞ with $\liminf_{k \to \infty} (\log|F(\xi_k)|)/\xi_k \geq h$, and that would contradict the known fact that functions $F(z)$ of the kind under consideration here have zero exponential growth along the real axis ([1], p. 174; [4], p. 97; [5], p. 315).

We thus have $\mathcal{O} = \bigcup_k I_k$ with the disjoint finite intervals $I_k = (a_k, b_k)$. Their formation is best visualized by imagining parallel rays of light, all of slope h, shining *downwards* upon the graph of $\log|F(x)|$ vs x. Certain disjoint portions of that graph will thus be *cast in shadow*, and the intervals I_k lie precisely *under* them.

From this observation it is clear that for each $I_k = (a_k, b_k)$ we have

$$\log|F(b_k)| = \log|F(a_k)| + h(b_k - a_k). \tag{5}$$

Let us now define a function $\omega_+(x)$ by putting

$$\omega_+(x) = \begin{cases} \log|F(x)| & \text{for } x \notin \mathcal{O}, \\ \log|F(a_k)| + h(x - a_k) & \text{if } a_k < x < b_k. \end{cases}$$

Then $\omega_+(x)$ is continuous, piecewise smooth, and $\geq \log|F(x)|$ on $\mathbf{R}$. All the discontinuities of $\omega'_+(x)$ are at the *left endpoints* a_k of the intervals of $\mathcal{O}$; elsewhere, $\omega'_+(x)$ exists and is $\leq h$. At any a_k, $\omega'_+(a_k-)$ and $\omega'_+(a_k+)$ both exist, with the *former* equal to the derivative of $\log|F(x)|$ at a_k and hence $\leq h$; the *latter* is just equal to h.

We have, besides, the

Lemma.

$$\int_{-\infty}^{\infty} \frac{\omega_+(x)}{1+x^2}\, dx < \infty.$$

P r o o f: Because $F(z)$ has all its zeros in $\mathscr{I}z < 0$, we may just as well assume $|F(x+iy)|$ to be an *increasing* function of $y \ge 0$ for each real x; that will in any event be so if we replace $F(z)$ by $e^{-icz}F(z)$ with c sufficiently large > 0. (Verification by logarithmic differentiation of the Hadamard product; see [4], p. 226; [5], p. 457.)

We have $\int_{\mathbb{R}\sim\mathcal{O}} (\omega_+(x)/(1+x^2))\, dx = \int_{\mathbb{R}\sim\mathcal{O}} (\log|F(x)|/(1+x^2))\, dx < \infty$, so it is only necessary to show that

$$\int_{\mathcal{O}} \frac{\omega_+(x)}{1+x^2}\, dx < \infty .$$

For that purpose, we apply the lemma of §3 to the function

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathscr{I}z \log|F(t)|}{|z-t|^2}\, dt,$$

harmonic and (without loss of generality) strictly positive in $\mathscr{I}z > 0$, noting that for suitable choice of the number $A > 0$, we have in that half plane

$$\log|F(z)| = A\mathscr{I}z + U(z) \tag{6}$$

([1], p. 47; [4], p. 92; [5], p. 311).

Consider any interval $I_k = (a_k, b_k)$. Since $\log|F(b_k+iy)|$ increases for $y > 0$, we have, for

$$y_k = \frac{1}{2A}\log|F(b_k)|, \tag{7}$$

$$U(b_k + iy_k) \ge \frac{1}{2}\log|F(b_k)| = Ay_k$$

by (6). From (5), we have $Ay_k \ge \frac{1}{2}h(b_k - a_k)$, and Harnack's theorem hence gives us a constant c depending on h and A with

$$U(x + iy_k) \geq cAy_k \text{ for } a_k \leq x \leq b_k.$$

The rectangle of height y_k with base on the interval I_k therefore belongs to the set Ω_a figuring in the lemma of §3 if a is taken equal to cA. By that lemma, we thus have

$$\sum_k \int_0^{y_k} \int_{a_k}^{b_k} \frac{dxdy}{1 + x^2 + y^2} < \infty.$$

When $a_k \geq 0$, the denominator in the corresponding integral appearing on the left is, by (7), $\leq 1 + b_k^2 + (\log |F(b_k)|)^2 / 4A^2 \leq 1 + \text{const.} b_k^2$, $F(z)$ being of exponential type. Referring again to (7), we see that

$$\sum_{a_k \geq 0} \frac{(b_k - a_k) \log |F(b_k)|}{1 + b_k^2} < \infty. \tag{8}$$

This, and (5) imply in particular that $\sum_{a_k \geq 0} (b_k - a_k)^2 / (1 + b_k^2) < \infty$, but then there can only be a *finite number* of $b_k \geq 1$ with $b_k > 2a_k$. Thence, by (8),

$$\sum_{b_k \geq 1} (b_k - a_k) \log |F(b_k)| / (1 + a_k^2) < \infty,$$

so, by the definition of $\omega_+(x)$, we have

$$\sum_{b_k \geq 1} \int_{I_k} \frac{\omega_+(x)}{1 + x^2} dx < \infty.$$

The corresponding sum over the I_k with $a_k \leq -1$ is seen in like manner to be convergent, and the remaining I_k (if there are any) lie in [-1,1] on which $\omega_+(x)$ is certainly bounded. Therefore,

$$\int_{\mathcal{O}} \frac{\omega_+(x)}{1 + x^2} dx = \sum_k \int_{I_k} \frac{\omega_+(x)}{1 + x^2} dx < \infty.$$

The lemma is proved.

Our next step is to obtain a continuous piecewise smooth $\omega_-(x) \geq \log |F(x)|$ with $\omega'_-(x-)$ and $\omega'_-(x+)$ both $\geq -h$, by a construction analogous to the one of $\omega_+(x)$. The function $\omega_-(x)$ may even be gotten by applying the latter procedure directly to

$\log |F(-x)|$ instead of $\log |F(x)|$ and afterwards changing the sign of x. It is readily seen that

$$\int_{-\infty}^{\infty} \frac{\omega_-(x)}{1+x^2}\, dx < \infty;$$

for that it suffices to repeat the proof of the lemma with $\overline{F(-\bar{z})}$ in place of $F(z)$.

With $\omega_+(x)$ and $\omega_-(x)$ at hand, one takes finally

$$\omega(x) = \max\,(\omega_+(x), \omega_-(x))$$

and puts

$$W(x) = e^{\omega(x)}.$$

One then has the

Theorem. *For the function $W(x)$ just defined, we have*

$$|F(x)| \le W(x), \quad x \in \mathbf{R}\ ,$$

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\, dx < \infty\ ,$$

and

$$|\log W(x) - \log W(x')| \le h|x - x'| \ \text{ on } \ \mathbf{R}\ .$$

P r o o f: The first two relations follow directly from the corresponding properties of $\omega_+(x)$ and $\omega_-(x)$.

The last one is a geometrically evident consequence of the inequalities $\omega'_+(x) \le h$, $\omega'_-(x) \ge -h$ and the definition of $\omega(x)$; let us nevertheless give a formal proof in order that there may be no doubt.

For any given x_0, $\omega(x_0) = \max(\omega_+(x_0), \omega_-(x_0))$ is either *equal* to $\log |F(x_0)|$ or $>$ that quantity. In the former case, $\omega_-(x_0)$ and $\omega_+(x_0)$ must both *equal* $\log |F(x_0)|$, so x_0 lies outside the two open sets where either of the first two functions is $>$ the last one. That makes $d \log |F(x)| / dx$ both $\le h$ and $\ge -h$ at x_0 (see the description of $\mathcal{O}$ above). At such a point x_0, $\omega'_-(x+)$ has *the same* value as the preceding derivative, while $\omega'_+(x+)$ is *at least* as large, being, however, $\le h$. We see that $\omega'(x_0+)$ exists and lies between $-h$ and h.

When $\omega(x_0) > \log|F(x_0)|$ and $\omega(x)$ coincides either with $\omega_+(x)$ or with $\omega_-(x)$ on a *neighbourhood* of x_0, $\omega'(x_0)$ exists and is either h or $-h$. The remaining possibility here is that $\omega(x) = \omega_-(x)$ for $x_0 - \eta < x \le x_0$ and $\omega(x) = \omega_+(x)$ for $x_0 \le x < x_0 + \eta$, where $\eta > 0$. Then $\omega'(x_0+)$ exists and is equal to h.

The continuous function $\omega(x)$ thus has a right-hand derivative $\omega'(x+)$ at every point x, with $-h \le \omega'(x+) \le h$. This implies that $|\omega(x) - \omega(x')| \le h|x - x'|$ on $\mathbf{R}$. We are done.

The construction just carried out enables us to do another one, interesting in its own right.

Theorem. *If $F(z)$ is entire, of exponential type, with $|F(x)| \ge 1$ on* $\mathbf{R}$ *and* $\int_{-\infty}^{\infty} (\log|F(x)|/(1+x^2))dx < \infty$, *there are entire functions $H(z)$ of arbitrarily small exponential type with $|H(x)| \ge |F(x)|$ on* $\mathbf{R}$ *and* $\int_{-\infty}^{\infty} (\log|H(x)|/(1+x^2))dx < \infty$.

P r o o f: Fixing any $h > 0$, we form the majorant $W(x)$ of $|F(x)|$ having the properties ensured by the preceding theorem and show how to obtain an entire function $H(z)$ of exponential type $\le 2h$ fulfilling the above condition on $\log|H(x)|$, with $|H(x)| \ge W(x)$ on $\mathbf{R}$. The procedure has been used elsewhere ([8], pp. 302-303), and we explain it here for the reader's convenience.

Put

$$\Omega(x) = \pi(x^2 + 1)(W(x))^2,$$

and then define $M(z_0)$ (for any complex z_0) as the *supremum* of $|f(z_0)|$ for the entire functions $f(z)$ of exponential type $\le h$, bounded on $\mathbf{R}$, with

$$\int_{-\infty}^{\infty} (|f(x)|^2/\Omega(x))dx \le 1. \tag{9}$$

For such f, we have

$$\log|f(z)| \le h|\mathcal{I}z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\mathcal{I}z|\log|f(t)|}{|z-t|^2}dt \tag{10}$$

([1], pp. 47-52; [4], pp. 92-93; [5], p. 311), so by (9) and the inequality between arithmetic and geometric means,

$$\log|f(z)| \le h|\mathcal{I}z| + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\mathcal{I}z|}{|z-t|^2}\log\left(\frac{\Omega(t)}{\pi}\right)dt,$$

provided that $|\mathcal{I}z| \ge 1$.We have, furthermore,

$$\log|f(x)| \le h + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(t+i)|}{(x-t)^2+1}dt$$

for $x \in \mathbf{R}$, so by the preceding relation and Fubini's theorem,

$$\log|f(x)| \le 2h + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log(\Omega(t)/\pi)}{(x-t)^2+4}dt, \qquad x \in \mathbf{R}\ .$$

This holds for all f of the kind under consideration satisfying (9), so by the definition of M,

$$\log M(x) \le 2h + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log((t^2+1)(W(t))^2)dt}{(x-t)^2+4}$$

on $\mathbf{R}$, and thence

$$\int_{-\infty}^{\infty} \frac{\log M(x)}{1+x^2}dx < \infty. \tag{11}$$

Substituting $\log|f(t)| \le \log M(t)$ in the right side of (10), we see by the last relation that the collection of our entire functions $f(z)$ satisfying (9) is a normal family in the complex plane. That means that to get the supremum $M(z_0)$ we do not need to use the *whole* collection of those functions f, but may take any *subset* of it, *dense* therein in the

$$\text{norm}\left(\int_{-\infty}^{\infty} (|f(t)|^2/\Omega(t))\,dt\right)^{1/2}.$$

One can now show that $(M(x))^2$ coincides on $\mathbf{R}$ with an entire function of exponential type $\le 2h$. Take any countable subset of our collection of functions $f(z)$, *dense* therein in the above norm. By applying Schmidt's orthogonalization process to that subset, we arrive at a *complete* sequence $\{p_n(z)\}$ of entire functions of exponential type $\le h$ bounded on

$\mathbf{R}$, *orthonormal* for the inner product $\int_{-\infty}^{\infty} (p(x)\,\overline{q(x)} / \Omega(x))dx$. According to the observation just made, $M(x)$ can then be obtained as a supremum of the finite linear combinations of the $p_n(x)$ satisfying (9). From this, a simple computation based on Schwarz' inequality shows that

$$(M(x))^2 = \sum_n |p_n(x)|^2, \qquad x \in \mathbf{R}.$$

For each N, $G_N(z) = \sum_{n \le N} p_n(z)\overline{p_n(\bar{z})}$ is entire, bounded on $\mathbf{R}$, and of exponential type $\le 2h$. Moreover, $G_N(x) \le (M(x))^2$, so by an analogue of (10),

$$\log |G_N(z)| \le 2h|\Im z| + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z| \log M(t)}{|z-t|^2}\,dt \tag{12}$$

for all N. We see from this and (11) that the $G_N(z)$ form a normal family in the complex plane. When $N \to \infty$, they converge on $\mathbf{R}$ (to $(M(x))^2$); they hence tend everywhere to an entire function $G(z)$ also satisfying (12), and it is thence readily verified that $G(z)$ is of exponential type $\le 2h$. (To estimate the growth of $G(z)$ inside sectors of the form $|\arg z| < \delta$, $|\arg z - \pi| < \delta$, use first (12) to get uniform estimates for the $G_N(z)$ on the *boundaries* of those sectors and then apply Phragmén-Lindelöf to deduce uniform estimates *inside* them.)

Since $G(x) = (M(x))^2$, we have

$$\int_{-\infty}^{\infty} (\log G(x) / (1 + x^2))\,dx < \infty$$

by (11).

It remains finally to show that a suitable multiple of $(M(x))^2$ dominates $W(x)$; *that is where we use the property*

$$|\log W(x) - \log W(x')| \le h|x - x'|. \tag{13}$$

Fixing any $x_0 \in \mathbf{R}$, we take the test function

$$f_0(z) = \cos h\sqrt{(z-x_0)^2 - R_0^2}$$

with $R_0 = (\log W(x_0)) / \sqrt{2}h$. Because the Taylor series of $\cos w$ involves only even powers of w, $f_0(z)$ is entire; it is of exponential type h and bounded on the real axis. We have $\log f_0(x_0) \le (\log W(x_0)) / \sqrt{2}$, so $\log |f_0(x)| \le \log W(x)$ for $|x - x_0| \le R_0$ by (13). The same holds good when $|x - x_0| > R_0$, for then $|f_0(x)| \le 1 \le W(x)$. Thus, $|f_0(x)| \le W(x)$ on $\mathbf{R}$, so by the definition of $\Omega(x)$, $\int_{-\infty}^{\infty} (|f_0(x)|^2 / \Omega(x))dx \le 1$. Thence, $M(x_0) \ge f_0(x_0) \ge \frac{1}{2}(W(x_0))^{1/\sqrt{2}}$, and finally $G(x) = (M(x))^2 \ge \frac{1}{4}(W(x))^{\sqrt{2}}$ on $\mathbf{R}$, making $4G(x) \ge W(x) \ge |F(x)|$ there since $|F(x)| \ge 1$. The theorem follows on taking $H(z) = 4G(z)$.

5. The theorem on the multiplier

Theorem. (Beurling and Malliavin, [3]). *If $f(z)$ is entire, of exponential type, and $\int_{-\infty}^{\infty} (\log^+|f(x)| / (1 + x^2))\, dx < \infty$, there is, for any $\eta > 0$, an entire function $\psi(z) \not\equiv 0$ of exponential type $\le \eta$ with $(1 + |f(x)|)\psi(x)$ bounded on $\mathbf{R}$.*

P r o o f: Let $G(z) = 1 + f(z)\overline{f(\bar{z})}$; then $G(z)$ is entire, of exponential type and $G(x) = 1 + |f(x)|^2 \ge 1$ on $\mathbf{R}$. Also, $\int_{-\infty}^{\infty} (\log G(x) / (1 + x^2))\, dx < \infty$, so the theorem of Akhiezer already used in §2 gives us an entire function $g(z)$ of exponential type with all its zeros in $\Im z < 0$, such that $G(z) = g(z)\overline{g(\bar{z})}$. Given $\eta > 0$, we put

$$F(z) = g(\pi z / \eta);$$

$F(z)$ satisfies the hypothesis of the second theorem in §4 so there is, for any $h > 0$, an entire function $H(z)$ of exponential type $\le 2h$ with $|H(x)| \ge |F(x)|$ on $\mathbf{R}$ and $\int_{-\infty}^{\infty} (\log |H(x)| / (1 + x^2))dx < \infty$.

Taking $h > 0$ small enough we get from the lemma of §2 an entire function $\varphi(z) \not\equiv 0$ of exponential type $\le \pi$ with $H(x)\varphi(x)$ bounded on $\mathbf{R}$. The product $F(x)\varphi(x)$ is thus bounded on $\mathbf{R}$ and the desired conclusion holds with $\psi(z) = \varphi(\eta z / \pi)$. We are done.

R e m a r k. Beurling and Malliavin also proved in [3] that if $W(x) \geq 1$ has the last two of the properties enumerated in the first theorem of §4, there are entire functions $\psi(z) \not\equiv 0$ of arbitrarily small exponential type with $W(x)\psi(x)$ bounded on $\mathbf{R}$. At the end of [8] is was shown that this result *follows* from the theorem just proved; for that purpose the construction made to establish the second theorem of §4 was used. Now we see by the first theorem of §4 that the result just stated also *implies* the theorem of the present §. These two results are thus equivalent.

References

1. *P. Koosis*, The Logarithmic Integral. I. Cambridge University Press, (1988).
2. *P. Koosis*, Weighted polynomial approximation on arithmetic progressions of intervals or points.— Acta Math. (1966), v. 116, pp. 223–277.
3. *A. Beurling and P. Malliavin*, On Fourier transforms of measures with compact support.— Acta Math. (1962), v. 107, pp. 291–309.
4. *R. Boas*, Entire Functions. Academic Press, New York (1954).
5. *B. Ia. Levin*, Raspredelenie kornìei tselykh funktsii. Gostekhizdat, Moscow (1956).
6. *A. Beurling and P. Malliavin*, On the closure of characters and the zeros of entire functions.— Acta Math. (1967), v. 118, pp. 79–93.
7. *W. Fuchs*, Topics in the Theory of Functions of one Complex Variable. Van Nostrand, Princeton (1967).
8. *P. Koosis*, Harmonic estimation in certain slit regions and a theorem of Beurling and Malliavin. — Acta Math. (1979), v. 142, pp. 275–304.

Соотношение между двумя результатами о целых функциях экспоненциального типа

Пол Кусис

Теорема о полиномах, сформулированная в начале статьи, распространена на случай целых функций малого экспоненциального типа. Этот результат использован для вывода теоремы Берлинга–Маллявэна о мультипликаторах. Приведена также точная характеризация порядка убывания коэффициентов Фурье ненулевой функции, исчезающей на некотором интервале.

Співвідношення між двома результатами про цілі функції експоненціального типу

Пол Кусіс

Теорему про поліноми, що сформульована на початку статті, розповсюджено на випадок цілих функцій малого експоненціального типу. Цей результат застосовано для виведення теореми Бьорлінга–Маллявена про мультиплікатори. Приведено також точну характеризацію порядку спадання коэфіцієнтів Фурь'є ненульової функції, яка зникає на деякому інтервалі.