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On a class of kinetic equations for reacting gas mixtures

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We consider a general class of kinetic equations for real gases with (possibly) multiple inelastic collisions and chemical reactions. We prove the existence, uniqueness and positivity of solutions for the Cauchy problem and obtain the conservation relations for mass, momentum and energy, the H-Theorem as well as the law of the mass action.

1. Introduction

We investigate the mathematical properties of a class of Boltzmann-type kinetic equations for a model of reacting gas composed of several species of mass points with well-defined, unique internal energy state and multi-particle (in)elastic collisions (reactions). The number of species and the multiplicity of the collisions may be arbitrary. The gas particles move freely between collisions. The gas collisions occur with energy and momentum conservation according to the laws of the classical mechanics. For the one-component gas, with elastic binary collisions, the model kinetic equations can be reduced to the classical Boltzmann equation.

Our interest in this model is due to the following thing. Certain kinetic equations for the real gas, which are important for applications, but less understood mathematically, appear to belong to our class of Boltzmann-type kinetic equations, as soon as they are written in convenient form. The main example refers to the Wang Chang and Uhlenbeck [1] as well as the Ludwig and Heil [2] equations, describing the real gas with inelastic collisions and chemical reactions, respectively. The fact that the equations introduced in Ref.[1, 2] belong to the class examined in this paper is the consequence of the point

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of view, implicitly adopted in Ref. [1, 2] (see also [3, 4]): in certain situations a real gas particle (molecule, atom, etc.) with internal structure can be considered as a, mechanical system that differs from a mass point by a succession of internal states; each internal state has a well-defined value of the energy. It becomes convenient to treat different internal states of the gas particle with internal structure, as distinct, structureless point-objects, belonging to different species, of given mass and unique internal energy state, and described by different distribution functions. Consequently we can think of the gas of particles with internal structure as a gas mixture of different mass-points, with unique internal energy, and re-write the original kinetic equations in a suitable form by re-labeling the original distribution functions (each original distribution function, describing a succession of internal states of a particle with internal structure, is replaced by a sequence of distribution functions associated to each internal state).

The aim of the present paper is to solve the Cauchy problem for the aforementioned class of reactive Boltzmann-type equations and to prove the basic global conservation relations, the H-Theorem, as well as the law of the mass action. The analysis reveals new mathematical difficulties, in comparison with the classical Boltzmann equation and other rigorous models [5-9]. The difficulties are essentially due to the presence of the internal energy. They are introduced by the reaction thresholds, and are already visible in the case of the gas model with three-body collisions (reactions). The situations with more than three-body collisions (reactions) do not introduce additional conceptual problems. However, the mathematical difficulties are better understood by investigating the general model than particular cases that might contain irrelevant details.

The plan of the paper is as follows. In the next section we introduce the class of reactive Boltzmann-type kinetic equations. The main result, Theorem 1, obtained in Section 3, proves the existence, uniqueness and positivity of solutions (with small initial data) for the Cauchy problem associated to this class of equations. The solutions are global (in time) in the case in which the endo-energetic reactions are not present at the gas processes. In the case of the simple gas with elastic binary collisions, Theorem 1 reduces to known existence results on the classical Boltzmann equation [10]. The argument of Theorem 1 follows by fixed point techniques, due to estimations based on the (local) conservation relations for mass, momentum and energy. The key estimation is given in Lemma 1. In Section 4 we prove the bulk conservation relations for mass, momentum and energy as well as the H-Theorem.

Finally, the following fact should be remarked. The probability of multiparticle collisions is zero, in some sense ([11]), in the dynamics of the classical hard sphere gas with elastic collisions (which plays an essential role in the validation of the classical Boltzmann equation). The situation seems being different in the case of the reacting gas: reaction processes *producing more that two particles* could be important to the gas evolution.

Some of the results presented here, have been announced in [12].

2. The frame

Consider a model of reacting gas without external fields, composed of $N \geq 1$, distinct species of mass-points, with one-state internal energy. Each species of gas constituents will be labeled by some simple index $k = 1, \ldots, N$. The gas particles have a free classical motion, in the whole space, between (in)elastic, instant collisions. By hypothesis, at most, $M \geq 2$ identical partners may participate in some in (out) collision (reaction) channel. During the gas processes, the particles may change their chemical nature (in particular, mass and internal energy) and velocity. It is supposed that the collisions occur with the conservation of mass, momentum and energy, respectively, according to the laws of classical mechanics. The particles internal energies enter in the energy balance.

Let $\mathcal{M} := \{ \gamma = (\gamma_n)_{1 \leq n \leq N} \mid \gamma_n \in \{ 0, 1, \ldots, M \}\}$ be a multi-index set. A certain gas collision (reaction) process can be specified by a couple $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$. Here $\alpha = (\alpha_1, \ldots, \alpha_N)$ is the "in" channel. It designates the precollision configuration, with $\alpha_n \in \{ 0, 1, \ldots, M \}$ participants of the species n, $1 \leq n \leq N$. Further, $\beta = (\beta_1, \ldots, \beta_N)$ denotes the "out" channel. It refers to the post-collision configuration, with $\beta_n \in \{ 0, 1, \ldots, M \}$ participants of the species n, $1 \leq n \leq N$. For some $\gamma \in \mathcal{M}$, the total numbers of the particles in channel γ is $|\gamma| := \sum_{n=1}^{N} \gamma_n$. The family of those species present in the channel $\gamma \in \mathcal{M}$ can be identified by $\mathcal{N}(\gamma) := \{ n \mid 1 \leq n \leq N, \gamma_n \geq 1\}$. Consequently, if $\gamma \in \mathcal{M}$, with $|\gamma| \geq 1$, for each $\mathcal{N}(\gamma)$, there are exactly γ_n identical particles of the species n, participating in γ. Their velocities will be denoted by $\mathrm{w}_{n,1}, \ldots, \mathrm{w}_{n,\gamma_n} \in \mathbf{R}^3$. Also set $\mathrm{w} = ((\mathrm{w}_{n,i})_{1 \leq i \leq \gamma_n})_{n \in \mathcal{N}(\gamma)}$, understanding that $\mathrm{w} \in \mathbf{R}^{3|\gamma|}$. By $m_n > 0$ and $E_n \in \mathbf{R}$, denote the mass and the internal energy, respectively of a mass-point of the species $n = 1, \ldots, N$.

Let $V_\gamma(\mathrm{w})$ and $W_\gamma(\mathrm{w})$ be the classical mass center velocity and the total energy, respectively, for the particles in channel γ, i.e.,

$$V_\gamma(\mathrm{w}) := \left(\sum_{n=1}^{N} \gamma_n m_n\right)^{-1} \sum_{n \in \mathcal{N}(\gamma)} \sum_{i=1}^{\gamma_n} m_n \mathrm{w}_{n,i},$$

$$W_\gamma(\mathrm{w}) := \sum_{n \in \mathcal{N}(\gamma)} \sum_{i=1}^{\gamma_n} (2^{-1} m_n \mathrm{w}_{n,i}^2 + E_n).$$

According to the previous conservation assumptions we are interested in those gas processes $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$, where

$$\sum_{n=1}^{N} m_n (\alpha_n - \beta_n) = 0, \quad V_\alpha (\mathbf{w}) = V_\beta (\mathbf{u}), \quad W_\alpha (\mathbf{w}) = W_\beta (\mathbf{u}), \tag{1}$$

with $\mathbf{w} = ((\mathbf{w}_{n,i})_{1 \le i \le \alpha_n})_{n \in \mathcal{N}(\alpha)}$ and $\mathbf{u} = ((\mathbf{u}_{n,i})_{1 \le i \le \beta_n})_{n \in \mathcal{N}(\beta)}$ defining the velocities of the particles in the channels α and β, respectively.

Suppose that one knows the transition law ([1, 2]) $K_{\alpha, \beta}$ of each reaction process (α, β). Following the standard Boltzmann procedure, we can formally write equations similar to those introduced in [1, 2]

$$\partial_t f_k + \mathbf{v} \cdot \nabla f_k = P_k (f) - S_k (f), \quad 1 \le k \le N. \tag{2}$$

The unknowns are the functions $f_k : \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}_+$, $1 \le k \le N$, where $\mathbf{R}_+ := [0, \infty)$. Here $f_k = f_k(t, \mathbf{v}, \mathbf{x})$ (t – time, $\mathbf{v}$ – velocity, $\mathbf{x}$ – position) is the distribution function for species k of mass-points and $f := (f_1, \dots, f_N)$. The collision processes are described by the nonlinear terms $P_k(f)$ and $S_k(f)$

$$P_k (f)(t, \mathbf{v}, \mathbf{x}) := \sum_{\alpha, \beta \in \mathcal{M}} \alpha_k \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} f_\beta (t, \mathbf{u}, \mathbf{x}) K_{\beta, \alpha} (\mathbf{u}, \mathbf{w}) \times$$

$$\times \delta(\mathbf{w}_{k, \alpha_k} - \mathbf{v}) \, \delta(V_\beta (\mathbf{u}) - V_\alpha (\mathbf{w})) \, \delta(W_\beta (\mathbf{u}) - W_\alpha (\mathbf{w})) \, d\mathbf{u} \otimes d\mathbf{w}, \tag{3}$$

$$S_k (f)(t, \mathbf{v}, \mathbf{x}) := \sum_{\alpha, \beta \in \mathcal{M}} \alpha_k \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} f_\alpha (t, \mathbf{w}, \mathbf{x}) K_{\alpha, \beta} (\mathbf{w}, \mathbf{u}) \times$$

$$\times \delta(\mathbf{w}_{k, \alpha_k} - \mathbf{v}) \, \delta(V_\beta (\mathbf{u}) - V_\alpha (\mathbf{w})) \, \delta(W_\beta (\mathbf{u}) - W_\alpha (\mathbf{w})) \, d\mathbf{u} \otimes d\mathbf{w}, \tag{4}$$

for all $t \ge 0$, $\mathbf{v}, \mathbf{x} \in \mathbf{R}^3$; $1 \le k \le N$. Here, $(K_{\alpha, \beta})_{(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}}$ is the family of transition functions $K_{\alpha, \beta} : \mathbf{R}^{3|\alpha|} \times \mathbf{R}^{3|\beta|} \to \mathbf{R}_+$, $\alpha, \beta \in \mathcal{M}$ and

$$f_\gamma (t, \mathbf{w}, \mathbf{x}) = \prod_{n \in \mathcal{N}(\gamma)} \prod_{i=1}^{\gamma_n} f_n (t, \mathbf{w}_{n,i}, \mathbf{x}).$$

We introduce the following general assumptions:

a) $K_{\alpha,\beta} \equiv 0$ if $|\alpha| \le 1$ or $|\beta| \le 1$.

b) If for some $\alpha, \beta \in \mathcal{M}$, $\sum_{n=1}^{N} \alpha_n m_n \neq \sum_{n=1}^{N} \beta_n m_n$, then $K_{\alpha,\beta} \equiv 0$.

c) For each $\mathbf{w}$, $\mathbf{u}$ and $n \in \mathcal{N}(\alpha)$ fixed, $K_{\alpha,\beta}(\mathbf{w},\mathbf{u})$ is invariant at the interchange of components $\mathbf{w}_{n,1}, \ldots, \mathbf{w}_{n,\alpha_n}$ of $\mathbf{w}$; a similar statement is true with respect to the interchange of the components of $\mathbf{u}$.

d) For each $a \in \mathbf{R}^3$, define the map $\mathbf{w} \to T(a)\mathbf{w}$ by setting $(T(a)\mathbf{w})_{n,i} := \mathbf{w}_{n,i} + a$, for all n, i; then $K_{\alpha,\beta}(\mathbf{w},\mathbf{u}) \equiv K_{\alpha,\beta}(T(a)\mathbf{w},\mathbf{u}) \equiv K_{\alpha,\beta}(\mathbf{w}, T(a)\mathbf{u})$, for all $a \in \mathbf{R}^3$, $(\mathbf{w},\mathbf{u}) \in \mathbf{R}^{3|\alpha|} \times \mathbf{R}^{3|\beta|}$ and $\alpha, \beta \in \mathcal{M}$.

Assumption a) excludes the "spontaneous decay" ($|\alpha| \le 1$) and the "total fusion" ($|\beta| \le 1$). Condition b) states the mass conservation during the gas processes. Moreover, c) expresses the "indistinguishability" of identical collision partners. Finally, d) claims absence of external fields.

The presence of the Dirac δ-"functions" in (3) and (4) expresses the conservation of the total energy and momentum, respectively, during collisions.

It can be easily seen that the kinetic equations introduced in [1, 2] can be written in the form (2), by redefining the distribution functions according to the remarks in the previous section.

For some channel $\gamma \in \mathcal{M}$, let

$$W_{r,\gamma}(\mathbf{w}) := W_\gamma(\mathbf{w}) - 2^{-1}\Big(\sum_{n=1}^{N} \gamma_n m_n\Big) V_\gamma(\mathbf{w})^2 - \sum_{n=1}^{N} \gamma_n E_n, \quad \mathbf{w} \in \mathbf{R}^{3|\gamma|},$$

be the corresponding mass center kinetic energy. Obviously, $W_{r,\gamma}(\mathbf{w}) \ge 0$.

We suppose that, $\forall \alpha, \beta \in \mathcal{M}$, the transition law $K_{\alpha,\beta}$ is continuous on the set $\{(\mathbf{w},\mathbf{u}) \in \mathbf{R}^{3|\alpha|} \times \mathbf{R}^{3|\beta|} \mid W_{r,\alpha}(\mathbf{w}) > 0,\ W_{r,\beta}(\mathbf{u}) > 0\}$.

We introduce the following hypothesis, extending a class of cut-off conditions for elastic binary collisions [10].

Assumption. *There are some constants $C > 0$, $0 \le q \le 1$, such that for all α, β, $\mathbf{w} \in \mathbf{R}^{3|\alpha|}$, $\mathbf{u} \in \mathbf{R}^{3|\beta|}$, we have*

$$K_{\alpha,\beta}(\mathbf{w},\mathbf{u}) \le C \frac{1 + W_{r,\alpha}(\mathbf{w})^{q/2} + W_{r,\beta}(\mathbf{u})^{q/2}}{W_{r,\alpha}(\mathbf{w})^{(3|\alpha|-5)/2} + W_{r,\beta}(\mathbf{u})^{(3|\beta|-5)/2}}. \tag{5}$$

In the rest of this section we give a meaning to (3), (4). Let $C_c(\mathbf{R}^3 \times \mathbf{R}^3)$ denote the space of continuous functions with compact support on $\mathbf{R}^3 \times \mathbf{R}^3$. For each $\tau \geq 0$, $n = 1, \ldots, N$, fixed, let $\dot{C}_{n,\tau}$ be the closure of $C_c(\mathbf{R}^3 \times \mathbf{R}^3)$ – real in the norm

$$| h |_{n\tau} = \sup \{ \exp [\tau m_n (\mathbf{x}^2 + \mathbf{v}^2)] \, | h(\mathbf{v}, \mathbf{x}) | : \mathbf{v}, \mathbf{x} \in \mathbf{R}^3 \}, \ h \in C_c(\mathbf{R}^3 \times \mathbf{R}^3).$$

Set $\dot{C}_\tau = \prod\limits_{1 \leq n \leq N} \dot{C}_{n,\tau}$, with norm $| h |_\tau := \max\limits_{1 \leq n \leq N} | h_n |_{n,\tau}$, for $h = (h_1, \ldots, h_N) \in \dot{C}_\tau$.

Let $\delta_\varepsilon : \mathbf{R} \to \mathbf{R}_+$, $\varepsilon > 0$, be an even mollifier with $\operatorname{supp} \delta_\varepsilon = [-\varepsilon, \varepsilon]$ (i.e. $\delta_\varepsilon(t) =: \varepsilon^{-1} J(t/\varepsilon)$, for some even function $J \in C_c(\mathbf{R}; \mathbf{R}_+)$, with *supp* $J = [-1, 1]$ and $\| J \|_{L^1} = 1$). Set $\delta^3_\varepsilon(y) := \delta_\varepsilon(y_1) \cdot \delta_\varepsilon(y_2) \cdot \delta_\varepsilon(y_3)$, with $y = (y_1, y_2, y_3) \in \mathbf{R}^3$. For some $\tau > 0$ and $f = (f_1, \ldots, f_N) \in \dot{C}_\tau$, define

$$P_{k\varepsilon\eta}(f)(\mathbf{v}, \mathbf{x}) := \sum_{\alpha, \beta \in \mathcal{M}} \alpha_k \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|-3}} d\mathbf{u} \otimes d\tilde{\mathbf{w}}_{(k)} \, [f_\beta(\mathbf{u}, \mathbf{x}) \times$$
$$\times K_{\beta,\alpha}(\mathbf{u}, \mathbf{w}) \, \delta^3_\varepsilon(V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w})) \, \delta_\eta(W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))]_{\mathbf{w}_{k,\alpha_k} = \mathbf{v}}, \tag{6}$$

$$R_{k\varepsilon\eta}(f)(\mathbf{v}, \mathbf{x}) := \sum_{\alpha, \beta \in \mathcal{M}} \alpha_k \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|-3}} d\mathbf{u} \otimes d\tilde{\mathbf{w}}_{(k)} \, [f_{\alpha,k}(\mathbf{w}, \mathbf{x}) \times$$
$$\{\times K_{\alpha,\beta}(\mathbf{w}, \mathbf{u}) \, \delta^3_\varepsilon(V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w})) \, \delta_\eta(W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))]_{\mathbf{w}_{k,\alpha_k} = \mathbf{v}}, \tag{7}$$

for all $\mathbf{v}, \mathbf{x} \in \mathbf{R}^3$, $1 \leq k \leq N$. Here by definition, the terms with $\alpha_k = 0$, vanish identically, $d\tilde{\mathbf{w}}_{(k)}$ is the Euclidean element of area induced by $d\mathbf{w}$ on the hyperplane $\{ \mathbf{w} \in \mathbf{R}^{3|\alpha|} : \mathbf{w}_{k,\alpha_k} = \mathbf{v} \}$, while

$$f_\beta(\mathbf{u}, \mathbf{x}) := \prod_{n \in \mathcal{N}(\beta)} \prod_{i=1}^{\beta_n} f_n(\mathbf{u}_{n,i}, \mathbf{x}),$$

$$f_{\alpha,k}(\mathbf{w}, \mathbf{x}) = \prod_{n \in \mathcal{N}(\alpha) \setminus \{k\}} \prod_{i=1}^{\alpha_n} f_n(\mathbf{w}_{n,i}, \mathbf{x}) \cdot \prod_{p=1}^{\alpha_k - 1} f_k(\mathbf{w}_{k,p}, \mathbf{x}).$$

Proposition 1. *Let* $\tau > 0$ *and* $f \in \dot{C}_\tau$.

a) For each $k = 1,\dots, N$, *there exist the limits*

$$P_k(f)(\mathbf{v}, \mathbf{x}) = \lim_{\eta \to 0} \lim_{\varepsilon \to 0} P_{k\varepsilon\eta}(f)(\mathbf{v}, \mathbf{x})$$

and

$$R_k(f)(\mathbf{v}, \mathbf{x}) = \lim_{\eta \to 0} \lim_{\varepsilon \to 0} R_{k\varepsilon\eta}(f)(\mathbf{v}, \mathbf{x}),$$

$\forall (\mathbf{v}, \mathbf{x}) \in \mathbf{R}^3 \times \mathbf{R}^3$. *Also,* $P_k(f) \in C_{k,\mu}(\mathbf{R}^3 \times \mathbf{R}^3)$ *for all* $\mu \in [0, \tau)$, *while* $\sup_{\mathbf{v},\mathbf{x}} \{ (1 + \mathbf{v}^2)^{-q/2} | R_k(f)(\mathbf{v}, \mathbf{x}) | \} < \infty$.

b) Let $h \in C(\mathbf{R}^3)$ *with* $\sup_{\mathbf{v}} (1 + \mathbf{v}^2)^{-1} | h(\mathbf{v}) | < \infty$. *Then,* $\forall \mathbf{x} \in \mathbf{R}^3$,

$$\int_{\mathbf{R}^3} h(\mathbf{v})\, P_k(f)(\mathbf{v}, \mathbf{x})\, d\mathbf{v} = \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \int_{\mathbf{R}^3} h(\mathbf{v})\, P_{k\varepsilon\eta}(f)(\mathbf{v}, \mathbf{x})\, d\mathbf{v},$$

$$\int_{\mathbf{R}^3} h(\mathbf{v}) f_k(\mathbf{v}, \mathbf{x})\, R_k(f)(\mathbf{v}, \mathbf{x})\, d\mathbf{v} = \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \int_{\mathbf{R}^3} h(\mathbf{v})\, f_k(\mathbf{v}, \mathbf{x})\, R_{k\varepsilon\eta}(f)(\mathbf{v}, \mathbf{x})\, d\mathbf{v},$$

for each $k = 1,\dots, N$.

P r o o f. Set $T_\beta(\mathbf{u}) = W_\beta(\mathbf{u}) - \sum_{n=1}^{N} \beta_n E_n$. We associate Jacobi coordinates $(\underline{V}, \xi) \in \mathbf{R}^3 \times \mathbf{R}^{3|\beta|-3}$ to the form $T_\beta(\mathbf{u})$ on $\mathbf{R}^{3|\beta|}$, with $\xi := (\xi_1, \dots, \xi_{|\beta|-1})$, $\xi_i \in \mathbf{R}^3$, $i = 1, \dots, |\beta| - 1$ (see (A.2) Appendix A). Consider a representation of ξ in spherical coordinates on $\mathbf{R}^{3|\beta|-3}$, $\xi = r\,\mathbf{n}$, with $(r, \mathbf{n}) \in [0, \infty) \times \Omega_{3|\beta|-4}$, where $\Omega_{3|\beta|-4}$ is the unit sphere in $\mathbf{R}^{3|\beta|-3}$. In (6) and (7) we choose $(\underline{V}, r, \mathbf{n})$ as new integration variables such that $\mathbf{u} = \mathbf{u}(\underline{V}, r, \mathbf{n})$. Then the limits of Prop.1 follow by repeated application of Lebesgue's dominated convergence theorem, using the properties of $K_{\alpha,\beta}$, δ_ε^3 and δ_η. The continuity of $P_k(f)$ and $R_k(f)$ is a consequence of the continuity of $K_{\alpha,\beta}$. □

The proof of Prop.1 provides the limits (6), (7) in explicit form. Define

$$t_\beta(\mathrm{w}) = \begin{cases} \left[W_{r,\alpha}(\mathrm{w}) + \sum_{n=1}^{N} (\alpha_n - \beta_n) E_n \right]^{1/2} & \text{if } W_{r,\alpha}(\mathrm{w}) + \sum_{n=1}^{N} (\alpha_n - \beta_n) E_n \geq 0, \\ 0, \textit{ otherwise.} & \end{cases}$$

If

$$W_{r,\alpha}(\mathrm{w}) + \sum_{n=1}^{N} (\alpha_n - \beta_n) E_n \geq 0, \tag{8}$$

then set

$$\mathbf{u}_{\beta\alpha}(\mathrm{w}, \mathbf{n}) = \mathbf{u}(\underline{V}, r, \mathbf{n})|_{\underline{V} = V_\alpha(\mathrm{w}),\, r = t_\beta(\mathrm{w})}. \tag{9}$$

For the sake of simplicity, $\mathbf{u}_{\beta\alpha}$ will replace the notation $\mathbf{u}_{\beta\alpha}(\mathrm{w}, \mathbf{n})$. Define

$$p_{\beta\alpha}(\mathrm{w}, \mathbf{n}) := 2^{-1} \Delta_\beta \cdot t_\beta(\mathrm{w})^{3|\beta|-5} K_{\beta,\alpha}(\mathbf{u}_{\beta\alpha}, \mathrm{w}), \tag{10}$$

$$r_{\beta\alpha}(\mathrm{w}, \mathbf{n}) := 2^{-1} \Delta_\beta \cdot t_\beta(\mathrm{w})^{3|\beta|-5} K_{\alpha,\beta}(\mathrm{w}, \mathbf{u}_{\beta\alpha}), \tag{11}$$

where the constant Δ_β is introduced by the Jacobian of $\mathrm{w} \to (\underline{V}, r, \mathbf{n})$. With the definitions (6), (7), we can write

$$P_k(f)(\mathbf{v}, \mathbf{x}) =$$

$$= \sum_{\alpha,\beta \in \mathcal{M}} \alpha_k \int_{\mathbf{R}^{3|\alpha|-3} \times \Omega_{3|\beta|-4}} d\tilde{\mathrm{w}}_{(k)} \otimes d\mathbf{n} \, [p_{\beta\alpha}(\mathrm{w}, \mathbf{n}) f_\beta(\mathbf{u}_{\beta\alpha}, \mathbf{x})]_{\mathrm{w}_{k,\alpha_k} = \mathbf{v}}, \tag{12}$$

$$R_k(f)(\mathbf{v}, \mathbf{x}) =$$

$$= \sum_{\alpha,\beta \in \mathcal{M}} \alpha_k \int_{\mathbf{R}^{3|\alpha|-3} \times \Omega_{3|\beta|-4}} d\tilde{\mathrm{w}}_{(k)} \otimes d\mathbf{n} \, [r_{\beta\alpha}(\mathrm{w}, \mathbf{n}) f_{\alpha,k}(\mathrm{w}, \mathbf{x})]_{\mathrm{w}_{k,\alpha_k} = \mathbf{v}}. \tag{13}$$

For f as in Prop. 1, we define $S_k(f)(\mathbf{v}, \mathbf{x}) = f_k(\mathbf{v}, \mathbf{x}) R_k(f)(\mathbf{v}, \mathbf{x})$ with $R_k(f)$ given by (13).

We point out the following simple relations resulting from the definition of $\mathbf{u}_{\beta\alpha}$, provided that, condition (8), is fulfilled:

$$V_\beta(\mathbf{u}_{\beta\alpha}) = V_\alpha(\mathbf{w}), \quad W_\beta(\mathbf{u}_{\beta\alpha}) = W_\alpha(\mathbf{w}),$$
$$W_{r,\beta}(\mathbf{u}_{\beta\alpha}) = W_{r,\alpha}(\mathbf{w}) + \sum_{n=1}^{N} (\alpha_n - \beta_n) E_n . \tag{14}$$

By (5) and (14), there exists some constant $K > 0$ such that if condition (8) is fulfilled, then (for $q \in [0,1]$ introduced in (5)),

$$p_{\beta\alpha}(\mathbf{w}, \mathbf{n}) \le K[1 + W_{r,\alpha}(\mathbf{w})^{q/2}], \quad r_{\beta\alpha}(\mathbf{w}, \mathbf{n}) \le K[1 + W_{r,\alpha}(\mathbf{w})^{q/2}]. \tag{15}$$

R e m a r k: In the definitions of $p_{\beta\alpha}$ and $r_{\beta\alpha}$, the presence of t_β exhibits the contributions of the reaction thresholds.

3. Existence theory

In this paper we are interested to solve Eq.(2) in $\dot{C}_0$ (the space of continuous distribution functions, vanishing at infinity in the velocity and position variables). With the notations of Prop.1, for some $\tau > 0$ fixed, set $P(f) = (P_1(f), \dots, P_N(f))$ and $S(f) = (S_1(f), \dots, S_N(f))$, $\forall f \in \dot{C}_\tau \subset \dot{C}_0$. Then $f \to P(f)$ and $f \to S(f)$, considered as maps in $\dot{C}_0$, have extensions (also denoted P and S) to their natural domains in $\dot{C}_0$.

The Cauchy problem for Eq. (2) formulated in $\dot{C}_0$ is

$$d_t f = Af + P(f) - S(f), \quad f(t=0) = f_0 , \tag{16}$$

with A the infinitesimal generator of the positivity preserving, continuous group $\{U^t\}_{t\in\mathbf{R}}$ of isometries of $\dot{C}_0$, given by its $\dot{C}_{n,0}$ components, $1 \le n \le N$,

$$(U^t f)_n(\mathbf{v}, \mathbf{x}) := U_n^t f_n(\mathbf{v}, \mathbf{x}) = f_n(\mathbf{v}, \mathbf{x} - t\mathbf{v}), \quad (t, \mathbf{v}, \mathbf{x}) \in \mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^3. \tag{17}$$

We call $f \in C(0, T; \dot{C}_0)$ a mild solution, on $[0, T]$, of Eq. (16) (in $\dot{C}_0$) if $P(f), S(f) \in C(0, T; \dot{C}_0)$, and f satisfies

$$f(t) = U^t f_0 + \int_0^t U^{t-s} P(f(s))\, ds - \int_0^t U^{t-s} S(f(s))\, ds, \tag{18}$$

(the integral being in $\dot{C}_0$ in the sense of Riemann).

Our main result states the existence, uniqueness and positivity of mild solutions, for initial data close to the vacuum state. These solutions are (time) global in the case of the gas with purely exo-energetic reactions and/or elastic (multiple) collisions.

For $T > 0$ fixed, consider $C(0, T; \dot{C}_\tau)$ with the usual sup norm, denoted $\| \circ \|_\tau$. If $g = (g_1, \dots, g_N) \in C(0, T; \dot{C}_\tau)$ and $t \in [0, T]$, by $g_n(t, \mathbf{v}, \mathbf{x})$, denote the value of $g_n(t) \in \dot{C}_{n,\tau}$, $1 \le n \le N$, at $(\mathbf{v}, \mathbf{x}) \in \mathbf{R}^3 \times \mathbf{R}^3$. Let $\dot{C}_\tau^+ :=$ $:= \{ g = (g_1, \dots, g_N) \in \dot{C}_\tau : g_n(\mathbf{v}, \mathbf{x}) \ge 0, \ \ \forall\ (\mathbf{v}, \mathbf{x}) \in \mathbf{R}^3 \times \mathbf{R}^3;\ n = 1, \dots, N \}$. Finally, for some $R > 0$, put $\mathcal{H}_\tau(R) = \{ h \mid h \in C(0, T; \dot{C}_\tau^+), \| h \|_\tau \le R \}$.

Theorem 1. *Let* $\tau > 0$ *and* $f_0 \in \dot{C}_\tau^+$.

a) For each $T > 0$, $\exists\, R_T, R_T^* > 0$ *such that if* $| f_0 |_\tau \le R_T$, *then Eq. (16) in* $\dot{C}_0$ *has a unique mild solution* f *on* $[0, T]$, *satisfying* $U^{-t} f \in \mathcal{H}_\tau(R_T^*)$. *The map* $f_0 \to f$ *is continuous from* $\{ h \in \dot{C}_\tau^+ : | h |_\tau \le R_T \}$ *to* $C(0, T; \dot{C}_0)$.

b) Assume that $K_{\alpha,\beta} \equiv 0$ *for each couple* (α, β) *that yields* $\sum_{n=1}^{N} (\alpha_n - \beta_n) E_n < 0$ *(exo-energetic reactions). In this case,* $\exists\, R, R^* > 0$ *such that if* $| f_0 |_\tau \le R$, *then for each* $T > 0$, *Eq. (16) in* $\dot{C}_0$, *has a unique mild solution* f, *on* $[0, T]$, *satisfying* $U^{-t} f \in \mathcal{H}_\tau(R^*)$. *The map* $f_0 \to f$ *is continuous from* $\{ h \in \dot{C}_\tau^+ : | h |_\tau \le R_T \}$ *to* $C(0, T; \dot{C}_0)$.

c) In each situation, $U^{-t} f \in C^1(0, T; \dot{C}_0)$.

R e m a r k. In the case of the simple gas, with binary elastic collisions, the statements of Theorem 1.b) reduce to known results on the classical Boltzmann equation.

The proof of Theorem 1 will be given in several steps. We would like to apply the Banach fixed point theorem to Eq. (18) in $C(0, T; \dot{C}_0)$. This is not, directly, possible since, P and S may be unbounded. However, writing Eq. (18) more conveniently, the smothering properties of the time integrals appear to play a compensating role. The argument uses the following key estimation, extending certain energetic inequalities, obtained for the classical Boltzmann equation in [10]. For some $\gamma \in \mathcal{M}$, define

$$\Phi_\gamma (t, \mathbf{w}, \mathbf{x}, \mathbf{v}) := \sum_{n \in \mathcal{N}(\gamma)} \sum_{i=1}^{\gamma_n} m_n \{ [\mathbf{x} - t(\mathbf{w}_{n,i} - \mathbf{v})]^2 + \mathbf{w}_{n,i}^2 \}, \tag{19}$$

for all $\mathbf{w} \in \mathbf{R}^{3|\gamma|}$, $\mathbf{x}, \mathbf{v} \in \mathbf{R}^3$, $t > 0$. Also set

$$\Gamma_{k\gamma} (t, \mathbf{v}, \mathbf{x}) := \gamma_k \exp [\tau m_k (\mathbf{v}^2 + \mathbf{x}^2)] \times$$

$$\times \int_{\mathbf{R}^{3|\gamma|-3}} d\tilde{\mathbf{w}}_{(k)} \int_0^t ds \Big[(1 + W_{r,\gamma} (\mathbf{w})^{q/2}) \exp(-\tau \Phi_\gamma (s, \mathbf{w}, \mathbf{x}, \mathbf{v})) \Big]_{\mathbf{w}_{k,\gamma_k} = \mathbf{v}}, \tag{20}$$

for all $\mathbf{v}, \mathbf{x} \in \mathbf{R}^3$, $t > 0$; $q \in [0, 1]$.

Lemma 1. *a) Under conditions (1),*

$$\Phi_\beta (t, \mathbf{u}, \mathbf{x}, \mathbf{v}) = \Phi_\alpha (t, \mathbf{w}, \mathbf{x}, \mathbf{v}) + 2(1 + t^2) \sum_{n=1}^{N} (\alpha_n - \beta_n) E_n . \tag{21}$$

b) $\max\limits_{\gamma \in \mathcal{M},\, 1 \le k \le N} \sup \{ \Gamma_{k\gamma} (t, \mathbf{v}, \mathbf{x}) \mid (t, \mathbf{v}, \mathbf{x}) \in \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 \} = const < \infty.$

The proof is given in Appendix B.

Let $T, \tau > 0$ and $f_0 \in \dot{C}_\tau^+$. With the substitution $g(t) := U^{-t} f(t)$, Eq.(18) becomes,

$$g(t) = f_0 + \int_0^t P^\# (g)(s)\, ds - \int_0^t S^\# (g)(s)\, ds, \quad 0 \le t \le T. \tag{22}$$

Here $P^\#$ and $S^\#$ are considered as operators in $C(0, T; \dot{C}_0)$, defined on their natural domains $D(P^\#)$ and $D(S^\#)$, respectively, by

$$P^\# (g)(t) := U^{-t} P(U^t g(t)), \quad S^\# (g)(t) := U^{-t} S(U^t g(t)).$$

It follows that, we can prove Theorem 1, by looking for those $g \in C(0, T; \dot{C}_0^+)$ solving Eq. (22) in $\dot{C}_0$. Since U^t leaves $C(0, T; \dot{C}_0^+)$ invariant, we may equivalently look for those $g = (g_1, \ldots, g_N) \in D(P^\#) \cap D(S^\#) \cap C(0, T; \dot{C}_0^+)$, solving the system

$$g_k (t, \mathbf{v}, \mathbf{x}) = I_k (g)(t, \mathbf{v}, \mathbf{x}), \quad k = 1, \ldots, N, \tag{23}$$

$(t, \mathbf{v}, \mathbf{x}) \in [0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$. Here $I_k(g) \in C([0, T] \times \mathbf{R}^3 \times \mathbf{R}^3)$ is given by

$$I_k(g)(t, \mathbf{v}, \mathbf{x}) = f_{k,0}(t, \mathbf{v}, \mathbf{x}) \exp\left[-\int_0^t R_k^{\#}(g)(\lambda, \mathbf{v}, \mathbf{x})\, d\lambda\right] +$$

$$+ \int_0^t \exp\left[-\int_s^t R_k^{\#}(g)(\lambda, \mathbf{v}, \mathbf{x})\, d\lambda\right] P_k^{\#}(g)(s, \mathbf{v}, \mathbf{x})\, ds, \tag{24}$$

with $R_k^{\#}(g)(t, \mathbf{v}, \mathbf{x}) := U_k^{-t} R_k(U^t g(t))(\mathbf{v}, \mathbf{x})$, $k = 1, \ldots, N$ (the integrals being in the classical sense). Obviously, the system (23) represents a weak form of Eq.(22). Due to the assumptions on f_0, it will appear that Eq.(23) has solutions given by elements of $C(0, T; \dot{C}_\tau^+)$. Let $I(g) := (I_1(g), \ldots, I_N(g))$. We show that $g \to I(g)$ fulfills the conditions for applying the Banach fixed point theorem in $\mathcal{H}_\tau(R)$, for R small enough.

Proposition 2. *a) If $g \in C(0, T; \dot{C}_\tau^+)$, then also $I(g) \in C(0, T; \dot{C}_\tau^+)$.*

b) For each $T > 0$, there exist $R_T, R_T^* > 0$, with $R_T^* \to 0$, as $R_T \to 0$, such that if $|f_0|_\tau \le R_T$, then $g \to I(g)$ leaves $\mathcal{H}_\tau(R_T^*)$ invariant. Moreover, if $K_{\beta,\alpha} \equiv 0$, whenever $\sum_{n=1}^{N} (\alpha_n - \beta_n) E_n < 0$, then there exist $R, R^* > 0$, independent of T, with $R^* \to 0$, as $R \to 0$, such that if $|f_0|_\tau \le R_T$, then $g \to I(g)$ leaves $\mathcal{H}_\tau(R^*)$ invariant.

P r o o f. a) First, remark that $C(0, T; \dot{C}_{k,\tau})$, $k = 1, \ldots, N$, can be identified with the set of those $h \in C([0, T] \times \mathbf{R}^3 \times \mathbf{R}^3)$ (real) with the property

$$\sup_{|\mathbf{x}| + |\mathbf{v}| \ge r} \{\exp[\tau m_k (\mathbf{x}^2 + \mathbf{v}^2)]\, |h(t, \mathbf{v}, \mathbf{x})|\} \to 0 \ \text{ as } \ r \to 0, \tag{25}$$

uniformly in $t \in C[0, T]$. We verify (25). If $g \in C(0, T; \dot{C}_\tau^+)$, $\gamma \in \mathcal{M}$, denote

$$G_\gamma^{\#}(t, \mathbf{w}, \mathbf{x}, \mathbf{v}) = \prod_{n \in \mathcal{N}(\gamma)} \prod_{i=1}^{\gamma_n} g_n(t, \mathbf{w}_{n,i}, \mathbf{x} - t(\mathbf{w}_{n,i} - \mathbf{v})) \exp[\tau \Phi_\gamma(t, \mathbf{w}, \mathbf{x}, \mathbf{v})].$$

Using the definitions of $P^{\#}$ and P, as well as Rel.(15) and Lemma 1.a), we estimate (24): since $R_k^{\#}(g)(t, \mathbf{v}, \mathbf{x}) \geq 0$, $1 \leq k \leq N$, the exponents are negative in (24); moreover, $P_k^{\#}(g)(t, \mathbf{v}, \mathbf{x}) \geq 0$; then, with the notations of Rel. (12), (13), for some constant $K > 0$,

$$0 \leq I_k(g)(t, \mathbf{v}, \mathbf{x}) \leq$$

$$\leq f_{k,0}(t, \mathbf{v}, \mathbf{x}) + K \sum_{\alpha, \beta \in \mathcal{M}} \alpha_k \int_{\mathbf{R}^{3|\alpha|-3} \times \Omega_{3|\beta|-4}} d\,\tilde{\mathbf{w}}_{(k)} \otimes d\,\mathbf{n} \Big[(1 + W_{r,\alpha}(\mathbf{w})^{q/2}) \times$$

$$\times \int_0^t ds\, \Lambda_{\beta\alpha}(s)\, G_\beta^{\#}(s, \mathbf{u}_{\beta\alpha}, \mathbf{x}, \mathbf{v}) \exp\left[-\tau\, \Phi_\alpha(s, \mathbf{w}, \mathbf{x}, \mathbf{v})\right] \Big]_{\mathbf{w}_{k,\alpha_k} = \mathbf{v}}, \tag{26}$$

for all $(t, \mathbf{v}, \mathbf{x}) \in ([0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$. Here,

$$\Lambda_{\beta\alpha}(t) = \begin{cases} \exp\left[-2\tau\,(1 + t^2) \sum_{n=1}^{N} (\alpha_n - \beta_n) E_n\right], & \text{if } K_{\beta,\alpha} \not\equiv 0, \\ 0, & \text{if } K_{\beta,\alpha} \equiv 0. \end{cases} \tag{27}$$

Since $g \in C(0, T; \dot{C}_\tau^+)$, by (25), (14), there exists $r \geq 0$ such that $G_\beta^{\#}(t, \mathbf{u}, \mathbf{x}, \mathbf{v}) \leq \varepsilon \, \| g \|_\tau^{|\beta|-1}$, provided that $\Phi_\beta(t, \mathbf{u}, \mathbf{v}, \mathbf{x}) \geq r$. Observe that $m_k(\mathbf{x}^2 + \mathbf{v}^2) \leq \Phi_\alpha(t, \mathbf{w}, \mathbf{v}, \mathbf{x}) \,|_{\mathbf{w}_{k,\alpha_k} = \mathbf{v}}$. Consequently, by Lemma 1.a), for each $\varepsilon > 0$, there exists $r_0 > 0$ (possibly depending on T) such that if $(\mathbf{x}^2 + \mathbf{v}^2) \geq r_0$, then

$$0 \leq G_\beta^{\#}(s, \mathbf{u}_{\beta\alpha}, \mathbf{x}, \mathbf{v}) \,|_{\mathbf{w}_{k,\alpha_k} = \mathbf{v}} \leq \varepsilon \, \| g \|_\tau^{|\beta|-1},$$

uniformly in the rest of variables. We introduce the last inequality in (26). There exist two constants $K_1 > 0$ and $r_1 > 0$, such that, if $\mathbf{x}^2 + \mathbf{v}^2 \geq r_1$, then

$$0 \leq I_k(g)(\mathbf{v}, \mathbf{x}) \leq f_{k,0}(\mathbf{v}, \mathbf{x}) +$$

$$+ \varepsilon\, K_2\, \Lambda(T) \exp\left[-\tau\, m_k (\mathbf{x}^2 + \mathbf{v}^2)\right] \sum_{|\alpha| \geq 2, |\beta| \geq 2} \Gamma_{k\alpha}(t, \mathbf{v}, \mathbf{x}) \, \| g \|_\tau^{|\beta|-1}, \tag{28}$$

for all $(t, \mathbf{v}, \mathbf{x}) \in ([0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$. Here,

$$\Lambda(T) := \sup_{\alpha, \beta \in \mathcal{M}} [\sup_{0 \le t \le T} \Lambda_{\beta\alpha}(t)]. \tag{29}$$

Since $f_{k,0}$ is in $\dot{C}_{k,\tau}$, it is now sufficient to apply Lemma 1.b) to obtain that (25) is satisfied. This concludes the proof of a).

b) Since $0 \le G^{\#}_{\beta} \le \| g \|^{|\beta|}_{\tau}$, the same procedure as before implies

$$| I_k(g)(t) |_{k,\tau} \le | f_0 |_\tau + K_2 \Lambda(T) \sum_{|\beta| \ge 2} \| g \|^{|\beta|}_{\tau},$$

for some constant $K_2 > 0$. Now the argument can be easily concluded. ❒

Let I denote the map $g \to I(g)$, according to Prop. 2.b). Clearly, Eq.(23) can be formulated in $C(0, T; \dot{C}^{+}_{\tau})$ as

$$g = I(g). \tag{30}$$

Proposition 3. *For each $T > 0$ there exist R_T, $R^*_T > 0$, with $R^*_T \to 0$, as $R_T \to 0$, such that if $|f_0|_\tau \le R_T$, then I is a strict contraction on $\mathcal{H}_\tau(R^*_T)$. Assume that $K_{\beta\alpha} \equiv 0$, whenever $\sum_{n=1}^{N} (\alpha_n - \beta_n) E_n \le 0$. In this case, there exist R, $R^* > 0$, with $R^* \to 0$, as $R \to 0$, independent of T, such that if $|f_0|_\tau \le R$, then I is a strict contraction on $\mathcal{H}_\tau(R^*)$.*

P r o o f. By (24), for $g, h \in C(0, T; \dot{C}^{+}_{\tau})$, we can write

$$| I_k(g)(t, \mathbf{v}, \mathbf{x}) - I_k(h)(t, \mathbf{v}, \mathbf{x}) | \le$$

$$\le Q^A_k(g, h)(t, \mathbf{v}, \mathbf{x}) + Q^B_k(g, h)(t, \mathbf{v}, \mathbf{x}) + Q^C_k(g, h)(t, \mathbf{v}, \mathbf{x}), \tag{31}$$

with

$$Q^A_k(t, \mathbf{v}, \mathbf{x}) := f_{k,0}(\mathbf{v}, \mathbf{x}) \times$$

$$\times \left| \exp\left[-\int_0^t R^{\#}_k(g)(\lambda, \mathbf{v}, \mathbf{x}) d\lambda\right] - \exp\left[-\int_0^t R^{\#}_k(h)(\lambda, \mathbf{v}, \mathbf{x}) d\lambda\right] \right|,$$

$$Q_k^B(t, \mathbf{v}, \mathbf{x}) := \int_0^t ds\, P_k^\#(g)(s, \mathbf{v}, \mathbf{x}) \times$$

$$\times \left| \exp\left[-\int_s^t R_k^\#(g)(\lambda, \mathbf{v}, \mathbf{x})\, d\lambda \right] - \exp\left[-\int_s^t R_k^\#(h)(\lambda, \mathbf{v}, \mathbf{x})\, d\lambda \right] \right|,$$

$$Q_k^C(t, \mathbf{v}, \mathbf{x}) :=$$

$$:= \int_0^t \exp\left[-\int_s^t R_k^\#(h)(\lambda, \mathbf{v}, \mathbf{x}) d\lambda \right] \left| P_k^\#(g)(s, \mathbf{v}, \mathbf{x}) - P_k^\#(h)(s, \mathbf{v}, \mathbf{x}) \right| ds,$$

for all $(t, \mathbf{v}, \mathbf{x}) \in ([0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$; $k = 1, \ldots, N$.

First we estimate $Q_k^A(t, \mathbf{v}, \mathbf{x})$. Since $g_k(t, \mathbf{v}, \mathbf{x})$, $h_k(t, \mathbf{v}, \mathbf{x}) \geq 0$, then $R_k^\#(g)(t, \mathbf{v}, \mathbf{x}) \geq 0$ and $R_k^\#(h)(t, \mathbf{v}, \mathbf{x}) \geq 0$, hence we can write

$$Q_k^A(g, h)(t, \mathbf{v}, \mathbf{x}) \leq f_{k,0}(\mathbf{v}, \mathbf{x}) \int_0^t \left| R_k^\#(g)(\lambda, \mathbf{v}, \mathbf{x}) - R_k^\#(h)(\lambda, \mathbf{v}, \mathbf{x}) \right| d\lambda,$$

for all $(t, \mathbf{v}, \mathbf{x}) \in [0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$.

Using the definitions of $R_k^\#$, by arguments similar to those in the proof of Prop. 2, there exists a constant $C_1 > 0$ such that

$$Q_k^A(g, h)(t, \mathbf{v}, \mathbf{x}) \leq$$

$$\leq C_1 f_{k,0}(\mathbf{v}, \mathbf{x}) \| g - h \|_\tau \left(\sum_{|\alpha| \geq 2} \sum_{n=0}^{|\alpha|-2} \| g \|_\tau^{|\alpha| - n - 2} \| h \|_\tau^n \Gamma_{k\alpha}(t, \mathbf{v}, \mathbf{x}) \right), \tag{32}$$

for all $(t, \mathbf{v}, \mathbf{x}) \in [0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$. Since

$$Q_k^B(g, h)(t, \mathbf{v}, \mathbf{x}) \leq$$

$$\leq \int_0^t P_k^\#(g)(s, \mathbf{v}, \mathbf{x})\, ds \left(\int_0^t \left| R_k^\#(g)(\lambda, \mathbf{v}, \mathbf{x}) - R_k^\#(h)(\lambda, \mathbf{v}, \mathbf{x}) \right| d\lambda \right),$$

similar estimations give (for some constant $C_2 > 0$)

$$Q_k^B(g, h)(t, \mathbf{v}, \mathbf{x}) \le$$

$$\le C_2 \Lambda(T) \exp[-\tau m_k (\mathbf{x}^2 + \mathbf{v}^2)] \| g - h \|_\tau \left(\sum_{|\beta| \ge 2} \| g \|_\tau^{|\beta|} \right) \times$$

$$\times \left(\sum_{|\alpha| \ge 2} \Gamma_{k\alpha}(t, \mathbf{v}, \mathbf{x}) \right) \left(\sum_{|\alpha| \ge 2} \sum_{n=0}^{|\alpha|-2} \| g \|_\tau^{|\alpha|-n-2} \| h \|_\tau^n \Gamma_{k\alpha}(t, \mathbf{v}, \mathbf{x}) \right), \quad (33)$$

with $\Lambda(T)$ defined by (29); $(t, \mathbf{v}, \mathbf{x}) \in ([0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$.

In the same way, for some constant $C_3 > 0$, we obtain

$$Q_k^C(g, h)(t, \mathbf{v}, \mathbf{x}) \le C_3 \Lambda(T) \exp[-\tau m_k (\mathbf{x}^2 + \mathbf{v}^2)] \times$$

$$\times \| g - h \|_\tau \left(\sum_{|\alpha| \ge 2} \Gamma_{k\alpha}(t, \mathbf{v}, \mathbf{x}) \right) \left(\sum_{|\beta| \ge 2} \sum_{n=1}^{|\beta|-1} \| g \|_\tau^{|\beta|-n-1} \| h \|_\tau^n \right), \quad (34)$$

$(t, \mathbf{v}, \mathbf{x}) \in ([0, T] \times \mathbf{R}^3 \times \mathbf{R}^3$.

By inequalities (32)-(34), applying Lemma 1.b), one can find a constant $C_0 > 0$ and a polynomial $p(\cdot)$ with positive coefficients, such that $\forall\, r > 0$,

$$\| I(g) - I(h) \|_\tau \le C_0 [|f_0|_\tau + r\Lambda(T)]\, p(r) \| g - h \|_\tau, \quad (35)$$

provided that $g, h \in \mathcal{H}_\tau(r)$.

Now, by Prop.2 and Rel. (35), we can choose R_T, $R_T^* > 0$ such that if $|f_0|_\tau \le R_T$ then I is a strict contraction on $\mathcal{H}_\tau(R_T^*)$. By (29), if $K_{\beta,\alpha} \equiv 0$, whenever $\sum_{n=1}^{N} (\alpha_n - \beta_n) E_n \le 0$, then $\Lambda(T) = 1$, $\forall T > 0$. Consequently there exist R, $R^* > 0$, independent of T, such that if $|f_0|_\tau \le R$, then I is a strict contraction on $\mathcal{H}_\tau(R^*)$. This concludes the proof of Prop. 3. ❐

The existence and uniqueness part in Theorem 1.a) follows by Prop. 2, Prop. 3 and the Banach fixed point theorem: for R_T, $R_T^* > 0$, small enough, Eq.(30), with $|f_0| \le R_T$, can be uniquely solved in $\mathcal{H}_\tau(R_T^*)$. To conclude the argument it is sufficient to remark that $\mathcal{H}_\tau(R_T^*) \subset D(P^\#) \cap D(S^\#) \subset C(0, T, \dot{C}_0)$. To prove the rest of Theorem 1.a), namely the continuity of the solution in the initial datum, first remark by

Prop.2, that, for each $g \in \mathcal{H}_\tau(R^*_T)$, fixed, the map $f_0 \to I(g)$ is continuous from $\{h \in \dot{C}^+_\tau : |h|_\tau \le R_T\}$ to $C(0, T, \dot{C}_\tau)$. Then the proof follows by means of the inequality (35).

Part b) of Theorem 1 can be similarly proved.

Part c) of Theorem 1 is immediate: $g \in C^1(0, T, \dot{C}_\tau)$, by Eq.(22), while the solution f of Eq. (18) is related to g by $f = U^t g$. □

4. Conservation relations, H-Theorem and the law of the mass action

In this section we prove the global conservations relations for mass, momentum and energy and a H-Theorem analogous to the results obtained in the case of the classical Boltzmann equation with elastic binary collisions.

Set $\Psi^0_k(\mathbf{v}) = m_k$, $\Psi^4_k(\mathbf{v}) = 2^{-1}m_k\mathbf{v}^2 + E_k$ and $\Psi^i_k(\mathbf{v}) = m_k\mathbf{v}_i$ for all $\mathbf{v} = (\mathbf{v}_1\, \mathbf{v}_2, \mathbf{v}_3) \in \mathbf{R}^3$, with $\mathbf{v}_i \in \mathbf{R}$, $i = 1, 2, 3$; $1 \le k \le N$. The following result states the bulk momentum and energy conservation relations.

Theorem 2. *a) Let* $\tau > 0$ *and* $f \in \dot{C}_\tau$. *Then for* $i = 0, 1, \dots, 4$,

$$\sum_{k=1}^{N} \int_{\mathbf{R}^3} \Psi^i_k(\mathbf{v})(P_k(f)(\mathbf{v}, \mathbf{x}) - S_k(f)(\mathbf{v}, \mathbf{x}))\, d\mathbf{v} \equiv 0,$$

$\forall\, \mathbf{x} \in \mathbf{R}^3$.

b) *If* f_0 *and* f *are as in Theorem 1, then, for each* $t \in [0, T]$,

$$\sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Psi^i_k(\mathbf{v}) f_k(t, \mathbf{v}, \mathbf{x})\, d\mathbf{v} \otimes d\mathbf{x} \equiv \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Psi^i_k(\mathbf{v}) f_{k,0}(\mathbf{v}, \mathbf{x})\, d\mathbf{v} \otimes d\mathbf{x}.$$

P r o o f. a) We give the argument for Ψ^4_k, the other cases being similar. By Prop.1.b),

$$\sum_{k=1}^{N} \int_{\mathbf{R}^3} \Psi^4_k(\mathbf{v}) P_k(f)(\mathbf{v}, \mathbf{x})\, d\mathbf{v} =$$

$$= \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \sum_{k \in \mathcal{N}(\alpha)} \sum_{i=1}^{\alpha_k} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} (2^{-1}m_k \mathbf{w}^2_{k,i} + E_k) \times$$

$$\times f_\beta(\mathbf{u}, \mathbf{x}) K_{\beta,\alpha}(\mathbf{u}, \mathbf{w})\, \delta^3_\varepsilon(V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w}))\, \delta_\eta(W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))\, d\mathbf{u} \otimes d\mathbf{w} =$$

$$= \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \sum_{k \in \mathcal{N}(\beta)} \sum_{i=1}^{\beta_k} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} (2^{-1} m_k \mathbf{u}^2_{k,i} + E_k) \times$$

$$\times f_\alpha(\mathbf{w}, \mathbf{x}) K_{\alpha,\beta}(\mathbf{w}, \mathbf{u})\, \delta^3_\varepsilon(V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w}))\, \delta_\eta(W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))\, d\mathbf{u} \otimes d\mathbf{w},$$

where the last equality results by interchanging α and $\mathbf{w}$ with β and $\mathbf{u}$ respectively, and using the symmetry of δ^3_ε, and δ_η, respectively as well as the invariance of $K_{\beta,\alpha}$ at permutations. Then it is sufficient to remark that

$$\sum_{k=1}^{N} \int_{\mathbf{R}^3} \Psi^4_k(\mathbf{v})(P_k(f)(\mathbf{v}, \mathbf{x}) - S_k(f)(\mathbf{v}, \mathbf{x}))\, d\mathbf{v} =$$

$$= \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} (W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w})) \times$$

$$\times f_\alpha(\mathbf{w}, \mathbf{x}) K_{\alpha,\beta}(\mathbf{w}, \mathbf{u})\, \delta^3_\varepsilon(V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w}))\, \delta_\eta(W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))\, d\mathbf{u} \otimes d\mathbf{w} \equiv 0.$$

b) Let f be as in Theorem 1. Note that for each t fixed, U^t_k, introduced in Rel. (17) is a positivity preserving, linear isometry on $L^1(\mathbf{R}^3 \times \mathbf{R}^3, d\mathbf{v} \otimes d\mathbf{x})$; $k = 1, \ldots, N$. Further, by Theorem 1.c), $g = U^{-t} f$ is of class C^1 and verifies Eq. (22). Then for each $i = 1, \ldots, 4$,

$$\frac{d}{dt} \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Psi^i_k(\mathbf{v}) f_k(t, \mathbf{v}, \mathbf{x})\, d\mathbf{v} \otimes d\mathbf{x} =$$

$$= \frac{d}{dt} \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Psi^i_k(\mathbf{v})\, g_k(t, \mathbf{v}, \mathbf{x})\, d\mathbf{v} \otimes d\mathbf{x} =$$

$$= \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Psi^i_k(\mathbf{v})\, (P^{\#}_k(g)(t, \mathbf{v}, \mathbf{x}) - S^{\#}_k(g)(t, \mathbf{v}, \mathbf{x}))\, d\mathbf{v} \otimes d\mathbf{x} =$$

$$= \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Psi^i_k(\mathbf{v})(P_k(f)(t, \mathbf{v}, \mathbf{x}) - S_k(f)(t, \mathbf{v}, \mathbf{x}))\, d\mathbf{v} \otimes d\mathbf{x} \equiv 0,$$

using again the L^1-properties of U_k^t and Part a). This concludes the proof. □

Let $C_1, \dots, C_N > 0$ be constants and $C^{\alpha} := C^{\alpha_1} \times \dots \times C^{\alpha_N}$, $\alpha \in \mathcal{M}$. In the rest of this section, we suppose the following detailed balance condition

$$C^{\beta} K_{\alpha,\beta}(\mathbf{w}, \mathbf{u}) \equiv C^{\alpha} K_{\beta,\alpha}(\mathbf{u}, \mathbf{w}), \ \forall \ \mathbf{w}, \ \mathbf{u}, \ \alpha, \ \beta. \tag{36}$$

First remark that if $f = (f_1, \dots, f_N) \in \dot{C}_{\tau}^{+}$, $\tau > 0$, with $f_n > 0$, then $f_n \log f_n \in L^1(\mathbf{R}^3 \times \mathbf{R}^3, d\mathbf{v} \otimes d\mathbf{x})$, for all $n = 1, \dots, N$. The argument is standard ([10]): let $\log^+(\log^-)$ denote the positive (negative) part of the function log; clearly $f_n \log^+ f_n \in L^1$; it is sufficient to prove the same for $f_n \log^- f_n$; to this end, in the inequality $\xi \log^- \xi \le \eta - \xi \log \eta$, valid for $\xi > 0$ and $0 < \eta \le 1$ ([10]), we take $\xi = f_n(\mathbf{v}, \mathbf{x})$ and $\eta = \exp(-\mathbf{v}^2 - \mathbf{x}^2)$, obtaining

$$(f_n \log^- f_n)(\mathbf{v}, \mathbf{x}) \le \exp(-\mathbf{v}^2 - \mathbf{x}^2) + (\mathbf{v}^2 + \mathbf{x}^2) f_n(\mathbf{v}, \mathbf{x}). \tag{37}$$

Therefore we can define the H-function

$$H(f) = \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} f_k(\mathbf{v}, \mathbf{x})[\log(C_k f_k(\mathbf{v}, \mathbf{x})) - 1] \, d\mathbf{v} \otimes d\mathbf{x}. \tag{38}$$

Proposition 4. *a) Let* $\tau > 0$, $f = (f_1, \dots, f_N) \in \dot{C}_{\tau}^{+}$, *such that* $f_n > 0$ *and* $\sup(1 + \mathbf{x}^2 + \mathbf{v}^2)^{-1} |\log f_n(\mathbf{v}, \mathbf{x})| < \infty$ *for all* $n = 1, \dots, N$. *Then*

$$D(f) := \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} (P_k(f)(\mathbf{v}, \mathbf{x}) - S_k(f)(\mathbf{v}, \mathbf{x})) \log(C_k f_k(\mathbf{v}, \mathbf{x})) \, d\mathbf{v} \otimes d\mathbf{x} \le 0.$$

Moreover $D(f) \equiv 0$ *iff for each couple* (α, β) *such that* $K_{\alpha,\beta} \not\equiv 0$, *it follows that*

$$C^{\alpha} f_{\alpha}(\mathbf{w}, \mathbf{x}) \equiv C^{\beta} f_{\beta}(\mathbf{u}, \mathbf{x}), \tag{39}$$

$\forall \mathbf{x} \in \mathbf{R}^3$, *provided that* $\mathbf{w}, \mathbf{u}$ *satisfy* (1).

b) Let f_0 *and* f *be as in Theorem 1. In addition, suppose that* $f_{n,0} > 0$ *and* $\sup(1 + \mathbf{x}^2 + \mathbf{v}^2)^{-1} |\log f_{n,0}(\mathbf{v}, \mathbf{x})| < \infty$ *for all* $n = 1, \dots, N$. *Then* $t \to H(f)$ *is of class* C^1 *and*

$$\frac{d}{dt} H(f)(t) = D(f(t)).$$

P r o o f. a) In our case, $D(f)$ is well defined. Then, by Prop.1.b), taking $h = \log f_k$ and $f = (f_1, \dots, f_N) \in \dot{C}_\tau^+$, with $f_n > 0$, for all $n = 1, \dots, N$,

$$\sum_{k=1}^{N} \int_{\mathbf{R}^3} P_k(f)(\mathbf{v}, \mathbf{x}) \log (C_k f_k)(\mathbf{v}, \mathbf{x}))\, d\mathbf{v} =$$

$$= \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} f_\beta(\mathbf{u}, \mathbf{x}) \times$$

$$\times \log (C^\alpha f_\alpha(\mathbf{w}, \mathbf{x})) K_{\beta, \alpha}(\mathbf{u}, \mathbf{w})\, \delta_\varepsilon^3 (V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w})) \times$$

$$\times \delta_\eta (W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))\; d\mathbf{u} \otimes d\mathbf{w} =$$

$$= \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} f_\alpha(\mathbf{w}, \mathbf{x}) \log (C^\beta f_\beta(\mathbf{u}, \mathbf{x})) \times$$

$$\times K_{\alpha, \beta}(\mathbf{w}, \mathbf{u})\, \delta_\varepsilon^3 (V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w}))\, \delta_\eta (W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))\; d\mathbf{u} \otimes d\mathbf{w} \tag{40}$$

where the last equality results by interchanging α and $\mathbf{w}$ with β and $\mathbf{u}$ respectively, and using the symmetry of δ_ε^3, and δ_η, respectively as well as the invariance of $K_{\beta, \alpha}$ at permutations.

Similarly

$$\sum_{k=1}^{N} \int_{\mathbf{R}^3} S_k(f)(\mathbf{v}, \mathbf{x}) \log (C_k f_k)(\mathbf{v}, \mathbf{x}))\, d\mathbf{v} =$$

$$= \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} f_\alpha(\mathbf{w}, \mathbf{x}) \times$$

$$\times \log (C^\alpha f_\alpha(\mathbf{w}, \mathbf{x})) K_{\alpha, \beta}(\mathbf{w}, \mathbf{u})\, \delta_\varepsilon^3 (V_\beta(\mathbf{u}) - V_\alpha(\mathbf{w})) \times$$

$$\times \delta_\eta (W_\beta(\mathbf{u}) - W_\alpha(\mathbf{w}))\; d\mathbf{u} \otimes d\mathbf{w} =$$

$$= \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} f_\beta (\mathbf{u}, \mathbf{x}) \log (C^\beta f_\beta (\mathbf{u}, \mathbf{x})) \times$$

$$\times K_{\beta, \alpha} (\mathbf{u}, \mathbf{w}) \, \delta^3_\varepsilon (V_\beta (\mathbf{u}) - V_\alpha (\mathbf{w})) \, \delta_\eta (W_\beta (\mathbf{u}) - W_\alpha (\mathbf{w})) \, d\mathbf{u} \otimes d\mathbf{w}. \tag{41}$$

Then a few algebraic manipulations involving Rel. (40), (41) imply that

$$D(f) = 2^{-1} \int_{\mathbf{R}^3} d\mathbf{x} \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \sum_{\alpha, \beta \in \mathcal{M}} \int_{\mathbf{R}^{3|\beta|} \times \mathbf{R}^{3|\alpha|}} H_{\alpha, \beta} (\mathbf{w}, \mathbf{u}, \mathbf{x}) \times$$

$$\times \delta^3_\varepsilon (V_\beta (\mathbf{u}) - V_\alpha (\mathbf{w})) \, \delta_\eta (W_\beta (\mathbf{u}) - W_\alpha (\mathbf{w})) \, d\mathbf{u} \otimes d\mathbf{w}, \tag{42}$$

with

$$H_{\alpha, \beta} (\mathbf{w}, \mathbf{u}, \mathbf{x}) =$$

$$= [K_{\beta, \alpha} (\mathbf{u}, \mathbf{w}) f_\beta (\mathbf{u}, \mathbf{x}) - K_{\alpha, \beta} (\mathbf{w}, \mathbf{u}) f_\alpha (\mathbf{w}, \mathbf{x})] \log \frac{C^\alpha f_\alpha (\mathbf{w}, \mathbf{x})}{C^\beta f_\beta (\mathbf{u}, \mathbf{x})}.$$

Assuming that condition (36) is fulfilled, it follows that

$$H_{\alpha, \beta} (\mathbf{w}, \mathbf{u}, \mathbf{x}) = - K_{\beta, \alpha} (\mathbf{u}, \mathbf{w}) f_\beta (\mathbf{u}, \mathbf{x}) \left(\frac{C^\alpha f_\alpha (\mathbf{w}, \mathbf{x})}{C^\beta f_\beta (\mathbf{u}, \mathbf{x})} - 1 \right) \log \frac{C^\alpha f_\alpha (\mathbf{w}, \mathbf{x})}{C^\beta f_\beta (\mathbf{u}, \mathbf{x})} \leq 0,$$

hence $D(f) \leq 0$. Rel.(39) is now obvious.

b) Since U^t_k, $1 \leq k \leq N$, is a positivity preserving isometry on $L^1 (\mathbf{R}^3 \times \mathbf{R}^3, d\mathbf{v} \otimes d\mathbf{x})$, using Rel.(38), we obtain $H(f(t)) = H(g(t))$. But $g(t) = U^{-t} f(t)$ is the unique solution of Eq.(22) and $g = U^{-t} f$ is of class C^1. From the definition of I and the fact that f_0 satisfies the conditions of Prop. 4. a) it follows that $\forall \, t \in [0, T]$, $f(t)$ satisfies the conditions of Prop. 4. a), so that $D(f(t))$ is well defined. Moreover, using again the L^1-properties of U^t_k,

$$\sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} d\mathbf{v} \otimes d\mathbf{x} \, (P^\#_k (g)(t, \mathbf{v}, \mathbf{x}) - S^\#_k (g)(t, \mathbf{v}, \mathbf{x})) \log (C_k \, g_k (t, \mathbf{v}, \mathbf{x})) =$$

$$= \sum_{k=1}^{N} \int_{\mathbf{R}^3 \times \mathbf{R}^3} d\mathbf{v} \otimes d\mathbf{x} \, (P_k (f)(t, \mathbf{v}, \mathbf{x}) - S_k (f)(t, \mathbf{v}, \mathbf{x})) \log C_k f_k (t, \mathbf{v}, \mathbf{x}).$$

Putting all these together and then using part a), it follows that $\frac{d}{dt} H(f)(t) = \frac{d}{dt} H(g)(t) = D(f)(t) \leq 0$, concluding the proof. □

The main result of this section follows from Prop. 4 using techniques similar to those of [10].

Theorem 3. *Let* $f_0 = (f_{1,0}, \dots, f_{N,0})$ *and* $f = (f_1, \dots, f_N)$ *be as in Theorem 1 with* $f_{n,0} > 0$, $1 \leq n \leq N$. *Then under condition* (36), $H(f(t_2)) \leq H(f(t_1))$ *for all* $t_2 \geq t_1$.

P r o o f. Define the sequence $f_0^{(l)} = (f_{1,0}^{(l)}, \dots, f_{N,0}^{(l)})$, $l = 1, 2, \dots$ by setting for each $k = 1, \dots, N$,

$$f_{k,0}^{(l)}(\mathbf{v}, \mathbf{x}) = \max \left\{ f_{k,0}(\mathbf{v}, \mathbf{x}), \frac{|f_0|_\tau}{l(1 + \mathbf{v}^2 + \mathbf{x}^2)} \exp[-\tau m_k (\mathbf{v}^2 + \mathbf{x}^2)] \right\}.$$

Let $f(l)$ and f be the mild solutions of Eq.(16), provided by Theorem 1, for initial data $f_0^{(l)}$ and f_0, respectively. Obviously, $H(f(t))$ and $H(f^{(l)}(t))$ are well defined. Clearly $|f_0^{(l)} - f_0|_\tau \to 0$, as $l \to \infty$. Then $\|f^{(l)} - f\|_0 \to 0$, as $l \to \infty$, by the continuity in initial datum, stated in Theorem 1. Moreover, for each $t \in [0, T]$, the sequence $\left(U^{-t} f^{(l)}(t)\right)_{l \in \mathbf{N}}$ is bounded in $\dot{C}_\tau$. Using (37), it follows that for each t, the sequence $(f^{(l)}(t) \log f^{(l)}(t))_{l \in \mathbf{N}}$ is bounded by some function in $L^1(\mathbf{R}^3 \times \mathbf{R}^3, d\mathbf{v} \otimes d\mathbf{x})$, hence $H(f^{(l)}(t)) \to H(f(t))$, as $l \to \infty$, by the dominated convergence theorem. By (24), and the definition of $f_{k,0}^{(l)}$, the conditions of Prop. 4.b) are fulfilled by $f_{k,0}^{(l)}$, for each l. Then the function $t \to H(f^{(l)}(t))$ is non decreasing. Consequently, the same is true for $H(f)$, concluding the proof. □

R e m a r k. Rel. (39) is satisfied by local maxwellians ([10]) and it provides a generalization of the law of the mass action.

Indeed, under condition (36), let the local maxwellians solving Eq. (2) be given by

$$\omega_n = \omega_n(q_n, u, T) := q_n (m_n / 2\pi k T)^{3/2} \exp[-m_n (\mathbf{v} - \mathbf{u})^2 / 2kT], \tag{43}$$

for all $n = 1, \dots, N$. Here $q_n = q_n(t, \mathbf{x})$ is the concentration of species n, while $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ and $T = T(t, \mathbf{x})$ are the notations for the gas bulk velocity and the equilibrium temperature, respectively (k denotes the Boltzmann constant).

Corollary 1. *The concentrations* q_n , $n = 1, \ldots, N$, *satisfy non trivially the law of the mass action, i.e., for all* α, β

$$\sum_{n=1}^{N} (\alpha_n - \beta_n) \left[\frac{3}{2} \log (m_n / 2\pi kT) + \log (C_n q_n) + E_n / kT \right] \equiv 0. \tag{44}$$

P r o o f. The result is immediate from (39), by the mass conservation condition, namely $K_{\alpha, \beta} = 0$ when $\sum_n (\alpha_n - \beta_n) m_n = 0$. $\square$

5. Final remarks

Theorem 1 is applicable to the reactive, expanding gas. Our global existence results do not cover the case in which endo-energetic chemical reactions (collisions) are present in the gas processes. In the latter case, one should avoid possible pathologies, introduced by the particles which may loose completely their relative kinetic energy during the endo-energetic reactions.

The balance conditions (36) play no role in Theorems 1, 2, but it is essential for the validity of the results stated in Theorem 3.

Theorem 1 can be analogously proved considering instead of $\dot{C}_0$ the space of those $f = (f_1, \ldots, f_N)$ with $f_n \in L^1 (\mathbf{R}^3 \times \mathbf{R}^3; d\mathbf{v} \otimes d\mathbf{x})$, equipped with the norm $\max_{1 \le k \le N} |f_k|_{L^1}$. In the latter case, the continuity condition on $K_{\alpha, \beta}$ can be replaced by measurability. Then $\dot{C}_\tau$ (with $\tau > 0$) need be replaced by a space of measurable functions (e.g. $\dot{C}_{n, \tau}$, can be replaced by the space of those $h \in L^\infty (\mathbf{R}^3 \times \mathbf{R}^3; d\mathbf{v} \otimes d\mathbf{x})$, with $ess \sup \exp [\tau (\mathbf{x}^2 + m_k \mathbf{v}^2)] \, | h(\mathbf{v}, \mathbf{x}) | < \infty$).

Some of the results of this paper have been announced in [12], (where the main theorem is actually valid for $\tau = 0$, since $\{ U^t \}_{t \in \mathbf{R}}$ is not a continuous group of isometries on $\dot{C}_\tau$ for $\tau > 0$).

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Appendix A

Let n be non-negative integer and $a_1, \ldots, a_n > 0$, constants. Consider a positive quadratic form $T := T(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \sum_{i=1}^{n} a_i \mathbf{v}_i^2$ on $\mathbf{R}^{3n}$, $r_i \in \mathbf{R}^3$, $1 \le i \le n$. Consider the transformation

$$\mathbf{R}^{3n} \ni (\mathbf{v}_1, \ldots, \mathbf{v}_n) \to (\underline{V}, \zeta) \in \mathbf{R}^3 \times \mathbf{R}^{3n-3}, \tag{A.1}$$

defined by

$$\underline{V} := \Big(\sum_{i=1}^{n} a_i\Big)^{-1} \sum_{i=1}^{n} a_i \mathbf{v}_i,$$

$$\zeta := (\zeta_1, \ldots, \zeta_{n-1}),$$

$$\zeta_i := \mathbf{v}_{i+1} - \Big(\sum_{j=1}^{i} a_j\Big)^{-1} \sum_{j=1}^{i} a_j \mathbf{v}_j, \quad i = 1, \ldots, n-1.$$

By transformation (A.1), the form T becomes

$$T = T(\underline{V}, \zeta) = \Big(\sum_{i=1}^{n} a_i\Big) \underline{V}^2 + \sum_{i=1}^{n-1} \mu_i \zeta_i^2 ,$$

with

$$\mu_i^{-1} = a_{i+1}^{-1} + \Big(\sum_{j=1}^{i} a_j\Big)^{-1}, \quad i = 1, \ldots, n-1.$$

The system of coordinates on $\mathbf{R}^{3n}$ resulting from (A.1) will be called a Jacobi system of coordinates associated to $T(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Moreover, the same term will designate the system of coordinates obtained by the transformation

$$\mathbf{R}^{3n} \ni (\mathbf{v}_1, \ldots, \mathbf{v}_n) \to (\underline{V}, \xi) \in \mathbf{R}^3 \times \mathbf{R}^{3n-3}, \tag{A.2}$$

where $\xi := (\xi_1, \ldots, \xi_{n-1})$ and $\xi_i := \mu_i^{1/2} \zeta_i$, with $\underline{V}$ and ζ_i as in (A.1); $1 \le i \le n-1$. Obviously, by (A.2), $T = T(\underline{V}, \xi) = (\sum_{i=1}^{n} a_i) \underline{V}^2 + \xi^2$.

Appendix B

Part a) of Lemma 1 is straightforward. To prove b), we first consider $|\gamma| = 2$ (with $\gamma_k \geq 1$). Then, by (15) clearly, there exist two non-negative constants c_0 and c such that

$$\Gamma_{k\gamma}(t, \mathbf{v}, \mathbf{x}) \leq$$

$$\leq c_0 \int_{\mathbf{R}^3} d\,\mathbf{y}(1 + |\mathbf{y} - \mathbf{v}|^q) \exp(-c\,\mathbf{y}^2) \int_0^t ds \exp\{-c\,[\mathbf{x} - (\mathbf{y} - \mathbf{v})s]^2\} \qquad \text{(B.1)}$$

for all $t \geq 0$, $\mathbf{x}, \mathbf{v} \in \mathbf{R}^3$, $q \in [0, 1]$.

By (B.1), the sup estimation on $\Gamma_{k\gamma}(t, \mathbf{v}, \mathbf{x})$ reduces to known inequalities of Lemma 2.5 in [10]. Briefly, in this case, for each $t \geq 0$,

$$\Gamma_0(t, \mathbf{v}, \mathbf{x}) := \int_0^t ds \exp\{-c\,[\mathbf{x} - (\mathbf{y} - \mathbf{v})s]^2\} \leq$$

$$\leq \int_0^t ds \exp\{-c\,[|\mathbf{x}| - |\mathbf{y} - \mathbf{v}|\,s]^2\} \leq |\mathbf{y} - \mathbf{v}|^{-1}. \qquad \text{(B.2)}$$

Introducing (B.2) in the right side of (B.1), and integrating with respect to the reference frame with the $\mathbf{y}_3$ axis oriented in the direction of $\mathbf{v}$, we get

$$\Gamma_{k\gamma}(t, \mathbf{v}, \mathbf{x}) \leq \int_{-\infty}^{+\infty} d\,\mathbf{y}_3 \exp(-c\,\mathbf{y}_3^2) \int_0^\infty \rho \frac{1 + [\rho^2 + (\mathbf{y}_3 - \mathbf{v})^2]^{q/2}}{[\rho^2 + (\mathbf{y}_3 - \mathbf{v})^2]^{1/2}} \exp[-c\,\rho^2]\, d\rho \leq$$

$$\leq const \int_{-\infty}^{+\infty} d\,\mathbf{y}_3 \exp(-c\,\mathbf{y}_3^2) \int_0^\infty (1 + \rho^q) \exp[-c\,\rho^2]\, d\rho \leq const.$$

The case $|\gamma| > 2$ (with $\gamma_k \geq 1$) can be reduced to $|\gamma| = 2$ as follows. With the notations of (20), consider the form $T_\gamma(\tilde{\mathbf{w}}_{(k)}) := W_\gamma(\mathbf{w}) - \sum_{n=1}^{N} \gamma_n E_n - 2^{-1} m_k \mathbf{w}^2_{k,\gamma_k}$, representing the kinetic energy of $|\gamma| - 1$ particles in the channel γ (more precisely, the kinetic energy of all the particles in channel γ, except the particle with velocity

w_{k,γ_k}). To $T_\gamma(\tilde{\mathbf{w}}_{(k)})$, we associate a Jacobi system of coordinates $\mathbf{R}^{3|\gamma|-3} \ni \tilde{\mathbf{w}}_{(k)} \to (\underline{V}, \xi) \in \mathbf{R}^3 \times \mathbf{R}^{3|\gamma|-6}$, of type (A.2) in Appendix A. Then, in the new variables, $m^{-1}(\sum_{n \in \mathcal{N}(\gamma)} \sum_{i=1}^{\gamma_n} m_n w_{n,i} - m_k w_{k,\gamma_k}) = \underline{V}$, and $T_\gamma(\tilde{\mathbf{w}}_{(k)}) = \frac{m}{2}\underline{V}^2 + \xi^2$ with $m := \sum_{n=1}^{N} \gamma_n m_n - m_k$. A few simple manipulations show that on $\{\mathbf{w} \in \mathbf{R}^{3|\gamma|} \,|\, \mathbf{w}_{k,\gamma_k} = \mathbf{v}\}$, both Φ_γ, given by (19), and the relative energy $W_{r,\gamma}$ of *all* the particles in channel γ (see Section 2), can be written in terms of $(\underline{V}, \xi) \in \mathbf{R}^3 \times \mathbf{R}^{3(|\gamma|-2)}$ as

$$\Phi_\gamma(t, \mathbf{w}, \mathbf{x}, \mathbf{v})_{|\mathbf{w}_{k,\gamma_k} = \mathbf{v}} = m_k(\mathbf{x}^2 + \mathbf{v}^2) + m\underline{V}^2 + $$

$$+ 2(1 + t^2)\xi^2 + m[\mathbf{x} - (\underline{V} - \mathbf{v})t]^2, \tag{B.3}$$

and

$$W_{r,\gamma}(\mathbf{w})_{|\mathbf{w}_{k,\gamma_k} = \mathbf{v}} = \xi^2 + 2^{-1}(m_k^{-1} + m^{-1})^{-1}(\underline{V} - \mathbf{v})^2. \tag{B.4}$$

Then we choose $(\underline{V}, \xi)$ as new integration variables for the integral upon $d\tilde{\mathbf{w}}_{(k)}$ in (20) and we introduce (B.3) and (B.4) in (20). One obtains that there exist two constants c_0 and c such that

$$\Gamma_{k,\gamma}(t, \mathbf{v}, \mathbf{x}) \le c_0 \int_{\mathbf{R}^3} d\underline{V} \exp(-c\underline{V}^2) \int_{\mathbf{R}^{3(|\gamma|-2)}} d\xi \exp(-c\xi^2) \times$$

$$\times \{1 + [\xi^2 + (\underline{V} - \mathbf{v})^2]^{q/2}\} \int_0^t ds \exp[-c[x - (\underline{V} - \mathbf{v})s]^2], \tag{B.5}$$

$t \ge 0$, $\mathbf{v}, \mathbf{x} \in \mathbf{R}^3$, $q \in [0, 1]$.

Since for some constant $c_1 > 0$,

$$1 + [\xi^2 + (\underline{V} - \mathbf{v})^2]^{q/2} \le c_1 \cdot (1 + |\xi|^q)(1 + |\underline{V} - \mathbf{v}|^q), \tag{B.6}$$

we introduce (B.6) in (B.5) and integrating with respect to ξ we obtain

$$\Gamma_{k,\gamma}(t, \mathrm{v}, \mathrm{x}) \leq$$

$$\leq c_2 \int_{\mathbb{R}^3} d\,\underline{V}\,(1 + |\,\underline{V} - \mathrm{v}|^q)\exp(-c\underline{V}^2)\int_0^t ds \exp\left\{-c\,[x - (\,\underline{V} - \mathrm{v})s\,]^2\right\},$$

with c_1 , c_2 > 0, constants. This is exactly (B.I), concluding the proof. □

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О классе кинетических уравнений для реагирующих смесей газов

Рассматривается широкий класс кинетических уравнений для реальных газов с, возможно, многократными неупругими взаимодействиями и химическими реакциями. Мы доказываем существование, единственность и положительность решений задачи Коши и получаем законы сохранения массы, момента и энергии, Н-теорему, а также закон действующих масс.

Про клас кінетичних рівнянь для реагуючих сумішей газів

Розглянуто великий клас кінетичних рівнянь для дійсних газів з, можливо, багаторазовими непружними взаємодіями та хімічними реакціями. Ми доводимо існування, єдність та позитивність розв'язків задачі Коші та одержуємо закони збереження маси, моменту та енергії, Н-теорему, а також закон діючих мас.