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Analytic and asymptotic properties of multivariate Linnik's distribution

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The paper deals with properties of k-variate $(k \geq 2)$ Linnik's distribution defined by the characteristic function

$$\varphi_{\alpha k}(t) = 1/(1 + |t|^{\alpha}), \quad 0 < \alpha < 2, \quad t \in \mathbf{R}^k,$$

where $|t|$ denotes Euclidean norm of vector $t \in \mathbf{R}^k$. This distribution is absolutely continuous with respect to the Lebesgue measure in $\mathbf{R}^k$. Expansions of the density of the distribution into asymptotic and convergent series in powers of $|t|$, $|t|^{\alpha}$ are obtained. The forms of these expansions depend substantially on the arithmetical nature of the parameter α.

1. Introduction

In 1953, Ju.V. Linnik had proved [1] that the function

$$\varphi_{\alpha}(t) = \frac{1}{1 + |t|^{\alpha}}, \quad 0 < \alpha < 2, \quad -\infty < t < \infty,$$

is a characteristic function of an absolutely continuous distribution. Recently, this distribution attracted attention of a number of researchers who discovered some its interesting probabilistic properties and applications [2-9]. Analytic and asymptotic properties of the density $p_{\alpha}(x)$ of Linnik's distribution have been studied in [10]. The results of [10] show that the asymptotic expansion of $p_{\alpha}(x)$ as $x \to \infty$ is similar, to some extent, to those of stable densities. However, the expansion of the density $p_{\alpha}(x)$ into convergent series is quite different from that of the stable one. It depends substantially on the arithmetic nature of the parameter α.

This paper is devoted to multivariate generalizations of the results of [10]. The multivariate Linnik's distribution was introduced by D.N. Anderson [5] who proved that the function

$$\varphi_{\alpha k \Sigma}(t) = \frac{1}{1 + |t' \Sigma t|^{\alpha/2}}, \quad 0 < \alpha < 2, \quad t \in \mathbf{R}^k,$$

where Σ is a positive $k \times k$ matrix, is a characteristic function of a k-variate probability distribution which he called *the k-variate Linnik distribution*. It is evident that without loss

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of generality we can restrict our attention to the case when the matrix Σ coincides with the identity matrix I. We set

$$\varphi_{akI}(t) = \varphi_{ak}(t) = \frac{1}{1 + |t|^{\alpha}}, \quad 0 < \alpha < 2, \quad t \in \mathbf{R}^k, \tag{1.1}$$

where $|t|$ denotes the Euclidean norm of the vector $t \in \mathbf{R}^k$. We shall show that the distribution defined by characteristic function (1.1) is absolutely continuous with respect to k-dimensional Lebesgue measure. The corresponding density will be denoted by $p_{\alpha k}(x)$, $x \in \mathbf{R}^k$. It possesses spherical symmetry since $\varphi_{\alpha k}(t)$ does. Therefore we can set

$$p_{\alpha k}(x) = q_{\alpha k}(|x|), \quad x \in \mathbf{R}^k$$

where $q_{\alpha k}(r)$ is a function defined on the half-axis $0 \leq r < \infty$. Our aim is a description of analytic and asymptotic properties of the function $q_{\alpha k}(r)$ and its representations by integrals and series.

The method of this work seems to be much simpler than that of [10]. It is based on the representation of $q_{\alpha k}(x)$ by the contour integral (see Theorem 3 below). In the case $k = 1$ this representation has been obtained in [10] by the method based on the properties of Cauchy type integrals. Though that method is of interest by its own since it establishes connections between Linnik's distributions and the Riemann–Hilbert boundary problem of the Theory of Analytic Functions, it seems to be impossible to use it when investigating the case $k \geq 2$. In this work, we prove the above mentioned representation directly and then use it to investigate $q_{\alpha k}(r)$ for any $\alpha \in (0, 2)$ and any $k \geq 1$. In [10], the partial case $k = 1$ of that representation was derived from the expansions of $q_{\alpha 1}(r)$ for rational α's and then was used to study of $q_{\alpha 1}(r)$ in the case of irrational α. Thus, the method of the present paper seems to be more straightforward than that of [10] and, moreover, we do not use any result of [10]. Newertheless, it should be mentioned that the present paper does not contain any new result concerning the case $k = 1$.

2. Statement of results. We start with absolute continuity of multivariate Linnik's distribution and integral representations of its density.

Theorem 1. *The function* (1.1) *is the characteristic function of a k-variate absolutely continuous spherically symmetric distribution whose density* $p_{\alpha,k}(x)$ *can be represented by the formula*

$$p_{\alpha k}(x) = q_{\alpha k}(|x|) =$$

$$= \frac{\sin \frac{\pi\alpha}{2}}{2^{(k/2)-1}\pi^{(k/2)+1}|x|^{(k/2)-1}} \int_0^{\infty} \frac{K_{(k/2)-1}(|x|u)\, u^{(k/2)+\alpha}\, du}{|1 + u^{\alpha} e^{i\pi\alpha/2}|^2}, \quad x \in \mathbf{R}^k, \tag{2.1}$$

where $K_{\nu}(z)$ *is the Bessel function of the third kind (the Macdonald function).*

The definition of the function $K_\nu(z)$ can be found in [11], p. 78. Note that ([11], p. 79, Eq. (8), p. 80, Eq. (13))

$$K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}.$$

Therefore, if $k = 1$, formula (2.1) coincides with Linnik's formula [1]

$$p_{\alpha 1}(x) = \frac{\sin\frac{\pi\alpha}{2}}{\pi} \int_0^\infty \frac{e^{-|x|u} u^\alpha \, du}{|1 + u^\alpha e^{i\pi\alpha/2}|^2}, \quad x \in \mathbf{R}^1, \tag{2.2}$$

which plays the basic role in [10]. The role of formula (2.1) in the present paper is more modest. We use it only to prove the following theorem, which concerns the function $q_{\alpha k}(r)$ connected with $p_{\alpha k}(x)$ by the equation $p_{\alpha k}(x) = q_{\alpha k}(|x|)$.

Theorem 2. *The function $q_{\alpha k}(r)$ is completely monotonic on the half-axis $[0, \infty)$ that is it can be extended analytically to the half-plane $\{r : \operatorname{Re} r > 0\}$ and, for any $r > 0$ and any $n = 0, 1, 2, \ldots$, we have $(-1)^n q_{\alpha k}^{(n)}(r) > 0$.*

The base of the rest of results of our paper is a representation of $q_{\alpha k}(r)$ by a contour integral. To write down that representation, we need the following function of a complex variable

$$f_{\alpha k}(z) = -\frac{i}{\alpha 2^k \pi^{(k/2)-1}} \cdot \frac{2^{-z}}{\sin\frac{\pi(z+k-1)}{\alpha} \cos\frac{\pi z}{2} \Gamma\left(\frac{z+1}{2}\right) \Gamma\left(\frac{z+k-1}{2}\right)}. \tag{2.3}$$

This is a meromorphic function. The set of its poles lying in the half-plane $\{z : \operatorname{Re} z \geq -k+1\}$ coincides with the union

$$\{q\alpha - k + 1\}_{q=1}^\infty \cup \{2q+1\}_{q=0}^\infty.$$

The set of its poles lying in the half-plane $\{z : \operatorname{Re} z \leq -k+1\}$ is in general an infinite part of the sequence

$$\{-q\alpha - k + 1\}_{q=1}^\infty;$$

it can coincide with the whole sequence if e.g. α is an irrational number. Note that the point $z = -k+1$ is not a pole of $f_{\alpha k}(z)$. This function is analytic in the strip

$$\{z : -k + 1 - \alpha < \operatorname{Re} z < -k + 1 + \delta\}, \quad \delta = \min(1, \alpha).$$

Using the well-known formula $\Gamma(w)\Gamma(1-w) = \pi/\sin\pi w$, we can rewrite formula (2.3) in the form

$$f_{\alpha k}(z) = -\frac{i}{\alpha 2^k \pi^{k/2}} \frac{2^{-z}\Gamma\left(\frac{1-z}{2}\right)}{\sin\frac{\pi(z+k-1)}{\alpha} \Gamma\left(\frac{z+k-1}{2}\right)}. \tag{2.4}$$

The Stirling formula shows that in any fixed strip of the form: $\{z : |\operatorname{Re} z| < H\}$ the following asymptotic formula is valid

$$\log |f_{\alpha k}(z)| = -\left(\frac{\pi}{\alpha} + o(1)\right) |\operatorname{Im} z|, \quad z \to \infty, \tag{2.5}$$

and hence $f_{\alpha k}(z)$ decreases exponentially along vertical lines.

Theorem 3. *The following formula is valid*

$$q_{\alpha k}(r) = \int_{c - i\infty}^{c + i\infty} f_{\alpha k}(z) r^{z-1} dz, \tag{2.6}$$

where

$$-k + 1 - \alpha < c < -k + 1 + \delta, \quad \delta = \min(1, \alpha). \tag{2.7}$$

The integral in (2.6) *converges absolutely and uniformly with respect to r on any compact subset of* $(0, \infty)$.

Note that in virtue of the properties of $f_{\alpha k}(z)$ mentioned above, the integral in (2.6) does not depend on c under restriction (2.7).

The rest of the results of this paper (except Theorem 7) follows from representation (2.6) by the evaluation of the integral in the right hand side by the Cauchy Residue Theorem. In the case $r \to \infty$ we use closed increasing contours (more precisely, the boundaries of the rectangles $\{z : -k + 1 - (l + \frac{1}{2})\alpha < \operatorname{Re} z < c, \ |\operatorname{Im} z| < R\}$, $l, R \to \infty$, $l \in N$) lying to the left of the line of integration in (2.6). The poles of $f(z)$ in the half-plane $\{z : \operatorname{Re} z < c\}$ are simple and form a part of an arithmetical progression, therefore we do not meet any difficulties. In the case when r is bounded, we have to use the similar contour but lying to the right of the line of integration in (2.6). The situation with the poles of $f(z)$ lying in the half-plane $\{z : \operatorname{Re} z < c\}$ is more complicated and depends on the arithmetic nature of the parameter α. The following cases are possible: the poles are simple and separated (i.e. the infimum of distances between any of two poles is positive); the poles are simple but not separated; infinitely many of the poles are double. This situation generates the dependence of the expansions of $q_{\alpha k}(r)$ into convergent series on the arithmetic nature of α.

We start with the theorem dealing with the asymptotic behaviour of $q_{\alpha k}(r)$ as $r \to \infty$. The arithmetic nature of α does not play any role here.

Theorem 4. *The asymptotic behaviour of $q_{\alpha k}(r)$ as $r \to \infty$ can be described by the following asymptotic (divergent) series*

$$q_{\alpha k}(r) \sim \frac{1}{\pi^{(k/2)+1}} \sum_{q=1}^{\infty} \left\{(-1)^{q-1} 2^{q\alpha} \Gamma\left(\frac{k + q\alpha}{2}\right) \Gamma\left(1 + \frac{q\alpha}{2}\right) \sin \frac{\pi\alpha q}{2}\right\} r^{-q\alpha - k}, \quad r \to \infty. \tag{2.8}$$

Corollary. *The following asymptotic formula is valid*

$$q_{\alpha k}(r) \sim \left\{\frac{2^{\alpha}}{\pi^{(k/2)+1}} \Gamma\left(\frac{k + \alpha}{2}\right) \Gamma\left(1 + \frac{\alpha}{2}\right) \sin \frac{\pi\alpha}{2}\right\} r^{-\alpha - k}, \quad r \to \infty.$$

Now we shall consider representations of $q_{\alpha k}(r)$ by convergent series. As mentioned above the forms of such representations depend substantially on the arithmetic nature of the parameter α. In general, the form is more complicated for rational α. Therefore we start with the theorem dealing with all irrational α's and with a few of rational ones only. However, the series in this theorem is convergent in a rather special sense.

Theorem 5. *Suppose, one of the following conditions is satisfied:*
(i) *α is an irrational number;*
(ii) *the dimension k is odd, α is a rational number of the form $\alpha = m/n$ where m, n are relatively prime integers and m is even.*
Then the following formula is valid

$$q_{\alpha k}(r) = \frac{1}{r^k} \lim_{s \to \infty} \left\{ \frac{1}{\pi^{(k/2)-1}} \sum_{q=1}^{s} \frac{(-1)^{q+1} (r/2)^{q\alpha}}{\cos \frac{\pi(q\alpha - k + 1)}{2} \Gamma\left(\frac{q\alpha - k + 2}{2}\right) \Gamma\left(\frac{q\alpha}{2}\right)} + \right.$$

$$\left. + \frac{2}{\alpha \pi^{(k/2)-1}} \sum_{1 \le 2q+1 < -k+1+\left(s+\frac{1}{2}\right)\alpha} \frac{(-1)^q (r/2)^{q\alpha + k}}{\sin \frac{\pi(2q+k)}{\alpha} \Gamma(q+1) \Gamma\left(q + \frac{k}{2}\right)} \right\}. \qquad (2.9)$$

The limit is uniform with respect to r on any bounded subset of $[0, \infty)$.

The question arises if it is possible to take separate limits of two sums staying in the right hand side of (2.9). It turns out that this is the case for almost all values of $\alpha \in (0, 2)$ in the sense of the Lebesgue measure but not for all ones. To describe the situation more precisely, we need the Liouville numbers.

Recall that an irrational number β is called a Liouville number if, for any $\nu = 2, 3, \ldots$, there is a pair of integers (p, q) $(q \ge 2)$ such that

$$\left| \beta - \frac{p}{q} \right| < \frac{1}{q^\nu}.$$

It is well-known (see e.g. [12]) that the Liouville numbers are transcendent and form an everywhere dense set of cardinality continuum and zero Lebesgue measure.

Theorem 6. *If the number $\alpha \in (0, 2)$ satisfies the hypotheses of Theorem 5 and is not a Liouville number then the following formula is valid*

$$q_{\alpha k}(r) = \frac{1}{r^k} \left\{ \frac{1}{\pi^{(k/2)-1}} \sum_{q=1}^{\infty} \frac{(-1)^{q+1} (r/2)^{q\alpha}}{\cos \frac{\pi(q\alpha - k + 1)}{2} \Gamma\left(\frac{q\alpha - k + 2}{2}\right) \Gamma\left(\frac{q\alpha}{2}\right)} + \right.$$

$$\left. + \frac{2}{\alpha \pi^{(k/2)-1}} \sum_{q=0}^{\infty} \frac{(-1)^q (r/2)^{2q+k}}{\sin \frac{\pi(2q+k)}{\alpha} \Gamma(q+1) \Gamma\left(q + \frac{k}{2}\right)} \right\}, \qquad (2.10)$$

where both of the series in the right hand side converge absolutely and uniformly with respect to r on every bounded subset of $[0, \infty)$.

Corollary. *If the number $\alpha \in (0, 2)$ satisfies the hypotheses of Theorem 5 and is not a Liouville number, then the following representation is valid*

$$q_{\alpha k}(r) = \frac{1}{r^k} \Lambda_{\alpha k}(r^\alpha) + N_{\alpha k}(r^2),$$

where $\Lambda_{\alpha k}(z)$ and $N_{\alpha k}(z)$ are entire functions of the orders $1/\alpha$ and $1/2$ respectively.

Theorem 7. *The values of α, such that both of the series in the right hand side of* (2.10) *diverge, form a dense set in* (0,2) *of cardinality continuum.*

Now we shall consider the case when the parameter α does not satisfy the hypotheses of Theorem 5. In this case α is rational and can be represented in the form $\alpha = m/n$, where m, n are relatively prime integers. Taking into account that the case, when the dimension k is odd but m is even, is covered by Theorem 6, it remains to consider the following cases:
(i) k is odd, m is odd;
(ii) k is even, m is odd;
(iii) k is even, m is even.
It is easy to verify, by standard methods, that in these cases the Diophantine equation

$$pm = (2q + k)n \tag{2.11}$$

has infinitely many solutions (p, q). Denote by $D_{\alpha k}$ the set of all solutions (p, q) and put

$$P_{\alpha k} = \{ p : (p, q) \in D_{\alpha k}, p \geq 1 \}, \tag{2.12}$$

$$Q_{\alpha k} = \{ q : (p, q) \in D_{\alpha k}, q \geq 0 \}. \tag{2.13}$$

It is possible to describe the sets $P_{\alpha k}$, $Q_{\alpha k}$ in an explicit form (see Section 8 below), they both are infinite.

Theorem 8. *If $\alpha \in (0, 2)$ is a rational number satisfying one of conditions* (i)-(iii), *then the following formula is valid*

$$q_{\alpha k}(r) = \frac{1}{r^k} \left\{ \frac{1}{\pi^{(k/2)-1}} \sum_{\substack{p=1 \\ p \notin P_{\alpha k}}}^{\infty} \frac{(-1)^{p+1} (r/2)^{p\alpha}}{\cos \frac{\pi(p\alpha - k + 1)}{2} \Gamma\left(\frac{p\alpha - k + 2}{2}\right) \Gamma\left(\frac{p\alpha}{2}\right)} + \right.$$

$$+ \frac{2}{\alpha \pi^{(k/2)-1}} \sum_{\substack{q=0 \\ q \notin Q_{\alpha k}}}^{\infty} \frac{(-1)^q (r/2)^{2q+k}}{\sin \frac{\pi(2q+k)}{\alpha} \Gamma(q+1) \Gamma\left(q + \frac{k}{2}\right)} +$$

$$\left. + \frac{2}{\pi^{k/2}} \sum_{q \in Q_{\alpha k}} \frac{(-1)^{q+p} (r/2)^{2q+k}}{\Gamma(q+1)\Gamma\left(q + \frac{k}{2}\right)} \left\{ \log \frac{r}{2} - \frac{\Gamma'(q+1)}{2\Gamma(q+1)} - \frac{\Gamma'\left(q + \frac{k}{2}\right)}{2\Gamma\left(q + \frac{k}{2}\right)} \right\} \right\}, \tag{2.14}$$

where p and q in the third sum are connected by equation (2.11). *All three series in the right hand side of* (2.14) *converge absolutely and uniformly with respect to r on every bounded subset of* $[0, \infty)$.

Corollary. *If* $\alpha \in (0, 2)$ *is a rational number satisfying one of conditions* (i)-(iii), *then the following representation is valid*

$$q_{\alpha k}(r) = \frac{1}{r^k}\Lambda_{\alpha k}(r^\alpha) + M_{\alpha k}(r^2)\log\frac{r}{2} + N_{\alpha k}(r^2),$$

where $\Lambda_{\alpha k}(z)$, $M_{\alpha k}(z)$, $N_{\alpha k}(z)$ *are entire functions of the orders* $1/\alpha$, $1/2$, $1/2$ *respectively.*

The asymptotic expansions of $q_{\alpha k}(r)$ as $r \to \infty$ are immediate consequences of expansions (2.10) and (2.14) if α satisfies either the hypotheses of Theorem 6 or of Theorem 8 since any convergent power series can be considered as an asymptotic one. But it turns out that if α does not satisfy neither the hypotheses of Theorem 6 nor of Theorem 8, the asymptotic behaviuor of $q_{\alpha k}(r)$ as $r \to 0$ is the same as in the statement of Theorem 6. The following theorem covers this case.

Theorem 9. *Suppose, the parameter* $\alpha \in (0, 2)$ *satisfies the hypotheses of Theorem 5. Then formula* (2.10) *remains true if the both series in its right hand side are considered as asymptotic ones as* $r \to 0$.

3. The First Integral Representation of $q_{\alpha k}(r)$. **(Proofs of Theorems 1 and 2.)** To prove Theorem 1, we define the function $p_{\alpha k}(x)$ by formula (2.1) and shall show that it is a probability density in $\mathbf{R}^k$ and its characteristic function coincides with (1.1).

The formula (see [11], p. 172, Eq. (5))

$$K_\nu(z) = \frac{\Gamma\left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-z\cosh\theta}\sinh^{2\nu}\theta\, d\theta \quad \left(|\arg z| < \frac{\pi}{2},\ \mathrm{Re}\,\nu > -\frac{1}{2}\right) \tag{3.1}$$

shows that $K_{(k/2)-1}(v) \ge 0$ for $v > 0$, $k = 1, 2, \ldots$ (for $k = 1$ we use the property ([11], p. 79, Eq. (8)) $K_{-\nu}(z) = K_\nu(z)$). The asymptotic behaviour of $K_{(k/2)-1}(v)$ as $v \to 0$ and $v \to \infty$ can be described by the formulas

$$K_{(k/2)-1}(v) = O\left(v^{-|(k/2)-1|}\log\frac{1}{v}\right), \quad v \to +0, \tag{3.2}$$

$$K_{(k/2)-1}(v) = O(e^{-v}), \quad v \to \infty, \tag{3.3}$$

(the first one is an immediate consequence of Eq.'s (12)-(15) of [11], p. 80; the second formula is that of Eq. (1) of [11], p. 202). Therefore, the integral in the right hand side of (2.1) is convergent and positive.

Using Fubini's theorem, which allows to change the order of integration when the integrand is non-negative, we obtain

$$\int_{\mathbf{R}^k} p_{\alpha k}(x)\,dx = \frac{2\pi^{k/2}}{\Gamma(k/2)}\int_0^\infty q_{\alpha k}(r) r^{k-1}\,dr =$$

$$= \frac{\sin \frac{\pi\alpha}{2}}{2^{(k/2)-2}\pi\,\Gamma(k/2)} \int_0^\infty \frac{u^{(k/2)+\alpha}\,du}{|1 + u^\alpha e^{i\pi\alpha/2}|^2} \int_0^\infty K_{(k/2)-1}(ru)\, r^{k/2}\,dr =$$

$$= \frac{\sin \frac{\pi\alpha}{2}}{2^{(k/2)-2}\pi\,\Gamma(k/2)} \int_0^\infty \frac{u^{\alpha-1}\,du}{|1 + u^\alpha e^{i\pi\alpha/2}|^2} \int_0^\infty K_{(k/2)-1}(v)\, v^{k/2}\,dv < \infty.$$

Hence, $p_{\alpha k}(x)$ is summable in $\mathbf{R}^k$.

Using Schoenberg's formula of the Fourier Transform of a spherically symmetric distribution (see [13], Ch. 5, Theorem 5.4, Eq. (6)) and then substituting there expression (2.1) of $q_{\alpha k}(r)$, we obtain, for $t \in \mathbf{R}^k$,

$$\varphi_{\alpha k}(t) := \int_{\mathbf{R}^k} p_{\alpha k}(x)\, e^{i(t,x)}\,dx = \frac{(2\pi)^{k/2}}{|t|^{(k/2)-1}} \int_0^\infty J_{(k/2)-1}(|t|s) q_{\alpha k}(s)\, s^{k/2}\,ds =$$

$$= \frac{2\sin\frac{\pi\alpha}{2}}{\pi\,|t|^{(k/2)-1}} \int_0^\infty \frac{u^{(k/2)+\alpha}\,du}{|1 + u^\alpha e^{i\pi\alpha/2}|^2} \int_0^\infty J_{(k/2)-1}(|t|s) K_{(k/2)-1}(su)\, s\,ds, \qquad (3.4)$$

where $J_\nu(z)$ is the Bessel function of the first kind. Now we shall use the formula

$$\int_0^\infty K_\nu(as)\, J_\nu(bs) s\,ds = \frac{(b/a)^\nu}{a^2 + b^2} \qquad (\mathrm{Re}\,\nu > -1,\ a > 0,\ b > 0), \qquad (3.5)$$

which we obtain from the Weber-Schafheitlin formula ([11], p. 210, Eq. (2)), setting in the latter $\mu = -\nu$ and using the identity $K_{-\nu}(z) = K_\nu(z)$ ([11], p. 79, Eq. (8)). Substituting (3.5) into (3.4), we have

$$\varphi_{\alpha k}(t) = \frac{2\sin\frac{\pi\alpha}{2}}{\pi\,|t|^{(k/2)-1}} \int_0^\infty \frac{u^{(k/2)+\alpha}}{|1 + u^\alpha e^{i\pi\alpha/2}|^2} \frac{(|t|/u)^{(k/2)-1}}{|t|^2 + u^2}\,du =$$

$$= \frac{2}{\pi}\,\mathrm{Im} \int_0^\infty \frac{u\,du}{(1 + u^\alpha e^{-i\pi\alpha/2})(|t|^2 + u^2)} = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} \frac{z\,dz}{(1 + z^\alpha)(|t|^2 - z^2)} =$$

$$= -\mathrm{Res}_{z=|t|} \left(\frac{z}{(1 + z^\alpha)(|t|^2 - z^2)} \right) = \frac{1}{1 + |t|^\alpha}.$$

This shows the validity of Theorem 1.

Theorem 2 is a corollary of Theorem 1. We restrict our attention to the case $k \geq 2$ since the case $k = 1$ follows from Linnik's formula (2.2). Using the formula (3.1) with $\nu = (k/2) - 1$, $z = ru$, we obtain from (2.1)

$$q_{\alpha k}(r) = \frac{\sin \frac{\pi\alpha}{2}}{2^{k-2}\pi^{(k+1)/2}\Gamma\left(\frac{k-1}{2}\right)} \int_0^\infty \frac{u^{k+\alpha-1}\,du}{|1+u^\alpha e^{i\pi\alpha/2}|^2} \int_0^\infty e^{-r\cosh\theta} \sinh^{k-2}\theta\, d\theta =$$

$$= \frac{\sin \frac{\pi\alpha}{2}}{2^{k-2}\pi^{(k+1)/2}\Gamma\left(\frac{k-1}{2}\right)} \int_0^\infty e^{-rv} v^{k+\alpha-1}\,dv \int_0^\infty \frac{\cosh^{-k+\alpha}\theta \sinh^{k-2}\theta\, d\theta}{|\cosh^\alpha\theta + v^\alpha e^{i\pi\alpha/2}|^2}. \tag{3.6}$$

Hence, the function $q_{\alpha k}(r)$ admits a representation of the form

$$q_{\alpha k}(r) = \int_0^\infty e^{-ru} h(u)\,du, \quad r > 0, \tag{3.6}$$

where $h(u)$ is non-negative on $[0, \infty)$. Since the integral in (3.6) converges for any $r > 0$, it converges absolutely and uniformly in each half-plane $\{r : \operatorname{Re} r > \delta > 0\}$. Evidently, (3.6) yields the inequality $(-1)^n q_{\alpha k}^{(n)}(r) > 0$ for any $r > 0$ and any $n = 0, 1, 2, \ldots$.

4. The Second Integral Representation of $q_{\alpha k}(r)$. (Proof of Theorem 3). We restrict our attention to the case $k \geq 2$ since the case $k = 1$ has been considered in [10]. Though the latter case could be considered by our approach also, it needs a separate consideration.

To prove Theorem 3, we define the function $q_{\alpha k}(r)$ by formula (2.6) and set $p_{\alpha k}(x) = q_{\alpha k}(|x|)$, $x \in \mathbf{R}^k$. Then we shall show that $p_{\alpha k}(x)$ is summable in $\mathbf{R}^k$ and its Fourier transform $\varphi_{\alpha k}(t)$ coincides with (1.1). This will prove that the function $q_{\alpha k}(r)$ is the same as in Section 3.

From (2.6) and (2.7), it follows that $q_{\alpha k}(r)$ is a continuous function of r on $(0, \infty)$ and, for any c satisfying condition (2.7), we have

$$|q_{\alpha k}(r)| \leq Cr^{c-1},$$

where C is a positive constant depending on c. Choosing $c = -k + 1 + \varepsilon$ and $c = -k + 1 - \varepsilon$, where $0 < \varepsilon < \delta$, we obtain

$$q_{\alpha k}(r) = O(r^{-k+\varepsilon}), \quad r \to 0,$$

$$q_{\alpha k}(r) = O(r^{-k-\varepsilon}), \quad r \to \infty,$$

respectively, whence $p_{\alpha k}(x) = q_{\alpha k}(|x|)$ is summable in $\mathbf{R}^k$.

Using Schoenberg's formula and (2.6), we obtain for $t \in \mathbf{R}^k$

$$\varphi_{\alpha k}(t) = \int_{\mathbf{R}^k} p_{\alpha k}(x) e^{i(t,x)}\,dx = \frac{(2\pi)^{k/2}}{|t|^{(k/2)-1}} \int_0^\infty J_{(k/2)-1}(|t|s) q_{\alpha k}(s) s^{k/2}\,ds =$$

$$= \frac{2\pi^{k/2}}{|t|^{(k/2)-1}} \int_0^\infty J_{(k/2)-1}(|t|s)\, s^{k/2}\, ds \int_{c-i\infty}^{c+i\infty} f_{\alpha k}(z) s^{z-1}\, dz. \tag{4.1}$$

We shall show that if $c = -k + 1 + (\delta/2)$, then the hypotheses of Fubini's theorem allowing the change of the order of integration are fulfilled. For $s \in (0, 1)$, using (2.5) and the well-known bound of the Bessel function at zero, we obtain

$$\left| J_{(k/2)-1}(|t|s) s^{k/2} f_{\alpha k}(z) s^{z-1} \right| \le$$
$$\le Cs^{(k/2)-1} s^{k/2} e^{-\pi|z|/(2\alpha)} s^{c-1} = Cs^{\delta/2} e^{-\pi|z|/(2\alpha)},$$

where C depends neither on s nor on z. The last function is evidently summable over $(0, 1) \times (c - i\infty, c + i\infty)$. Further, using the well-known bound of the Bessel function at ∞, we obtain for $s \in [1, \infty)$

$$\left| J_{(k/2)-1}(|t|s) s^{k/2} f_{\alpha k}(z) s^{z-1} \right| \le$$
$$\le Cs^{1/2} s^{k/2} e^{-\pi|z|/(2\alpha)} s^{c-1} = Ds^{(-k-1+\delta)/2} e^{-\pi|z|/(2\alpha)},$$

where D depends neither on s nor on z. The last function is summable over $[1, \infty) \times (c - i\infty, c + i\infty)$ since $(-k - 1 + \delta)/2 < -1$ for $k \ge 2$. Thus, the change of the order of integration in (4.1) is justified.

Hence,

$$\varphi_{\alpha k}(t) = \frac{(2\pi)^{k/2}}{|t|^{(k/2)-1}} \int_{c-i\infty}^{c+i\infty} f_{\alpha k}(z)\, dz \int_0^\infty J_{(k/2)-1}(|t|s)\, s^{(k/2)+z-1}\, ds =$$

$$= \frac{(2\pi)^{k/2}}{|t|^{k-1}} \int_{c-i\infty}^{c+i\infty} f_{\alpha k}(z) |t|^{-z}\, dz \int_0^\infty J_{(k/2)-1}(v)\, v^{(k/2)+z-1}\, dv. \tag{4.2}$$

We shall use the following formula of Weber-Sonine-Schafheitlin ([11], p. 391, Eq.(1))

$$\int_0^\infty \frac{J_\nu(t)\, dt}{t^{\nu-\mu+1}} = \frac{\Gamma\left(\frac{1}{2}\mu\right)}{2^{\nu-\mu+1}\Gamma(\nu - \frac{1}{2}\mu + 1)}, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}\nu + \frac{1}{2}.$$

Setting $\nu = (k/2) - 1$, $\mu = z + k - 1$, we have

$$\int_0^\infty J_{(k/2)-1}(v)\, v^{(k/2)+z-1}\, dv = \frac{\Gamma((z + k - 1)/2)}{2^{-(k/2)-z+1}\Gamma((1 - z)/2)}.$$

Substituting this expression into (4.2) and recalling explicit expression (2.4) of $f_{\alpha k}(z)$, we obtain

$$\varphi_{\alpha k}(t) = -\frac{i}{2\alpha |t|^{k-1}} \int_{c-i\infty}^{c+i\infty} \frac{|t|^{-z}\, dz}{\sin \frac{\pi(z + k - 1)}{\alpha}}. \tag{4.3}$$

To evaluate the integral in the right hand side of (4.3) in the case $|t| \geq 1$, we consider the integral along the boundary of the half-disk $\left\{z : |z + k - 1| < \left(n + \frac{1}{2}\right)\alpha, \operatorname{Re} z > c\right\}$, apply to it the Cauchy residue theorem and then let $n \to \infty$. We obtain

$$\varphi_{\alpha k}(t) = -\frac{i}{2\alpha |t|^{k-1}} \left\{ -2\pi i \sum_{q=1}^{\infty} \operatorname{Res}_{z = q\alpha - k + 1} \left(\frac{|t|^{-z}}{\sin \frac{\pi(z + k - 1)}{\alpha}} \right) \right\} = \frac{1}{1 + |t|^{\alpha}}.$$

In the case $|t| < 1$ we consider the integral along the boundary of the half-disk $\left\{z : |z + k - 1| < \left(n + \frac{1}{2}\right)\alpha, \operatorname{Re} z < c\right\}$, apply to it the Cauchy residue theorem and let $n \to \infty$. We obtain

$$\varphi_{\alpha k}(t) = -\frac{i}{2\alpha |t|^{k-1}} \left\{ 2\pi i \sum_{q=1}^{\infty} \operatorname{Res}_{z = -q\alpha - k + 1} \left(\frac{|t|^{-z}}{\sin \frac{\pi(z + k - 1)}{\alpha}} \right) \right\} = \frac{1}{1 + |t|^{\alpha}}.$$

5. Expansion of $q_{\alpha k}(r)$ into Asymptotic Series at ∞. (Proof of Theorem 4). Denote by Π_{lR} the boundary of the rectangle $\left\{z : -k + 1 - \left(l + \frac{1}{2}\right)\alpha < \operatorname{Re} z < c, \ |\operatorname{Im} z| < R\right\}$ where c satisfies (2.7), $l = 1, 2, \ldots$. Noting that all poles of $f_{\alpha k}(z)$ situated in $\{z : \operatorname{Re} z < c\}$ are contained in $\{-q\alpha - k + 1\}_{q=1}^{\infty}$, we have, by the Cauchy residue theorem,

$$\oint_{\Pi_{lR}} f_{\alpha k} r^{z-1}\, dz = 2\pi i \sum_{q=1}^{l} \operatorname{Res}_{z = -q\alpha - k + 1} \left(f_{\alpha k}(z) r^{z-1} \right).$$

Let $R \to \infty$. The integrals along both of the horizontal sides tend to zero in virtue of bound (2.5). Therefore we obtain

$$\int_{c - i\infty}^{c + i\infty} f_{\alpha k}(z) r^{z-1}\, dz = 2\pi i \sum_{q=1}^{l} \operatorname{Res}_{z = -q\alpha - k + 1} \left(f_{\alpha k}(z) r^{z-1} \right) +$$

$$+ \int_{-k+1-\left(l+\frac{1}{2}\right)\alpha - i\infty}^{-k+1-\left(l+\frac{1}{2}\right)\alpha + i\infty} f_{\alpha k}(z) r^{z-1}\, dz.$$

The left hand side coincides with $q_{\alpha k}(r)$ by Theorem 3. Expression (2.4) yields that all poles of $f_{\alpha k}(z)$ lying in $\{z : \operatorname{Re} z < c\}$ are simple and

$$\operatorname{Res}_{-q\alpha - k + 1} \left(f_{\alpha k}(z) r^{z-1} \right) = \frac{(-1)^{q+1}\, i\, 2^{q\alpha}\, \Gamma\left(\frac{q\alpha + k}{2}\right)}{2\pi^{(k/2)+1}\, \Gamma\left(-\frac{q\alpha}{2}\right)} =$$

$$= \frac{(-1)^{q_i}}{2\pi^{(k/2)+2}} \left\{ 2^{q\alpha} \Gamma\left(\frac{q\alpha + k}{2}\right) \Gamma\left(1 + \frac{q\alpha}{2}\right) \sin\frac{\pi\alpha q}{2} \right\} r^{-q\alpha - k}.$$

Using bound (2.5), we obtain, for any fixed l,

$$\left| \int_{-k+1-\left(l+\frac{1}{2}\right)\alpha - i\infty}^{-k+1-\left(l+\frac{1}{2}\right)\alpha + i\infty} f_{\alpha k}(z) r^{z-1}\, dz \right| \le C r^{-k-\left(l+\frac{1}{2}\right)\alpha} = o(r^{-k-l\alpha}), \quad r \to \infty.$$

Thus, we have, for any $l = 1, 2, \ldots,\ r \to \infty$,

$$q_{\alpha k}(r) =$$

$$= \frac{1}{\pi^{(k/2)+1}} \sum_{q=1}^{l} \left\{ (-1)^{q-1} 2^{q\alpha} \Gamma\left(\frac{q\alpha + k}{2}\right) \Gamma\left(1 + \frac{q\alpha}{2}\right) \sin\frac{\pi\alpha q}{2} \right\} r^{-q\alpha - k} + o(r^{-l\alpha - k}),$$

which is equivalent to (2.8).

6. Expansion of $q_{\alpha k}(r)$ into Conditionally Convergent Series and its Asymptotic Behaviour at Zero. (Proofs of Theorem 5 and of Theorem 9). It is easy to check that, if α satisfies either condition (i) or condition (ii) of Theorem 5, then the sets

$$A_1 = \{-q\alpha - k + 1\}_{q=1}^{\infty}, \quad A_2 = \{2q + 1\}_{q=0}^{\infty}$$

are disjoint, since the Diophantine equation

$$p\alpha - k + 1 = 2q + 1$$

does not have any solution (p, q). Formula (2.3) yields that the set of all poles of $f_{\alpha k}(z)$ lying in $\{z : \operatorname{Re} z > c\}$ coincides with the union $A_1 \cup A_2$ and all poles are simple.

Let Π_{lR} be the boundary of the rectangle $\{z : c < \operatorname{Re} z < -k + 1 + Q_l,\ |\operatorname{Im} z| < R\}$ where the number Q_l belonging to the interval $(l\alpha, (l+1)\alpha)$, $l = 1, 2, \ldots$, is chosen in the following way. Each of the intervals $(l\alpha - k + 1, (l+1)\alpha - k + 1)$ between two neighbouring points of A_2 contains at most one point of A_1. If it contains none, we set $Q_l = \left(l + \frac{1}{2}\alpha\right)$. If it contains one, $2q_l + 1$ say, we choose Q_l such that the distance from $Q_l - k + 1$ to the nearest of three points $l\alpha - k + 1$, $2q_l + 1$, $(l+1)\alpha - k + 1$ is not less than $\alpha/4$.

Applying to the integral

$$\oint_{\Pi_{lR}} f_{\alpha k}(z)\, r^{z-1}\, dz$$

the Cauchy residue theorem and then letting $R \to \infty$, we obtain

$$q_{\alpha k}(r) = -2\pi i \left\{ \sum_{q=1}^{l} \operatorname{Res}_{z = q\alpha - k + 1} (f_{\alpha k}(z)\, r^{z-1}) + \right.$$

$$+\sum_{1\le 2q+1\le (l+\frac{1}{2})\alpha-k+1} \operatorname{Res}_{z=2q+1}(f_{\alpha k}(z)\,r^{z-1})\} + \int_{Q_l-k+1-i\infty}^{Q_l-k+1+i\infty} f_{\alpha k}(z)\,r^{z-1}\,dz. \tag{6.1}$$

Using formula (2.3), we obtain

$$\operatorname{Res}_{z=q\alpha-k+1}(f_{\alpha k}(z)\,r^{z-1}) =$$

$$=\frac{(-1)^{q+1}i}{2\pi^{k/2}}\,\frac{2^{-q\alpha}r^{q\alpha-k}}{\cos\frac{\pi(q\alpha-k+1)}{2}\,\Gamma\left(\frac{q\alpha-k+2}{2}\right)\Gamma\left(\frac{q\alpha}{2}\right)}, \tag{6.2}$$

$$\operatorname{Res}_{z=2q+1}(f_{\alpha k}(z)\,r^{z-1}) = \frac{(-1)^{q}i}{\alpha\,2^{k}\pi^{k/2}}\,\frac{2^{-2q}r^{2q}}{\sin\frac{\pi(2q+k)}{\alpha}\,\Gamma(q+1)\,\Gamma\left(q+\frac{k}{2}\right)}. \tag{6.3}$$

To estimate the integral entering into the right hand side of (6.1) we note that, for

$$z\notin\left[\bigcup_{q=-\infty}^{\infty}\{z:|z-(q\alpha-k+1)|<\tfrac{\alpha}{4}\}\right]\cup\left[\bigcup_{q=-\infty}^{\infty}\{z:|z-(2q+1)|<\tfrac{\alpha}{4}\}\right],$$

the following bounds are valid

$$\left|\sin\frac{\pi(z+k-1)}{\alpha}\right|\ge C\exp\{\tfrac{\pi}{2}|\operatorname{Im} z|\},\qquad \left|\cos\frac{\pi z}{2}\right|\ge C\exp\{\tfrac{\pi}{2}|\operatorname{Im} z|\},$$

where C is a positive constant independent of z. Moreover, for any $M>0$ and $H>0$, the Stirling formula yields the bound

$$\left|\frac{1}{\Gamma(z)}\right|\le C_{MH}\exp\{\tfrac{\pi}{2}|\operatorname{Im} z|-M\operatorname{Re} z\},\qquad z\in\{z:\operatorname{Re} z\ge -H\},$$

where C_{MH} does not depend on z. Hence, on any vertical line $\{z:\operatorname{Re} z=Q_l-k+1\}$ lying inside $\{z:\operatorname{Re} z>0\}$ the following improvement of (2.5) is valid

$$|f_{\alpha k}(z)|\le A\exp\{-\tfrac{\pi}{\alpha}|\operatorname{Im} z|-M\operatorname{Re} z\},$$

where A depends neither on z nor on l. Using this bound, we obtain

$$\left|r^{k}\int_{Q_l-k+1-i\infty}^{Q_l-k+1+i\infty} f_{\alpha k}(z)\,r^{z-1}\,dz\right|\le B\,(e^{-M}r)^{Q_l} \tag{6.4}$$

for l being large enough, where B depends neither on l nor on r. If D is a given positive number, we choose M such that $e^{-M}D<1$. Then, as $l\to\infty$, the right hand side of (6.4) tends to zero uniformly with respect to $r\in[0, D]$. This proves Theorem 5.

To prove Theorem 9, we choose l being the smallest number such that $Q_l-k+1>2N-1$, where N is a given integer. The desired result will follow from (6.1)-(6.3) if we show that the integral in the right hand side of (6.1) is $o(r^{2N-2})$ as

$r \to 0$. But bound (6.4) shows that this integral is $O\left(r^{Q_l - k}\right)$ as $r \to 0$. Since $Q_l - k > 2N - 2$, we have $O\left(r^{Q_l - k}\right) = o(r^{2N-2})$, $r \to 0$.

7. Expansion of $q_{\alpha k}(r)$ into Absolutely Convergent Series for almost all Values of α. (Proofs of Theorem 6 and of Theorem 7). In virtue of Theorem 5, Theorem 6 will be proved if we show that the both series in the right hand side of (2.10) converge absolutely for any $r > 0$.

If α satisfies condition (ii) of Theorem 5, then for any integer q we have

$$\left| \cos \frac{\pi (q\alpha - k + 1)}{2} \right| \geq \sin \frac{\pi}{n}, \quad \left| \sin \frac{\pi (2q + k)}{\alpha} \right| \geq \sin \frac{\pi}{m}.$$

This yields the absolute convergence of the both above mentioned series.

We shall consider the case when α satisfies condition (i) of Theorem 5 (i.e. is irrational) but it is not a Liuoville number.

Evidently, for any integer $q \geq 2$, there is an integer l_q such that

$$\left| \alpha - \frac{2l_q + k}{q} \right| < \frac{1}{q}.$$

Since α is not a Liuoville number, there exists $r \geq 2$ such that

$$\left| \alpha - \frac{2l_q + k}{q} \right| \geq \frac{1}{q^r}.$$

Therefore we have, for any $q \geq 2$,

$$q^{1-r} \leq | q\alpha - (2l_q + k) | < 1.$$

Hence,

$$\left| \cos \frac{\pi (q\alpha - k + 1)}{2} \right| = \left| \sin \frac{\pi (q\alpha - (2l_q + k))}{2} \right| \geq q^{1-r}, \quad q \geq 2. \tag{7.1}$$

Therefore the first of the series in (2.10) converges absolutely for any $r \geq 0$.

Evidently, for any $q \geq 2$, there is an integer m_q such that

$$\left| \frac{1}{\alpha} - \frac{m_q}{2q + k} \right| < \frac{1}{2(2q + k)}. \tag{7.2}$$

Hence,

$$\frac{m_q}{2q + k} \leq \frac{1}{\alpha} + \left| \frac{1}{\alpha} - \frac{m_q}{2q + k} \right| < \frac{1}{\alpha} + \frac{1}{2(2\alpha + k)} < \frac{5}{4\alpha}, \quad m_q \leq \frac{5}{4\alpha}(2q + k).$$

Note that inequality (7.2) yields $m_q \geq 2$ for q being large enough. Since α is not a Liuoville number, we have, for an integer $r \geq 2$,

$$\left| \alpha - \frac{2q + k}{m_q} \right| \geq \frac{1}{m_q^r}.$$

Multiplying this inequality by $\frac{m_q}{\alpha(2q+k)}$, we obtain

$$\left| \frac{1}{\alpha} - \frac{m_q}{2q+k} \right| \geq \frac{1}{m_q^r} \frac{m_q}{\alpha(2q+k)} \geq \frac{1}{\alpha} \left(\frac{5}{4\alpha}\right)^{1-r} \frac{1}{(2q+k)^r}. \tag{7.3}$$

Inequalities (7.2) and (7.3) yield

$$\frac{1}{\alpha} \left(\frac{5}{4\alpha}\right)^{1-r} \frac{1}{(2q+k)^{r-1}} \leq \left| \frac{2q+k}{\alpha} - m_q \right| < \frac{1}{2}$$

for q being large enough. Therefore for such q we have

$$\left| \sin \frac{\pi(2q+k)}{\alpha} \right| = \left| \sin \left(\pi\left(\frac{2q+k}{\alpha} - m_q\right)\right) \right| \geq \frac{2}{\alpha} \left(\frac{5}{4\alpha}\right)^{1-r} \frac{1}{(2q+k)^{r-1}} \tag{7.4}$$

and the absolute convergence of the second of series in (2.10) for any $r > 0$ follows.

To deduce Corollary to Theorem 6, we set

$$\Lambda_{\alpha k}(z) = \frac{1}{\pi^{(k/2)-1}} \sum_{q=1}^{\infty} \frac{(-1)^{q+1} z^q}{2^{q\alpha} \cos \frac{\pi(q\alpha - k + 1)}{2} \Gamma\left(\frac{q\alpha - k + 1}{2}\right) \Gamma\left(\frac{q\alpha}{2}\right)},$$

$$N_{\alpha k}(z) = \frac{2^{1-k}}{\alpha\pi^{(k/2)-1}} \sum_{q=0}^{\infty} \frac{(-1)^q z^q}{2^{2q} \sin \frac{\pi(2q+k)}{\alpha} \Gamma(q+1) \Gamma\left(q + \frac{k}{2}\right)}.$$

It is known (see e.g. [14], p. 4, Eq. (1.05)) that the power series

$$f(z) = \sum_{q=0}^{\infty} a_q z^q$$

represents an entire function of the order ρ iff

$$\rho = \limsup_{q \to \infty} \frac{q \log q}{\log(1/|a_q|)}.$$

Taking into account (7.1), (7.4) and the Stirling formula, we obtain the desired assertion.

Now we shall prove Theorem 7.

Let $\{\sigma_n\}_{n=1}^{\infty}$ be a very fast increasing sequence of integers defined by the equations

$$\sigma_1 = 2, \quad \sigma_{n+1} = 2^{3\sigma_n}, \quad n = 1, 2, \ldots. \tag{7.5}$$

Denote by Δ the set of all sequences $\{\delta_j\}_{j=1}^{\infty}$ with the terms δ_j taking values 0 or 1 only and satisfying the conditions

(i) δ_j is allowed to be equal to 1 if $j \in \{\sigma_n\}_{n=1}^{\infty}$ only;

(ii) infinitely many of δ_j's are equal to 1.

It is evident that the set

$$\Omega = \left\{ y : y = \sum_{j=1}^{\infty} \delta_j 2^{-j}, \ \{\delta_j\}_{j=1}^{\infty} \in \Delta \right\}$$

has cardinality continuum.

Denote by Λ the set of all numbers $x \in (0, 2)$ representable by *finite* binary fractions. Set

$$E = \{ \alpha \in (0, 2) : \alpha = x + y, x \in \Lambda, y \in \Omega \}.$$

It is easy to see that the set E is dense in (0,2) and has cardinality continuum.

We shall prove that, for any $\alpha \in E$, the both series in the right hand side of (2.10) diverge.

If $\alpha \in E$, then there is an integer m such that

$$\alpha = b + \sum_{j=1}^{m} a_j 2^{-j} + \sum_{j=m+1}^{\infty} \delta_j 2^{-j},$$

where b is either 0 or 1, a_j's take the values either 0 or 1, $\{\delta_j\}_{j=1}^{\infty} \in \Delta$. Denote by $\{\eta_n\}_{n=1}^{\infty}$ the subsequence of $\{\sigma_n\}_{n=1}^{\infty}$ such that $\delta_j = 1$ for $j \in \{\eta_n\}_{n=1}^{\infty}$ and $\delta_j = 0$ for $j \notin \{\eta_n\}_{n=1}^{\infty}$. Then, for any $\eta_n > m$, we have

$$0 < \alpha - \left(b + \sum_{j=1}^{m} a_j 2^{-j} + \sum_{j=m+1}^{\eta_n} \delta_j 2^{-j} \right) = \sum_{j=\eta_n+1}^{\infty} \delta_j 2^{-j} < 2^{-\eta_{n+1}+1}.$$

Multiplying this inequality by 2^{η_n}, we see that there is an odd integer p_n such that

$$0 < \alpha 2^{\eta_n} - p_n < 2^{\eta_n - \eta_{n+1} + 1} < 2^{-\frac{1}{2}\eta_{n+1}} \tag{7.6}$$

for sufficiently large n.

Suppose, the dimension k is odd and consider the terms of the first of the series in (2.10) possessing the numbers $q = q_n = 2^{\eta_n}$. From (7.6) we obtain

$$\left| \cos \frac{\pi (q_n \alpha - k + 1)}{2} \right| = \left| \cos \frac{\pi q_n \alpha}{2} \right| = \left| \sin \frac{\pi (q_n \alpha - p_n)}{2} \right| \leq \frac{\pi}{2} 2^{-\frac{1}{2}\eta_{n+1}}$$

Hence, for sufficiently large n, we have

$$\left| \frac{(-1)^{q_n+1} (r/2)^{q_n \alpha}}{\cos \frac{\pi (q_n \alpha - k + 1)}{2} \Gamma \left(\frac{q_n \alpha - k + 2}{2} \right) \Gamma \left(\frac{q_n \alpha}{2} \right)} \right| \geq$$

$$\geq \frac{2}{\pi} 2^{-\frac{1}{2}\eta_{n+1}} (r/2)^{q_n \alpha} 2^{-(q_n \alpha + 1)^2}. \tag{7.7}$$

Since $\{\eta_n\}_{n=1}^{\infty}$ is a subsequence of $\{\sigma_n\}_{n=1}^{\infty}$, the following inequality is valid

$$\eta_{n+1} \geq 2^{3\eta_n} = q_n^3, \tag{7.8}$$

and therefore the left hand side of (7.7) tends to infinity as $n \to \infty$.

If the dimension k is even, we consider the terms possessing the numbers $q = q_n = 2^{\eta_n + 1}$. Using (7.6), we obtain

$$\left| \cos \frac{\pi (q_n \alpha - k + 1)}{2} \right| = \left| \sin \frac{\pi q_n \alpha}{2} \right| = \left| \sin \pi (2^{\eta_n} - p_n) \right| < \pi 2^{-\frac{1}{2}\eta_n + 1}$$

and an estimation similar to (7.7) shows that the left hand side of (7.7) tends to infinity as $n \to \infty$.

Thus, the first of the series in the right hand side of (2.10) diverges.

Consider the second series in the right hand side of (2.10). If the dimension k is odd, we consider the terms possessing the numbers $q = q_n = (p_n - k)/2$, where p_n is the odd number defined by (7.6). From (7.6) we obtain

$$\left| \sin \frac{\pi (2q_n + k)}{\alpha} \right| = \left| \sin \frac{\pi p_n}{\alpha} \right| = \left| \sin \frac{\pi (\alpha 2^{\eta_n} - p_n)}{\alpha} \right| < \frac{\pi}{\alpha} 2^{-\frac{1}{2}\eta_n + 1}.$$

Hence, for sufficiently large n,

$$\left| \frac{(-1)^{q_n} (r/2)^{q_n + k}}{\sin \frac{\pi (2q_n + k)}{\alpha} \Gamma(q_n + 1) \Gamma\left(q_n + \frac{k}{2}\right)} \right| \geq \frac{\alpha}{\pi} 2^{\frac{1}{2}\eta_n + 1} (r/2)^{q_n + k} 2^{-(q_n + k)^2}. \quad (7.9)$$

Noting that (7.6) yields

$$q_n = \frac{p_n - k}{2} < \frac{p_n}{2} < \frac{1}{2} \alpha 2^{\eta_n} < 2^{\eta_n} \leq (\eta_{n+1})^{1/3},$$

we conclude that the left hand side of (7.9) tends to ∞.

If the dimension k is even, we consider the terms possessing the numbers $q = q_n = (2p_n - k)/2$, where p_n is defined by (7.6). Then (7.6) yields

$$\left| \sin \frac{\pi (2q_n + k)}{\alpha} \right| = \left| \sin \frac{2\pi p_n}{\alpha} \right| = \left| \sin \frac{2\pi (\alpha 2^{\eta_n} - p_n)}{\alpha} \right| < \frac{2\pi}{\alpha} 2^{-\frac{1}{2}\eta_n + 1}$$

and the estimation similar to (7.9) shows that the second series in the right hand side of (2.10) diverges.

8. Expansion of $q_{\alpha k}(r)$ into Absolutely Convergent Series for the Most of Rational α. (Proof of Theorem 8). Recall that the set of all poles of the function $f_{\alpha k}(z)$ lying in the half-plane $\{ z : \operatorname{Re} z \geq -k + 1 \}$ coincides with the union of the sets

$$A_1 = \{ q\alpha - k + 1 \}_{q=1}^{\infty}, \quad A_2 = \{ 2q + 1 \}_{q=0}^{\infty}.$$

Under the hypotheses of Theorem 5, these sets are disjoint since the Diophantine equation

$$x\alpha - k + 1 = 2y + 1 \quad (8.1)$$

does not have any solution (x, y). Under the hypotheses of Theorem 8, that is if $\alpha = m/n$ (m, n being relatively prime integers) and if one of three conditions (i)-(iii) mentioned before the statement of Theorem 8 in Section 2 are satisfied, the equation (8.1) possesses infinitely many of solutions. Rewriting equation (8.1) in the form

$$xm = (2y + k)n$$

and using the standard tools of Diophantine Analysis, we can describe the set $D_{\alpha k}$ of all solutions in the following way:

$D_{\alpha k} = \left\{x = (2t+1)n,\ y = \dfrac{(2t+1)m-k}{2} : t \in \mathbf{Z}\right\}$ in case (i);

$D_{\alpha k} = \left\{x = 2tn,\ y = \dfrac{2tm-k}{2} : t \in \mathbf{Z}\right\}$ in case (ii);

$D_{\alpha k} = \left\{x = tn,\ y = \dfrac{tm-k}{2} : t \in \mathbf{Z}\right\}$ in case (iii).

As an immediate consequence we obtain the following descriptions of the sets $P_{\alpha k}$ and $Q_{\alpha k}$ defined by (2.12), (2.13):

$$P_{\alpha k} = \{(2t+1)n\}_{t=t_{km}}^{\infty}, \quad Q_{\alpha k} = \left\{\frac{(2t+1)m-k}{2}\right\}_{t=t_{km}}^{\infty},$$

$$t_{km} = \max\left\{0, \left[\frac{1}{2}\left(\frac{k}{m}-1\right)\right]\right\} \text{ in case (i);}$$

$$P_{\alpha k} = \{2tn\}_{t=t_{km}}^{\infty}, \quad Q_{\alpha k} = \left\{\frac{2tm-k}{2}\right\}_{t=t_{km}}^{\infty}, \quad t_{km} = \max\left\{1, \left[\frac{k}{2m}\right]\right\} \text{ in case (ii);}$$

$$P_{\alpha k} = \{tn\}_{t=t_{km}}^{\infty}, Q_{\alpha k} = \left\{\frac{tm-k}{2}\right\}_{t=t_{km}}^{\infty}, \quad t_{km} = \max\left\{1, \left[\frac{k}{m}\right]\right\} \text{ in case (iii).}$$

Evidently, the following equations are true

$$A_1 \setminus A_2 = \{p\alpha - k + 1 : p \geq 1, p \notin P_{\alpha k}\},$$

$$A_2 \setminus A_1 = \{2q + 1 : q \geq 0, q \notin Q_{\alpha k}\},$$

$$A_1 \cap A_2 = \{2q + 1 :\ q \in Q_{\alpha k}\} = \{p\alpha - k + 1 :\ p \in P_{\alpha k}\}.$$

The poles of $f_{\alpha k}(z)$ situated in $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are simple, the poles of $f_{\alpha k}(z)$ situated in $A_1 \cap A_2$ are double. Note that the set $A_1 \cup A_2$ is a subset of the set $\{l/n\}_{l=-\infty}^{\infty}$.

Applying the Cauchy residue theorem to the rectangle $\left\{z : c < \operatorname{Re} z < \frac{1}{n}\left(l + \frac{1}{2}\right) - k + 1,\ |\operatorname{Im} z| < R\right\}$ and letting $R \to \infty$, we obtain

$$q_{\alpha k}(r) = -2\pi i \left\{ \sum_{\substack{-k+1 < p\alpha - k + 1 < \frac{1}{n}\left(l+\frac{1}{2}\right) - k + 1 \\ p \notin P_{\alpha k}}} \operatorname{Res}_{z = p\alpha - k + 1}\left(f_{\alpha k}(z) r^{z-1}\right) + \right.$$

$$+ \sum_{\substack{1 \leq 2q + 1 < \frac{1}{n}\left(l+\frac{1}{2}\right) - k + 1 \\ q \notin Q_{\alpha k}}} \operatorname{Res}_{z = 2q+1}\left(f_{\alpha k}(z) r^{z-1}\right) +$$

$$\left. + \sum_{\substack{2q + 1 < \frac{1}{n}\left(l+\frac{1}{2}\right) - k + 1 \\ q \in Q_{\alpha k}}} \operatorname{Res}_{z = 2q+1}\left(f_{\alpha k}(z) r^{z-1}\right) \right\} +$$

$$+\int_{\frac{1}{n}\left(l+\frac{1}{2}\right)-k+1-i\infty}^{\frac{1}{n}\left(l+\frac{1}{2}\right)-k+1+i\infty} f_{\alpha k}(z)r^{z-1}\,dz. \qquad (8.2)$$

The estimations similar to (6.4) show that the integral in the right hand side of (8.2) tends to 0 as $l \to \infty$. Using formulas (6.2), (6.3) for evaluation of the residues at simple poles and the formula

$$\operatorname{Res}_{z=2q+1}(f_{\alpha k}(z)r^{z-1}) =$$

$$=\frac{(-1)^{q+p+1}i}{2^k\pi^{(k/2)+1}}\frac{(r/2)^{2q}}{\Gamma(q+1)\,\Gamma\left(q+\frac{k}{2}\right)}\left\{\log\frac{r}{2}-\frac{\Gamma'(q+1)}{2\Gamma(q+1)}-\frac{\Gamma'\left(q+\frac{k}{2}\right)}{2\Gamma\left(q+\frac{k}{2}\right)}\right\}$$

for evaluation of the residues at the double poles, we obtain (2.14).

The absolute and uniform convergence of each of three series in the right hand side of (2.14) is evident since the moduli of the terms $\cos\frac{\pi(p\alpha-k+1)}{2}$ and $\sin\frac{\pi(2q+k)}{\alpha}$ are bounded from below by a positive constant not depending on p and q.

The proof of Corollary to Theorem 8 is similar to that of Corollary to Theorem 6 and therefore can be omitted.

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Аналитические и асимптотические свойства многомерного распределения Линника

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В статье изучаются свойства k-мерного ($k \geq 2$) распределения Линника, определяемого характеристической функцией

$$\varphi_{\alpha k}(t) = 1/(1 + |t|^{\alpha}), \quad 0 < \alpha < 2, \quad t \in \mathbf{R}^k,$$

где $|t|$ обозначает евклидову норму вектора $t \in \mathbf{R}^k$. Это распределение абсолютно непрерывно относительно меры Лебега в $\mathbf{R}^k$. Получены разложения его плотности в асимптотические и сходящиеся ряды по степеням $|t|$ и $|t|^{\alpha}$. Форма этих разложений существенно зависит от арифметической природы параметра α.

Аналітичні та асимптотичні властивості багатовимірного розподілу Лінника

Й.В. Островський

В статті вивчаються властивості k-вимірного ($k \geq 2$) розподілу Лінника, що визначається характеристичною функцією

$$\varphi_{\alpha k}(t) = 1/(1 + |t|^{\alpha}), \quad 0 < \alpha < 2, \quad t \in \mathbf{R}^k,$$

де через $|t|$ позначено евклідову норму вектора $t \in \mathbf{R}^k$. Цей розподіл є абсолютно неперервним відносно міри Лебега в $\mathbf{R}^k$. Здобуто розклади його щільності в асимптотичні і збіжні ряди за степенями $|t|$ і $|t|^{\alpha}$. Форма цих розкладів істотно залежить від арифметичної природи параметру α.