Matematicheskaya fizika, analiz, geometriya
1996, v. 3, No 1/2, pp. 3–17

On recurrence and superrecurrence of H-cocycles

Sergey I. Bezuglyi

B. Verkin Institute for Low Temperature Physics & Engineering, National Academy of Sciences of Ukraine, 47, Lenin Ave., 310164, Kharkov, Ukraine

Received November 5, 1994

A method of construction of H-cocycles taking values in an abelian l.c.s. group is studied. The necessary and sufficient conditions of superrecurrence of H-cocycles are found generalizing K. Schmidt's result on superrecurrence of cocycles.

1. Introduction

The present article is devoted to study of a kind of weighted cocycles called H-cocycles the interest in which became apparent recently in papers [1-3, 7]. The expression $\alpha(x, T^n) = \sum_{i=0}^{n-1} \rho(x, T^i) f(T^i x)$ can be considered as an example of a H-cocycle, where T is a nonsingular automorphism of a measure space (X, μ), and $\rho(x, T^i) = \frac{d\mu \circ T^i}{d\mu}(x)$ is the Radon-Nikodym cocycle, and $f: X \to \mathbf{R}$ is a measurable function. It is easily verified that $\alpha(x, T^n)$ satisfies the following relation: $\alpha(x, T^{n+m}) = \alpha(x, T^n) + \rho(x, T^n)\,\alpha(T^n x, T^m)$. In this relation ρ can be considered as an element of the group $\mathbf{R}_+^*$ acting on $\mathbf{R}$ by group automorphisms. It means that the pair (ρ, α) belongs to the semidirect product $\mathbf{R}_+^* \rtimes \mathbf{R}$. It is clear that this construction can be generalized to every group extension $E(G, A)$ of an amenable group G (instead of $\mathbf{R}_+^*$) by an abelian group A (instead of $\mathbf{R}$); this is a point of the present paper (see precise definitions in Section 1). Conversely, if we have a cocycle a with values in $E(G, A)$, then its component α whose values are in the subgroup A is a H-cocycle. Such a point of view at the structure of H-cocycles allows one to prove easily most of the results from [1, 2, 8].

The main result of this article is the affirmative solution of a problem formulated in [2]. Let $a = (c, \alpha)$ be a cocycle with values in the group $E(G, A)$. Then a is a recurrent cocycle if and only if c is a recurrent cocycle and α is a recurrent H-cocycle. This result is a generalization of the theorem of K. Schmidt [6] on the superrecurrence of cocycles to the case of H-cocycles. Roughly speaking, an H-cocycle is recurrent if and only if it is superrecurrent.

The author would like to thank Department of Mathematics of Orleans University for their hospitality while this paper was being done. Especially the author is grateful to Professors V.Ya. Golodets and J. Renault for the stimulating discussions.

2. Definitions and examples

In this paper we will use the following notations: (X, μ) is a standard measure space, A is an abelian locally compact separable (l.c.s.) group, G is an amenable l.c.s. group, Γ is a countable group of nonsingular automorphisms of (X, μ) acting freely and conservatively. A measurable map $c : X \times \Gamma \to G$ is called a cocycle if $c(x, 1) = e$ (1 is the identical map, e is the identity of G) and $c(x, \gamma_2 \gamma_1) = c(\gamma_1 x, \gamma_2)\, c(x, \gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$. We denote the set of all cocycles $c : X \times \Gamma \to G$ by $Z^1(X \times \Gamma, G)$.

Let Aut(A) be the group of all automorphisms of A being considered as an abstract group (algebraic automorphisms). Assume that there is a Borel action of G on A, that is a Borel map $(g, a) \to g(a) :\ G \times A \to A$ such that $a \to g(a)$ is an algebraic automorphism of A for every $g \in G$. By [5], the action map $(g, a) \to g(a)$ is in fact continuous. In the sequel we will also fix a Borel action of G on A.

Let f be a continuous map from $G \times G$ into A satisfying the conditions:

$$f(e, g) = f(g, e) = 0, \quad g \in G,$$

$$g_4^{-1}(f(g_1, g_2)) + f(g_1 g_2, g_4) = f(g_2, g_4) + f(g_1, g_2 g_4), \quad g_1,\ g_2,\ g_4 \in G. \quad (1)$$

We call such a map f an algebraic 2-cocycle (or simply 2-cocycle). The notation $Z^2(G \times G, A)$ will stand for the set of all algebraic 2-cocycles. Let $p : G \to A$ be a continuous map and $p(e) = 0$ (such a map we call normalized). Define $f_p \in Z^2(G \times G, A)$ putting

$$f_p(g_1, g_2) = -g_2^{-1}(p(g_1)) + p(g_1 g_2) - p(g_2), \quad g_1, g_2 \in G. \quad (2)$$

The map f_p is called a 2-coboundary, let $B^2(G \times G, A)$ denote the set of all 2-coboundaries.

Let $E = G \times A$, and the topology on E is defined as the topology of direct product. For every $f \in Z^2(G \times G, A)$ one can define a group structure on E by setting

$$(g, a)^{-1} = (g^{-1}, -f(g, g^{-1}) - g(a)), \quad (3)$$

$$(g_1, a_1)(g_2, a_2) = (g_1 g_2, f(g_1, g_2) + g_2^{-1}(a_1) + a_2). \tag{4}$$

The set E with the group structure defined according to (3) and (4) is called the group extension of G by means of A and the 2-cocycle f. We denote it by $E_f(G, A) = E_f$. It is known that in such a way one can describe a collection of so-called topologically trivial extensions [4]. The two groups E_{f_1} and E_{f_2} are isomorphic if and only if $f_2 - f_1$ is a 2-coboundary. With $f = 0$ we have the semidirect product $E_0 = G \rtimes A$ of groups G and A.

Now we formulate the definition of H-cocycles for the group Γ of automorphisms of (X, μ).

Definition 2. 1. *Let $f \in Z^2(G \times G, A)$ and $c \in Z^1(X \times \Gamma, G)$. Let also $\alpha : X \times \Gamma \to A$ be a measurable map satisfying the following conditions*:

$$\alpha(x, 1) = 0,$$
$$\alpha(x, \gamma_2 \gamma_1) = f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) + c(x, \gamma_1)^{-1}(\alpha(\gamma_1 x, \gamma_2)) + \alpha(x, \gamma_1) \tag{5}$$

for every $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$. The set of all such maps α will be denoted by $Z^1_{f,c}(X \times \Gamma, A)$ (or $Z^1_{f,c}(A)$). The elements $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$ are called H-cocycles.

Example 2. 2. (a) Let $c \in Z^1(X \times \Gamma, G)$ and $p : G \to A$ be a normalized continuous map. Then $\alpha(x, \gamma) = p(c(x, \gamma))$ is the H-cocycle from $Z^1_{f_p, c}(A)$, where f_p is defined by (2). If $a : X \to A$ is a measurable map, then $\alpha(x, \gamma) = c(x, \gamma)^{-1} \times (a(\gamma x)) - a(x)$ is a H-cocycle from $Z^1_{0, c}(A)$.

The H-cocycles of the form $p(c(x, \gamma)) + c(x, \gamma)^{-1}(a(\gamma x)) - a(x)$ are called H-coboundaries.

(b) Let $f \in Z^2(G \times G, A)$ and $c(x, \gamma) = g(\gamma x) g(x)^{-1}$, where $g : X \to G$ is a measurable function. Then

$$\alpha(x, \gamma) = f(g(\gamma x), g(x)^{-1}) - f(g(x), g(x)^{-1})$$

is the H-cocycle from $Z^1_{f,c}(X \times \Gamma, A)$.

This statement is proved by the routine calculations.

(c) Let T be a free nonsingular automorphism of (X, μ), $c \in Z^1(X \times \{T^n\}, G)$, and ψ be a measurable map from X into A. Put

$$\alpha(x, T) = \psi(x),$$

$$\alpha(x, T^2) = c(x, T)^{-1}(\psi(Tx)) + \psi(x),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\alpha(x, T^n) = c(x, T^{n-1})^{-1}(\psi(T^{n-1}x)) + \dots + c(x, T)^{-1}(\psi(Tx)) + \psi(x),$$

and

$$\alpha(x, T^{-n}) = -c(x, T^{-n})^{-1}(\alpha(T^{-n}x, T^m)), \quad n \geq 0.$$

Then $\alpha : X \times \{ T^n \} \to A$ is a H-cocycle from $Z^1_{0,c}(X \times \{ T^n \}, A)$. Similarly if $f \in Z^2(G \times G, A)$, then we can apply the above procedure to construct the H-cocycle $\beta \in Z^1_{f,c}(X \times \{ T^n \}, A)$:

$$\beta(x, T) = \psi(x),$$

$$\beta(x, T^2) = f(c(Tx, T), c(x, T)) + c(x, T)^{-1}(\psi(Tx)) + \psi(x)$$

etc.

R e m a r k 2.3. The H-cocycles belonging (in our classification) to the set $Z^1_{0,\rho}(X \times \{ T^n \}, \mathbf{R})$ were studied in [1, 2, 8], where $\rho(x, T) = \frac{d\mu cT}{d\mu}(x)$ is the Radon-Nikodym cocycle. In [3] the H-cocycles from $Z^1_{0,c}(X \times \Gamma, A)$ were considered. The set of usual cocycles $Z^1(X \times \Gamma, A)$ coincides with $Z^1_{f,id}(X \times \Gamma, A)$ for every 2-cocycle f, where *id* is the identity cocycle (as a matter of fact $Z^1_{f,id}(A)$ does not depend on f).

3. The H-cohomology group

Let us fix a cocycle $c \in Z^1(X \times \Gamma, G)$. Put

$$B^1_{p,c}(X \times \Gamma, A) = \{ p(c(x, \gamma)) + c(x, \gamma)^{-1}(a(\gamma x)) - a(x) \mid a : X \to A \},$$

where $p : G \to A$ is a normalized continuos map. Define

$$\mathcal{B}^1_c(X \times \Gamma, A) = \bigcup_p B^1_{p,c}(X \times \Gamma, A),$$

$$\mathcal{Z}^1_c(X \times \Gamma, A) = \bigcup_f Z^1_{f,c}(X \times \Gamma, A), \quad f \in Z^2(G \times G, A).$$

Lemma 3.1. $Z_c^1(X \times \Gamma, A)$ *is an abelian group and* $\mathcal{B}_c^1(X \times \Gamma, A)$ *is its subgroup.*

P r o o f. It follows from the following relations:

$$Z^1_{f_1, c}(A) + Z^1_{f_2, c}(A) = Z^1_{f_1 + f_2, c}(A),$$

$$B^1_{p_1, c}(A) + B^1_{p_2, c}(A) = B^1_{p_1 + p_2, c}(A),$$

where $Z^1_{f_1, c}(A) + Z^1_{f_2, c}(A) = \{\alpha + \beta : \alpha \in Z^1_{f_1, c}(A), \beta \in Z^1_{f_2, c}(A)\}$. ◆

Let us define

$$\mathcal{H}^1_c(X \times \Gamma, A) = \mathcal{Z}^1_c(X \times \Gamma, A)/\mathcal{B}^1_c(X \times \Gamma, A).$$

$\mathcal{H}^1_c$ is called the first H-cohomology group.

Proposition 3.2. 1) *Let* f_1 *and* f_2 *be cohomologous 2-cocycles from* $Z^2(G \times G, A)$, *i.e.*

$$f_1(h_1, h_2) = f_2(h_1, h_2) + f_p(h_1, h_2), \quad h_1, h_2 \in G,$$

where f_p *satisfies* (2). *Then*

$$\alpha_1(x, \gamma) \to \alpha_2(x, \gamma) = -p(c(x, \gamma)) + \alpha_1(x, \gamma) \tag{6}$$

is the one-to-one map from $Z^1_{f_1, c}(A)$ *onto* $Z^1_{f_2, c}(A)$.

2) *Assume that there exist* $\bar{\alpha}_1 \in Z^1_{f_1, c}(A)$ *and* $\bar{\alpha}_2 \in Z^1_{f_2, c}(A)$ *such that* $\bar{\alpha}_2(x, \gamma) = p(c(x, \gamma)) + \bar{\alpha}_1(x, \gamma)$ *for a normalized map* $p : G \to A$. *Then* f_1 *and* f_2 *satisfy the relation*

$$f_1(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) = (f_2 - f_p)(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)). \tag{7}$$

If $c : X \times \Gamma \to G$ *is a cocycle with dense range in* G, *then* $f_1 = f_2 - f_p$.

3) $Z^1_{f_1, c}(A) \cap Z^1_{f_2, c}(A) = \emptyset$ *with* $f_1 \neq f_2$.

P r o o f. 1) Let us check that $\alpha_2 \in Z^1_{f_2, c}(A)$:

$$\alpha_2(x, \gamma_2\gamma_1) = \alpha_1(x, \gamma_2\gamma_1) - p(c(x, \gamma_2\gamma_1)) =$$

$$= f_1(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) + c(x, \gamma_1)^{-1}(\alpha_1(\gamma_1 x, \gamma_2)) + \alpha_1(x, \gamma_1) - p(c(x, \gamma_2\gamma_1)) =$$

$$=f_2(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) - c(x, \gamma_1)^{-1}(p(c(\gamma_1 x, \gamma_2))) + p(c(x, \gamma_2\gamma_1)) -$$

$$- p(c(x, \gamma_1)) + \alpha_1(x, \gamma_1) - p(c(x, \gamma_2\gamma_1)) + c(x, \gamma_1)^{-1}(\alpha_1(\gamma_1 x, \gamma_2)) =$$

$$= f_2(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) + c(x, \gamma_1)^{-1}(\alpha_2(\gamma_1 x, \gamma_2)) + \alpha_2(x, \gamma_1).$$

The remainder of statement 1) is evident.

2) Consider $\overline{\alpha}_2(x, \gamma_2\gamma_1)$. We have

$$\overline{\alpha}_2(x, \gamma_2\gamma_1) = p(c(x, \gamma_2\gamma_1) + \overline{\alpha}_1(x, \gamma_2\gamma_1)) =$$

$$= p(c(x, \gamma_2\gamma_1)) + f_1(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) +$$

$$+ c(x, \gamma_1)^{-1}(\overline{\alpha}_1(\gamma_1 x, \gamma_2)) + \overline{\alpha}_1(x, \gamma_1). \tag{8}$$

On the other hand,

$$\overline{\alpha}_2(x, \gamma_2\gamma_1) = f_2(c(\gamma_1 x, \gamma_2, c(x, \gamma_1)) + c(x, \gamma_1)^{-1}(p(c(\gamma_1 x, \gamma_1))) +$$

$$+ c(x, \gamma_1)^{-1}(\overline{\alpha}_1(\gamma_1 x, \gamma_2)) + p(c(x, \gamma_1)) + \overline{\alpha}_1(x, \gamma_1). \tag{9}$$

The equality (7) follows from (8) and (9).

Let now c be a cocycle with dense range in G (see the definition in [6] or below). We will prove that for arbitrary neighborhoods V_{h_1} and V_{h_2} of the elements h_1 and h_2 from G and for any set $B \subset X$ with $\mu(B) > 0$ there exist a set $D \subset B$, $\mu(D) > 0$ and the automorphisms $\gamma_1, \gamma_2 \in \Gamma$ such that

$$(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) \in V_{h_1} \times V_{h_2}$$

for a.e. $x \in D$. It follows from density of c in G that there exist a set $D' \in B$ of positive measure and an automorphism $\gamma_1 \in \Gamma$ such that $\gamma_1 D' \subset B$ and $c(x, \gamma_1) \in V_{h_2}$ for a.e. $x \in D'$. Choose a subset C, $\mu(C) > 0$ in $\gamma_1 D'$ and an automorphism $\gamma_2 \in \Gamma$ such that $\gamma_2 C \subset \gamma_1 D'$ and $c(y, \gamma_2) \in V_{h_1}$ for a.e. $y \in C$. Put $D = \gamma_1^{-1} C$. It is easy to verify that D is the desired subset. In view of arbitrariness of V_{h_1} and V_{h_2} it follows from (7) and continuity of f_1, f_2 and f_p that $f_1(h_1, h_2) = (f_2 - f_p)(h_1, h_2)$, i.e. f_1 and f_2 are cohomologous.

3) This is obvious due to (5). ◆

Corollary 3.3. *Let there exists a H-cocycle* $\overline{\alpha}_1 \in Z^1_{f_1, c}(A)$ *such that* $\overline{\alpha}_1(x, \gamma) + p(c(x, \gamma)) \in Z^1_{f_2, c}(A)$ *for a normalized map* $p : G \to A$. *Then, for every* $\alpha_1 \in Z^1_{f_1, c}(A)$, *the H-cocycle* $\alpha_1(x, \gamma) + p(c(x, \gamma))$ *belongs to* $Z^1_{f_2, c}(A)$ *and* $f_2 = f_1 + f_p$ *when c is a cocycle with dense range in G.*

Let $H^2(G \times G, A) = Z^2(G \times G, A)/B^2(G \times G, A)$ be the 2-nd cohomology group. We will denote by $[f]$ the elements of $H^2(G \times G, A)$ and by $[\alpha]$ the elements of $\mathcal{H}^1_c(X \times \Gamma, A)$.

Proposition 3.4. *Let* $\alpha \in Z^1_{f, c}(X \times \Gamma, A)$. *Then the map* $\Phi : [\alpha] \to [f]$ *is a surjective homomorphism from* $\mathcal{H}^1_c(X \times \Gamma, A)$ *onto* $H^2(G \times G, A)$.

P r o o f. For $\alpha \in Z^1_{f, c}(A)$ we define the map $\Phi_0 : \alpha \to f$ from $\mathcal{Z}^1_c(A)$ onto $\mathcal{Z}^2(G \times G, A)$. We have proved in Proposition 3.2 that $\Phi_0(\mathcal{B}^1_c(X \times \Gamma, A)) = B^2(G \times G, A)$. Hence Φ_0 defines the quotient map Φ correctly. It follows from Lemma 3.1 that Φ is a surjective map. ◆

Let c be a cocycle from $Z^1(X \times \Gamma, G)$ and E_f the group extension of G by A for a 2-cocycle f as above.

The next simple theorem will play the important role later on.

Theorem 3.5. *Let* $f \in Z^2(G \times G, G)$ *and* $c \in Z^1(X \times \Gamma, G)$. *For every* $\alpha \in Z^1_{f, c}(X \times \Gamma, A)$ *we set*

$$\alpha_c(x, \gamma) = (c(x, \gamma), \alpha(x, \gamma)) : X \times \Gamma \to E_f.$$

Then $\alpha_c \in Z^1(X \times \Gamma, E_f)$. *Conversely, any cocycle* $\alpha_0 \in Z^1(X \times \Gamma, E_f)$ *defines a cocycle* $c \in Z^1(X \times \Gamma, G)$ *and a H-cocycle* $\alpha \in Z^1_{f, c}(X \times \Gamma, A)$. *If c is fixed, then the map* $\Psi : \alpha \to \alpha_c$ *is one-to-one.*

P r o o f. Let us check that $\alpha_c(x, \gamma)$ is a cocycle taking values in the group E_f. In fact,

$$\alpha_c(\gamma_1 x, \gamma_2)\, \alpha_c(x, \gamma_1) = (c(\gamma_1 x, \gamma_2), \alpha(\gamma_1 x, \gamma_2))(c(x, \gamma_1), \alpha(x, \gamma_1)) =$$

$$= \Big(c(x, \gamma_2\gamma_1), f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) + (c(x, \gamma_1)^{-1}(\alpha(\gamma_1 x, \gamma_2)) + \alpha(x, \gamma_1) \Big) =$$
$$= \alpha_c(x, \gamma_2\gamma_1).$$

Conversely, if $\alpha_0 \in Z^1(X \times \Gamma, E_f)$, then there are two coordinates c and α of α_0 taking values in G and A respectively. It is obvious that $c \in Z^1(X \times \Gamma, G)$ and α is a H-cocycle from $Z^1_{f,c}(X \times \Gamma, A)$. ◆

Let $f_2 = f_1 + f_p$, where f_1, $f_2 \in Z^2(G \times G, A)$ and p is a normalized map. Then the map $i_p : (h, a) \to (h, a + p(h))$ defines the homeomorphism of groups E_{f_1} and E_{f_2}. Let i_p^* be the isomorphism of $Z^1(X \times \Gamma, E_{f_1})$ and $Z^1(X \times \Gamma, E_{f_2})$ induced by i_p. For every fixed $c \in Z^1(X \times \Gamma, G)$ there is one-to-one map $\Psi: \alpha \to \alpha_c$: $Z^1_{f,c}(X \times \Gamma, A) \to Z^1(X \times \Gamma, E_f)$. Equality (6) defines another one-to-one map j_p: $Z^1_{f_1,c}(A) \to Z^1_{f_2,c}(A)$. It follows from the above statements that the following diagram is commutative (here $\delta_p = p(c(x, \gamma))$):

$$\begin{array}{ccc} \alpha & \to & (c, \alpha) = \alpha_c \\ j_p \downarrow & & \downarrow i_p^* \\ \alpha + \delta_p & \to & (c, \alpha + \delta_p) = (\alpha + \delta_p)_c \end{array} \quad : \quad \begin{array}{ccc} Z^1_{f_1,c}(A) & \to & Z^1(X \times \Gamma, E_{f_1}) \\ j_p \downarrow & & \downarrow i_p^* \\ Z^1_{f_2,c}(A) & \to & Z^1(X \times \Gamma, E_{f_2}). \end{array}$$

Theorem 3.6. *If the cocycles c and c_1 from $Z^1(X \times \Gamma, G)$ are cohomologous, then the groups $\mathcal{H}^1_c(X \times \Gamma, A)$ and $\mathcal{H}^1_{c_1}(X \times \Gamma, A)$ are isomorphic.*

P r o o f. According to the hypothesis of the theorem there is a measurable function $h: X \to G$ such that $c_1(x, \gamma) = h(\gamma x)c(x, \gamma)\, h(x)^{-1}$, $\gamma \in \Gamma$. Put $Q(x) = (h(x), 0) : X \to E_f$, where f is an arbitrary 2-cocycle from $Z^2(G \times G, A)$. For $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$, we define $\alpha_c = (c, \alpha) \in Z^1(X \times \Gamma, E_f)$ as above and consider the cocycle $Q(\gamma x)\, \alpha_c(x, \gamma)\, Q(x)^{-1}$ from $Z^1(X \times \Gamma, E_f)$. It is easy to obtain that

$$Q(\gamma x)\, \alpha_c(x, \gamma)\, Q(x)^{-1} = [h(\gamma x)\, c(x, \gamma)\, h(x)^{-1}, f(h(\gamma x)\, c(x, \gamma)\, h(x)^{-1}) +$$

$$+ h(x)(f(h(\gamma x)\, c(x, \gamma))) + h(x)\,(\alpha(x, \gamma)) - f(h(x), h(x)^{-1})] =$$
$$= (c_1(x, \gamma), \alpha_1(x, \gamma)).$$

Taking into account formula (1) we obtain

$$\alpha_1(x, \gamma) = f(h(\gamma x), c(x, \gamma)\, h(x)^{-1}) + f(c(x, \gamma)\, h(x)^{-1}) -$$
$$- f(h(x), h(x)^{-1}) + h(x)(\alpha(x, \gamma)). \tag{10}$$

Therefore $\alpha_1 \in Z^1_{f, c_1}(X \times \Gamma, A)$ (Theorem 3.5) and formula (10) defines the map θ : $\alpha \to \alpha_1$ from $Z^1_{f, c}(A)$ into $Z^1_{f, c_1}(A)$ for any fixed $f \in Z^2(G \times G, A)$. It is evident that θ is injective. Moreover, for any $\alpha_1 \in Z^1_{f, c_1}(A)$, one can use (10) to find a H-cocycle $\alpha \in Z^1_{f, c}(A)$ such that $\theta(\alpha) = \alpha_1$, i.e. θ is the isomorphism. Let now $\alpha \in \mathcal{B}^1_c(X \times \Gamma, A)$. It means that there exist a normalized map $p : G \to A$ and a measurable function $a : X \to A$ such that

$$\alpha(x, \gamma) = p(c(x, \gamma)) + c(x, \gamma)^{-1}(a(\gamma x)) - a(x).$$

Prove that $\alpha_1(x, \gamma) = \theta\,(\alpha(x, \gamma)) \in \mathcal{B}^1_c(X \times \Gamma, A)$. Since $B^1_{p, c}(A) \subset Z^1_{f_p, c}(A)$ then one can rewrite (10) in the form

$$\alpha_1(x, \gamma) = f_p(h(\gamma x), c(x, \gamma)\, h(x)^{-1}) + f_p(c(x, \gamma), h(x)^{-1}) - f_p(h(x), h(x)^{-1}) +$$
$$+ h(x)(p(c(x, \gamma))) + h(x)c(x, \gamma)^{-1}(a(\gamma x)) - h(x)(a(x)) =$$
$$= - h(x)c(x, \gamma)^{-1}(p(h(\gamma x))) + p\,(h(\gamma x)\, c(x, \gamma)\, h(x)^{-1}) - p(c(x, \gamma)h(x)^{-1}) -$$
$$- h(x)(p(c(x, \gamma))) + p(c(x, \gamma)h(x)^{-1}) - p(h(x)^{-1}) + h(x)(p(h(x))) - p(e) +$$
$$+ p(h(x)^{-1}) + h(x)(p(c(x, \gamma))) + c_1(x, \gamma)^{-1}h(\gamma x)(a(\gamma x)) - h(x)(a(x)) =$$
$$= p(c_1(x, \gamma)) + c_1(x, \gamma)^{-1}(a_1(\gamma x)) - a_1(x),$$

where $a_1(x) = h(x)(a(x) - p(h(x)))$. Thus, it is proved that $\alpha_1 = \theta(\alpha) \in B^1_{p, c_1}(A)$ and θ is the one-to-one map from $B^1_{p, c}(A)$ onto $B^1_{p, c_1}(A)$. Therefore θ is the isomorphism of groups $\mathcal{Z}^1_c(X \times \Gamma, A)$ and $\mathcal{Z}^1_{c_1}(X \times \Gamma, A)$ transforming $\mathcal{B}^1_c(X \times \Gamma, A)$ onto $\mathcal{B}^1_{c_1}(X \times \Gamma, A)$, i. e. $\mathcal{H}^1_c(X \times \Gamma, A)$ and $\mathcal{H}^1_{c_1}(X \times \Gamma, A)$ are isomorphic. ◆

4. Recurrence and superrecurrence of H-cocycles

Definition 4.1. 1) *A H-cocycle (or cocycle)* $\beta \in Z^1_{f,c}(X \times \Gamma, A)$ *is recurrent if for any set* $D \subset X$ *of positive measure and any neighborhood* U_0 *of the identity in* A

$$\mu\left(\bigcup_{\gamma \in \Gamma} (D \cap \gamma^{-1} D \cap \{x \in X : \beta(x, \gamma) \in U_0\})\right) > 0.$$

Otherwise the H-cocycle (or cocycle) β *is transient.*

2) *H-cocycle* $\beta \in Z^1_{f,c}(X \times \Gamma, A)$ *is superrecurrent if for any set* $D \subset X$, $\mu(D) > 0$ *and for any neighborhoods of the identity* V_e *and* U_0 *in* G *and* A *respectively*

$$\mu\left(\bigcup_{\gamma \in \Gamma} (D \cap \gamma^{-1} D \cap \{x \in X : c(x, \gamma) \in V_e\} \cap \{x \in X : \beta(x, \gamma) \in U_0\})\right) > 0.$$

Otherwise β *is supertransient.*

We remark that for H-cocycles $\beta \in Z^1_{0,\rho}(X \times \{T^n\}, \mathbf{R})$ Definition 4.1 is equivalent to the definition of H-recurrence and H-superrecurrence considered in papers [1-3, 8].

For a cocycle $c \in Z^1(X \times \Gamma, G)$ one can define a group of nonsingular automorphisms $\Gamma(c) \subset \mathrm{Aut}\,(X \times G, \mu \times m_G)$ (m_G is the Haar measure on G) acting by the formula:

$$\gamma(c)(x, h) = (\gamma x, c(x, \gamma)h), \quad \gamma \in \Gamma, \quad (x, h) \in X \times G.$$

It is known that $\Gamma(c)$ is conservative if and only if c is recurrent [7].

If $\Gamma(c)$ is an ergodic automorphism group, then c is called a cocycle with dense range in G.

Lemma 4.2. *Let* $f \in Z^2(G \times G, A)$, $c \in Z^1(X \times \Gamma, G)$ *and* $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$. *Then*

$$\beta_\alpha(x, h, \gamma(c)) = h^{-1}(\alpha(x, \gamma)) + f(c(x, \gamma), h)$$

is a cocycle from $Z^1(X \times G \times \Gamma(c), A)$.

Proof. We have

$$b_\alpha(x, h, \gamma_2(c)\gamma_1(c)) = h^{-1}(\alpha(x, \gamma_2\gamma_1)) + f(c(x, \gamma_2\gamma_1), h) =$$

$$= h^{-1}[f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) + c(x, \gamma_1)^{-1}(\alpha(\gamma_1 x, \gamma_2)) + \alpha(x, \gamma_1)] +$$

$$+f(c(\gamma_1 x, \gamma_2)c(x, \gamma_1), h). \tag{11}$$

On the other hand,

$$b_\alpha(\gamma_1(c)(x, h), \gamma_2(c)) + b_\alpha(x, h, \gamma_1(c)) =$$

$$= (c(x, \gamma)h)^{-1}(\alpha(\gamma_1 x, \gamma_2)) + f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)h) +$$

$$+ h^{-1}(\alpha(x, \gamma_1)) + f(c(x, \gamma_1), h). \tag{12}$$

From the relation

$$h^{-1}(f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1))) + f(c(\gamma_1 x, \gamma_2)c(x, \gamma_1), h) =$$

$$= f(c(x, \gamma_1), h) + f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1) h)$$

we obtain that (11) and (12) are the same expressions, i.e. $b_\alpha \in Z^1(X \times G \times \Gamma(c), A)$. ◆

Lemma 4.3. *The following relation is valid:*

$$b_\alpha(x, hg^{-1}, \gamma(c)) = g(b_\alpha(x, h, \gamma(c))) + f(c(x, \gamma)h, g^{-1}) - f(h, g^{-1}). \tag{13}$$

P r o o f. It follows from Lemma 4.2 that

$$h_1^{-1}h(b_\alpha(x, h, \gamma(c))) = h_1^{-1}(\alpha(x, \gamma)) + h_1^{-1}h(f(c(x, \gamma), h)) \tag{14}$$

and

$$h_1^{-1}(\alpha(x, \gamma)) = b_\alpha(x, h_1, \gamma(c)) - f(c(x, \gamma), h). \tag{15}$$

Now (15) allows one to transform (13) as follows:

$$h_1^{-1}h(b_\alpha(x, h, \gamma(c))) = b_\alpha(x, h_1, \gamma(c)) + h_1^{-1}h(f(c(x, \gamma), h)) - f(c(x, \gamma), h_1).$$

Since

$$h_1^{-1}h(f(c(x, \gamma), h)) + f(c(x, \gamma)h, h^{-1}h_1) = f(h, h^{-1}h_1) - f(c(x, \gamma), h_1),$$

then

$$h_1^{-1}h(b_\alpha(x, h, \gamma(c))) - b_\alpha(x, h_1, \gamma(c)) = f(h, h^{-1}h_1) - f(c(x, \gamma)h, h^{-1}h_1).$$

With $g = h_1^{-1}h$ we obtain (13). ◆

Theorem 4.4. *Let c be a recurrent cocycle and $f \in Z^2(G \times G, A)$. Then the following properties are equivalent:*

(i) *a H-cocycle* $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$ *is superrecurrent;*

(ii) *a cocycle* $\alpha_c = (c, \alpha) \in Z^1(X \times \Gamma, E_f)$ *is recurrent;*

(iii) $b_\alpha \in Z^1(X \times G \times \Gamma(c), A)$ *is recurrent.*

P r o o f. Equivalence of (i) and (ii) is direct consequence of the definitions. To prove equivalence of (ii) and (iii) we note that the groups of automorphisms $\Gamma(\alpha_c)$ and $\Gamma(c)(b_\alpha)$ are isomorphic. In fact, let $P : (x, h, a) \to (x, (h, a)) : X \times G \times A \to X \times E_f$ (we use here the natural correspondence between E_f and $G \times A$). Then

$$\gamma(\alpha_c)(P(x, h, a)) = (\gamma x, (c, \alpha)(h, a)) =$$

$$= (\gamma x, c(x, \gamma)h, f(c(x, \gamma), h) + h^{-1}(\alpha(x, \gamma)) + a) = P\gamma(c)(b_\alpha)(x, h, a).$$

Equivalence of (ii) and (iii) follows from the result of K. Schmidt mentioned above [7]. ◆

Theorem 4.5. *Let c be a recurrent cocycle from* $Z^1(X \times \Gamma, G)$*. Then a H-cocycle* $\alpha \in Z^1_{f,c}(X \times \Gamma, A)$ *is recurrent if and only if it is superrecurrent.*

P r o o f. In the case when c is the Radon-Nikodym cocycle and α is an usual cocycle taking values in $\mathbf{R}^n$ this theorem was proved by K. Schmidt in [6]. We will prove the following statement which is equivalent to the theorem in view of Theorem 4.4:

A H-cocycle α *is transient if and only if the cocycle* b_α *is transient.*

Let α be a transient cocycle from $Z^1_{f,c}(X \times \Gamma, A)$. It means that there is a set $D \subset X$, $\mu(D) > 0$ and a neighborhood U_0 of the identity in A such that $\alpha(x, \gamma) \notin U_0$ for every $x \in D$ and $\gamma \in \Gamma$ with $\gamma x \in D$. We will use

Lemma 4.6. *Let the group G acts continuously on A. Then for any neighborhood W of the identity in A and any compact set* $K \subset G$ *there exists a neighborhood* W_0 *of the identity in A such that* $h^{-1}(W_0) \subset W$ *for all* $h \in K$.

P r o o f. For any $h \in G$ the map $(h, a) \to h^{-1}(a)$ is continous and $h^{-1}(0) = 0$. There exist the neighborhoods V_h of $h \in G$ and $W_0(h)$ of $0 \in A$ such that $h^{-1}(a) \in W$ for all $(h, a) \in V_h \times W_0(h)$. Let $\{h_i\}$ be a finite set of elements of G such that

$\bigcup_i V_{h_i} \supset K$. Put $W_0 = \bigcap_i W_0(h_i)$. It is easy to see that $h^{-1}(W_0) \subset W$ for all $h \in K$. ◆

Let us continue the proof of Theorem 4.5. Prove that there exists a neighborhood U'_0 of $0 \in A$ such that $h^{-1}(\alpha(x, \gamma)) \notin U'_0$ for a.e. $(x, h) \in D \times V_e$ and $\gamma(c)$ such that $\gamma(c)(x, h) \in D \times V_e$, where V_e is a compact symmetric neighborhood of the identity in G. (Here we use the recurrence of c). In fact, it sufficies to apply Lemma 4.6 and find a neighborhood U'_0 of $0 \in A$ such that $h(U'_0) \subset U_0$ for all $h \in V_e$. (We will choose V_e later). Then, for $(x, h) \in D \times V_e$,

$$h^{-1}(\alpha(x, \gamma)) \in h^{-1}(A - U_0) \subset A - U'_0 .$$

Lemma 4.7. *Let $f\colon G \times G \to A$ be a continuous map and $f(g, e) = 0$ for every $g \in G$. Then for any compact set $K \subset G$ and any neighborhood W of $0 \in A$ there exists a neighborhood V of the identity in G such that $f(g_1, g_2) \in W$ for all $g_1 \in K$ and $g_2 \in V$.*

The proof of this lemma is similar to that of Lemma 4.6.

Thus, we have proved that if (x, h) and $(\gamma x, c(x, \gamma)h)$ are in $D \times V_e$, then $h^{-1}(\alpha(x, \gamma)) \notin U'_0$. This implies that $c(x, \gamma) \in V_e V_e^{-1} = V_e^2$, where V_e^2 is the compact neighborhood of the identity in G. Let $\bar{U}_0$ be a neighborhood of $0 \in A$. Choose, according to Lemma 4.7, a neighborhood $V'_e \subset V_e$ such that $f(c(x, \gamma), h) \in \bar{U}_0$ for all $h \in V'_e$ Consider the set $D \times V'_e$ and $\gamma(c)$ such that $\gamma(c)(x, h) \in D \times V'_e$ where $(x, h) \in D \times V'_e$. Let $\bar{U}_0$ be choosen so that $((A - U'_0) + \bar{U}_0) \cap \tilde{U} = \emptyset$, where $\tilde{U}$ is a neighborhood of $0 \in A$. It means that $b_\alpha(x, h, \gamma(c)) \notin \tilde{U}$ if (x, h) and $\gamma(c)(x, h)$ in $D \times V'_e$.

Conversely, let b_α be the transient cocycle. We will prove that α is also a transient H-cocycle. According to the assumption there are a set $B \subset X \times G$ of positive measure and a neighborhood W of $0 \in A$ such that $b_\alpha(x, h, \gamma(c)) \notin W$ if (x, h) and $\gamma(c)(x, h)$ are in B. We can state that B may be choosen so that the projection of B into G is a compact set. Let V_e be a compact symmetric neighborhood of the identity in G and we set

$B' = \bigcup_{g \in V_e} R(g)B$, where $R(g)(x, h) = (x, hg^{-1})$, $(x, h) \in X \times G$, $g \in G$. It is easy to see that $R(g)\,\gamma(c) = \gamma(c)R(g)$ for every $g \in G$ and $\gamma(c) \in \Gamma(c)$. In view of the commutativity of $R(G)$ and $\Gamma(c)$, for any $(x, h) \in B$ the point $\gamma(c)(x, h)$ is in B if and only if $\gamma(c)(x, hg^{-1}) \in B'$ for $g \in V_e$. We apply Lemmas 4.6 and 4.7 to deduce from (13) that V_e can be choosen small enough so that there is a neighborhood W' of $0 \in A$ such that

$$b_\alpha (x, h, \gamma(c)) \notin W', \tag{16}$$

when (x, h) and $\gamma(c)(x, h)$ are in B'. Notice that (16) is valid if we take any subset of B' instead of B' itself. In view of the inequality $(\mu \times m_G)(B') > 0$ there is an element $h_0 \in G$ such that $B_{h_0} = \{ x \in X : (x, h_0) \in B \}$ has a positive measure. It is evident that $B'_{h_0} = B_{h_0}$. Let $B'' = \bigcup_{g \in V_e} R(g)(B_{h_0} \times \{ h_0 \}) = B_{h_0} \times V_{h_0}$, where $V_{h_0} = h_0 V_e$ is the neighborhood of h_0. If (x, h) and $\gamma(c)(x, h)$ are in B'' then $c(x, \gamma)$ is in $V_{h_0} V_{h_0}^{-1}$. For arbitrary neighborhood U of $0 \in A$ one can take V_e so small that $f(c(x, \gamma), h) \in U$ when (x, h) and $\gamma(c)(x, h)$ are in B'' (by Lemma 4.7). From this and (16) we have that there is a neighborhood W'' of $0 \in A$ such that $h^{-1}(\alpha(x, \gamma)) \notin W''$ for every $(x, h) \in B''$ and $\gamma(c)(x, h) \in B''$. Using Lemma 4.6 we find a neighborhood W''' of $0 \in A$ such that $\alpha(x, \gamma) \notin W'''$ for (x, h) and $\gamma(c)(x, h)$ from B''. With $D = B_{h_0}$ we obtain that $\alpha(x, \gamma) \notin W'''$ when x and γx are in D. ◆

Just before publication of this article, I have been informed by A.Danilenko that there is a simple example showing that recurrence of H-cocycles does not imply their superrecurrence. It means that Theorem 4.5 holds in its trivial part only. I am grateful to A.Danilenko for his remark.

References

1. *K. Dajani,* Genericity of nontrivial H-superrecurrent H-cocycles. – Trans. Amer. Math. Soc. (1991), v. 323, pp. 111–132.
2. *K. Dajani,* On superrecurrence. – Canad. Math. Bull. (1991), v. 33, pp. 35–57.
3. *K. Dajani,* Generic results for cocycles with values in a semidirect product. – Can. J. Math. (1993), v. 45, pp. 497–516.
4. *D.V. Fuks,* Continuous cohomologies of topological groups and characteristic classes (Russian), Appendix to K. Brown, Cohomology of Groups. Nauka, Moscow (1987).
5. *C.C. Moore,* Group extensions and cohomology for locally compact groups,III. – Trans. Amer. Math. Soc. (1976), v. 221, pp. 1–33.
6. *K. Schmidt,* On recurrence. – Z. Wahrscheinlichkeitstheorie verw. Geb. (1983), v. 68, pp. 75–95.

7. *K. Schmidt*, Algebraic ideas in ergodic theory. – Regional conference series in mathematics (1990), v. 76, 93 p.
8. *D. Ullman*, A generalization of a theorem of Atkinson to non-invariant measures. – Pacific J. Math. (1987),v. 130, pp. 187–193.

О рекуррентности и суперрекуррентности H-коциклов

С.И. Безуглый

Изучен метод построения H-коциклов, принимающих значения в абелевых л.к.с. группах. Найдены необходимые и достаточные условия суперрекуррентности H-коциклов, обобщающие результат К. Шмидта о суперрекуррентности коциклов.

Про рекурентність та суперрекурентність H-коциклів

С.І. Безуглий

Вивчено метод побудови H-коциклів, які приймають значення в абелевих л.к.с. групах. Знайдено необхідні та достатні умови суперрекурентності H-коциклів, які узагальнюють результат К. Шмідта о суперрекурентності коциклів.