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A theorem on stability of the argument of characteristic function

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Let $f(t)$ be the characteristic function of a probability distribution on the line. If $1 - |f(t)| \le \varepsilon$ for $|t| \le a$ and, moreover, $\varepsilon \le C_1$, then

$$\min_{\beta \in R} \max_{|t| \le a} |\arg f(t) - \beta t| \le C_2 \varepsilon^{3/4},$$

where C_1, C_2 are suitable absolute constants.

1. Introduction and statement of results. We shall use the concepts of distribution function and of characteristic function in the sense generally accepted in the Probability Theory. The following abbreviations will be used without any explanations: d.f. = distribution function on the real line, ch.f. = characteristic function of d.f. We shall denote by C with indices positive absolute constants.

The following theorem is well known (see, e.g., [1, p. 13]).

Theorem A. *Let δ be a positive number and let $f(t)$ be a ch. f. If $|f(t)| = 1$ for $|t| \le \delta$, then $f(t) = \exp(i\beta t)$ where $\beta \in R$.*

Thus, if absolute value of a ch.f. is equal (precisely!) to 1 in a neighbourhood of zero, then its argument is a linear function. A.Ja. Khinchin ([1, p. 19]) obtained an estimation of stability of Theorem A. To state Khinchin's result we need the following definition. Let $f(t)$ be a ch.f., set $U = \sup\{u > 0 : f(t) \ne 0, |t| < u\}$. The function $\omega(t)$ is called the *argument* of $f(t)$ and is denoted by $\arg f(t)$ if satisfies the conditions:

(i) $\omega(t)$ is continuous on the open interval $(-U, U)$,
(ii) $\omega(0) = 0$,
(iii) $f(t) = |f(t)| \exp\{i\omega(t)\}$ for $t \in (-U, U)$.

Theorem B (Khinchin [1, p. 19]). *Let δ be a positive number and $\{f_n(t)\}_{n=1}^{\infty}$ be a sequence of ch. f.'s such that $|f_n(t)| \to 1$ as $n \to \infty$ uniformly with respect to $t \in [-\delta, \delta]$. Then, for arbitrarily large $T > 0$, arbitrarily small $\rho > 0$, and sufficiently large n, the inequality holds*

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$$|\omega_n(t) - t\omega_n(1)| \le \rho \{ \sqrt{1 - |f_n(t)|} + \sqrt{1 - |f_n(1)|} \} \text{ for } |t| \le T,$$

where $\omega_n(t) = \arg f_n(t)$.

The aim of this paper is to prove the following theorem.

Theorem. *Let* $a > 0$. *There exist absolute constants* C_1, C_2 *such that the following statement holds:*

If ch.f. $f(t)$ *satisfies the condition*

$$1 - |f(t)| \le \varepsilon \textit{ for } |t| \le a,$$

where $0 \le \varepsilon \le C_1$, *then*

$$\min_{\beta \in R} \max_{|t| \le a} |\arg f(t) - \beta t| \le C_2 \varepsilon^{3/4}.$$

This theorem is stronger in a sense than Khinchin's one. It can be viewed as a fact of Stability Theory of Stochastic Models (see [2-4]).

In section 2, we give a preliminary estimate of the distance between the argument of the ch.f. $f(t)$ satisfying the condition $1 - |f(t)| \le \varepsilon$ for $|t| \le a$ and the set of all linear functions. In section 3, we specify this estimate.

2. The preliminary estimate. Let us introduce some notations. Let $f(t)$ be a ch.f. with the d.f. $F(x)$, μ be medion of d.f. $F(x)$. Denote by $F^*(x)$, $F_\mu(x)$, $F_0(x)$ the d.f.'s with ch.f.'s $|f(t)|^2$, $f_\mu(t) = f(t)\exp(-i\mu t)$, $f_0(t) = f(t)\exp(-i\omega(1)t)$ respectively.

The proof of the theorem we divide into series of lemmas.

Lemma 1. *If the theorem is valid for* $a = 1$, *then it is valid for every* $a > 0$.

P r o o f. Let $f(t)$ be the ch. f. and $1 - |f(t)| \le \varepsilon$ for $|t| \le a$. Consider a ch. f. $\varphi(\tau) := f(a\tau)$. We have $1 - |\varphi(\tau)| \le \varepsilon$ for $|\tau| \le 1$. Then there is a real number β such that

$$\max_{|\tau| \le 1} |\arg \varphi(\tau) - \beta\tau| \le C\varepsilon^{3/4},$$

which is equivalent to

$$\max_{|t| \le a} |\arg f(t) - (\beta/a)t| \le C\varepsilon^{3/4}.$$

This completes the proof of the lemma. ■

Suppose, the ch. f. $f(t)$ satisfies the condition of the theorem to be proved with $a = 1$. Obviously, the condition

$$1 - |f(t)| \le \varepsilon \textit{ for } |t| \le 1 \tag{1}$$

yields

$$1 - |f(t)|^2 \le 2\varepsilon \textit{ for } |t| \le 1. \tag{2}$$

Lemma 2. *Inequality (1) with $\varepsilon \le C_3$ yields*

$$\int\limits_{|x| \ge \Delta} dF^*(x) \le \begin{cases} 1, & 0 < \Delta \le \Delta_\varepsilon, \\ C_4 \varepsilon \dfrac{1+\Delta^2}{\Delta^2}, & \Delta > \Delta_\varepsilon, \end{cases}$$

where

$$\Delta_\varepsilon = \sqrt{\frac{C_4 \varepsilon}{1 - C_4 \varepsilon}}. \tag{3}$$

Proof. Using well-known Raikov's inequality

$$1 - \operatorname{Re} \varphi(2t) \le 4(1 - \operatorname{Re} \varphi(t)),$$

which is valid for any ch.f. $\varphi(t)$, we obtain from (2) by induction

$$1 - |f(t)|^2 \le 4^n 2\varepsilon, \ for \quad |t| \le 2^n, \quad n = 1, 2, \ldots. \tag{4}$$

We shall use (4) for the values of n satisfying the inequality $4^n 2\varepsilon \le 1/2$, that is for $n \le N$, where

$$N = \left[\frac{\log(1/\varepsilon)}{2 \log 2}\right] - 1.$$

In the equality (see, e.g., [5, p. 85])

$$\int\limits_{-\infty}^{+\infty} \frac{x^2}{1+x^2} dG(x) = \int\limits_0^{+\infty} e^{-t}(1 - \operatorname{Re} g(t))\, dt, \tag{5}$$

which is valid for any d.f. $G(x)$ with the ch.f. $g(t)$, we set $G(x) = F^*(x)$, $g(t) = |f(t)|^2$. We obtain

$$\int\limits_{|x| \ge \Delta} dF^*(x) \le \frac{1+\Delta^2}{\Delta^2} \int\limits_0^{\infty} e^{-t}(1 - |f(t)|^2)\, dt. \tag{6}$$

Using inequalities (4) for $0 \le t \le 2^N$ and the trivial inequality $1 - |f(t)|^2 \le 1$ for $t > 2^N$, we obtain

$$\int\limits_0^{\infty} e^{-t}(1 - |f(t)|^2)\, dt \le \int\limits_0^1 e^{-t} 2\varepsilon\, dt + \sum_{k=1}^{N} \left(\int\limits_{2^{k-1}}^{2^k} e^{-t} 4^k 2\varepsilon\, dt \right) + \int\limits_{2^N}^{\infty} e^{-t}\, dt <$$

$$< 2\varepsilon \left[1 + 3 \sum_{j=0}^{N-1} 4^j e^{-2^j}\right] + \exp(-2^N). \tag{7}$$

We have

$$\sum_{j=0}^{\infty} 4^j \exp(-2^j) \le C_5 . \tag{8}$$

Moreover, by the definition of N, we have

$$\exp(-2^N) \le \exp\left(-\frac{1}{4}\varepsilon^{-1/2}\right) < \varepsilon \tag{9}$$

for $\varepsilon \le C_3$. Using (6)-(9), we obtain for any $\Delta > 0$

$$\int_{|x| \ge \Delta} dF^*(x) \le C_4 \varepsilon \frac{1+\Delta^2}{\Delta^2} . \tag{10}$$

The left hand side of (10) is not greater than 1, hence

$$\int_{|x| \ge \Delta} dF^*(x) \le \min\left\{1, C_4 \varepsilon \frac{1+\Delta^2}{\Delta^2}\right\}.$$

This is equivalent to the statement of Lemma 2. ■

We shall suppose, in addition, that $\varepsilon \le C_6$, where C_6 is so small that $\Delta_\varepsilon \le 1/2$ for $\varepsilon \le C_6$. For any d.f. $G(x)$, define

$$W_G(\Delta) = 1 - G(\Delta) + G(-\Delta), \quad \Delta > 0.$$

From the symmetrization inequality (see, e.g., [6, p. 177])

$$\int_{|x-\mu| \ge \Delta} dF(x) \le 2 \int_{|x| \ge \Delta} dF^*(x)$$

and from Lemma 2, it follows that

$$W_{F_\mu}(\Delta) \le \begin{cases} 1, & 0 < \Delta \le \Delta_\varepsilon, \\ C_7 \varepsilon \dfrac{1+\Delta^2}{\Delta^2}, & \Delta > \Delta_\varepsilon. \end{cases} \tag{11}$$

Lemma 3. *If a ch. f. $f(t)$ satisfies condition* (1) with $0 < \varepsilon \le C_8$, then

$$|f_\mu(t) - 1| \le C_9 \sqrt{\varepsilon} \;\; \textit{for} \;\; |t| \le 1. \tag{12}$$

P r o o f. For $|t| \le 1$, we have

$$|f_\mu(t) - 1| \le \left(\int_{|x| \le 1} + \int_{|x| > 1} \right) |e^{itx} - 1| \, dF_\mu(x) \le$$

$$< \int_{|x| \le 1} |x| \, dF_\mu(x) + 2W_{F_\mu}(1). \tag{13}$$

Integration by parts shows (see, e.g., [2, p. 35]) that

$$\int_{|x| \le 1} |x| \, dF_\mu(x) \le \int_0^1 W_{F_\mu}(x)\, dx,$$

therefore, (11) and (13) yield

$$|f_\mu(t) - 1| \le \int_0^1 W_{F_\mu}(x)\, dx + 2W_{F_\mu}(1) \le \Delta_\varepsilon + C_7 \varepsilon \int_{\Delta_\varepsilon}^1 x^{-2}\, dx + C_7 \varepsilon. \tag{14}$$

Assuming that $\varepsilon \le 1/(2C_4)$, we have

$$\Delta_\varepsilon \le \sqrt{2C_4\varepsilon}, \qquad \int_{\Delta_\varepsilon}^1 x^{-2}\, dx \le 1/\sqrt{C_4\varepsilon}.$$

Using these inequalities and (14), we obtain (12). ■

Lemma 4. *If a ch.f. $f(t)$ satisfies* (1) *with* $\varepsilon \le C_9$, *then*

$$|\omega_0(t)| \le C_{10}\sqrt{\varepsilon}, \text{ for } |t| \le 1, \tag{15}$$

where

$$\omega_0(t) = \arg f_0(t) = \omega(t) - \omega(1)t.$$

P r o o f. If $|z-1| \le \delta$, then $|\arg z| \le (\pi/2)\delta$. Using this fact with $z = f_\mu(t)$ and applying Lemma 3, we obtain

$$|\omega_\mu(t)| = |\omega(t) - \mu t| \le C_{11}\sqrt{\varepsilon}, \text{ for } |t| \le 1. \tag{16}$$

Setting $t = 1$, we obtain

$$|\omega(1) - \mu| \le C_{11}\sqrt{\varepsilon}. \tag{17}$$

Since

$$|\omega_0(t)| \le |\omega(t) - \mu t| + |\mu - \omega(1)|\,|t|,$$

we obtain (15) from (16) and (17). ■

3. The more precise estimate. Further, the number ε will be assumed so small that

$$C_{10}\sqrt{\varepsilon} \le \pi/2, \tag{18}$$

i.e., $\varepsilon \le C_{12}$. Using the inequalities $|\sin u| \ge (2/\pi)|u|$ ($|u| \le \pi/2$) and $\varepsilon \le 1/2$, we obtain

$$|\omega_0(t)| \le \frac{\pi}{2} |\sin \omega_0(t)| \le \pi\, |f_0(t)|\, |\sin \omega_0(t)| = \pi\, |\operatorname{Im} f_0(t)|.$$

Let us introduce two positive parameters κ and $\alpha < 1$ connected by the conditions

$$\int_{|x| > \kappa} dF_0(x) \le \alpha, \tag{19}$$

$$\int_{|x| \geq \kappa} dF_0(x) \geq \alpha. \tag{20}$$

We have

$$|\omega_0(t)| \leq \pi \left\{ \left| \int_{|x|>\kappa} \sin tx \, dF_0(x) \right| + \left| \int_{|x| \leq \kappa} \sin tx \, dF_0(x) \right| \right\} =: \pi(J_1 + J_2). \tag{21}$$

Now, we are going to estimate the values of J_1 and J_2 in (21).

Lemma 5. *Suppose, conditions* (1),(18),(19), *and* (20) *are satisfied. Then the following inequality is valid*

$$J_1 \leq \sqrt{\alpha}\,(\sqrt{2\varepsilon} + |\omega_0(t)|) \text{ for } |t| \leq 1. \tag{22}$$

This lemma and Lemma 6 below were proved by A.Ya. Khinchin ([1, p. 21]), we give the proof here for reader's convenience.

P r o o f. Using the Cauchy–Bunyakovskii inequality, inequalities (19), (1), and $1 - \cos u \leq u^2/2$, $u \in R$, we have

$$J_1^2 \leq \int_{|x|>\kappa} dF_0(x) \int_{|x|>\kappa} \sin^2 tx \, dF_0(x) \leq$$

$$\leq \alpha \int_R (1 - \cos tx)(1 + \cos tx) \, dF_0(x) \leq 2\alpha(1 - \operatorname{Re} f_0(t)) =$$

$$= 2\alpha \{ 1 - |f(t)| + |f(t)|(1 - \cos \omega_0(t)) \} \leq \alpha\,(2\varepsilon + \omega_0^2(t)).$$

Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$, it follows that inequality (22) is valid. ■

Lemma 6. *Suppose, conditions* (1),(18)-(20) *are satisfied. Then the following inequality is valid*:

$$J_2 \leq \sqrt{\alpha}\,\sqrt{2\varepsilon} + \frac{1}{6}\kappa^3, \text{ for } |t| \leq 1. \tag{23}$$

P r o o f. Replacing t by 1 in (22) and taking into account the equality $\omega_0(1) = 0$, we obtain

$$\left| \int_{|x|>\kappa} \sin x \, dF_0(x) \right| \leq \sqrt{\alpha}\,\sqrt{2\varepsilon}. \tag{24}$$

Note that

$$\int_R \sin x \, dF_0(x) = \operatorname{Im} f_0(1) = 0. \tag{25}$$

It follows from (24) and (25) that

$$\Big| \int\limits_{|x| \le \kappa} \sin x \, dF_0(x) \Big| \le \sqrt{\alpha}\,\sqrt{2\varepsilon}. \tag{26}$$

It is easy to shown that

$$|\sin tx - t \sin x| \le \frac{1}{6} |x|^3 |t| \le \frac{1}{6}\kappa^3 \ for \ |t| \le 1, \ |x| \le \kappa. \tag{27}$$

From (26) and (27) it follows that

$$J_2 \le |t| \Big| \int\limits_{|x| \le \kappa} \sin x \, dF_0(x) \Big| + \frac{1}{6}\kappa^3 \int\limits_{|x| \le \kappa} dF_0(x) \le \sqrt{\alpha}\,\sqrt{2\varepsilon} + \frac{1}{6}\kappa^3.$$

This completes the proof of the lemma. ■

Now we are going to estimate the value of κ via α.

Lemma 7. *Suppose that conditions* (1), (18)-(20) *and also the inequality*

$$\kappa \le \pi \tag{28}$$

hold. Then the following inequality is valid

$$\kappa \le C_{13} \sqrt{\frac{\varepsilon}{\alpha - C_{14}\sqrt{\varepsilon}}}. \tag{29}$$

The parameter α below will be chosen greater than $C_{14}\sqrt{\varepsilon}$.

P r o o f. First, we prove that

$$\varepsilon \ge \frac{2}{\pi^2}\kappa^2 \Bigg\{ \alpha - \int\limits_{|x| > 2\pi - \kappa} dF_0(x) \Bigg\}. \tag{30}$$

Using inequalities (28), (20) and $|\sin u| \ge (2/\pi)|u|$ (for $|u| \le \pi/2$), we obtain

$$\varepsilon \ge 1 - |f(1)| = 1 - \mathrm{Re} f_0(1) = \int\limits_R (1 - \cos x)\, dF_0(x) \ge$$

$$\ge 2 \int\limits_{\kappa \le |x| \le 2\pi - \kappa} \sin^2 \frac{x}{2} \, dF_0(x) \ge \sin^2 \frac{\kappa}{2} \int\limits_{\kappa \le |x| \le 2\pi - \kappa} dF_0(x) \ge$$

$$\ge \frac{2}{\pi^2}\kappa^2 \Bigg\{ \alpha - \int\limits_{|x| > 2\pi - \kappa} dF_0(x) \Bigg\}.$$

Now, we obtain an estimate from above of the integral in (30). Since $\kappa \le \pi$, we have

$$\int\limits_{-\infty}^{\infty} \frac{x^2}{1 + x^2} \, dF_0(x) \ge$$

$$\geq \frac{(2\pi-\kappa)^2}{1+(2\pi-\kappa)^2} \int\limits_{|x|>2\pi-\kappa} dF_0(x) \geq \frac{\pi^2}{1+\pi^2} \int\limits_{|x|>2\pi-\kappa} dF_0(x). \tag{31}$$

Using formula (5) with $G(x) = F_0(x)$ and inequality (31), we obtain

$$\int\limits_{|x|>2\pi-\kappa} dF_0(x) \leq \frac{1+\pi^2}{\pi^2} \int\limits_0^{\infty} e^{-t}(1-\operatorname{Re} f_0(t))\,dt. \tag{32}$$

We shall estimate the integral in the right hand side of (32) in the same manner as in the proof of Lemma 2. Using the elementary inequality

$$|1-z| \leq (1-|z|) + |\arg z| \text{ for } z \in C,\ |z| \leq 1,$$

we obtain from (1) and (15)

$$0 \leq 1-\operatorname{Re} f_0(t) \leq |1-f_0(t)| \leq C_{15}\sqrt{\varepsilon} \text{ for } |t| \leq 1. \tag{33}$$

Raikov's inequality and (33) yield

$$1-\operatorname{Re} f_0(t) \leq 4^k\delta \text{ for } |t| \leq 2^k, \tag{34}$$

where $\delta := C_{15}\sqrt{\varepsilon}$. Set $N = [(\log(1/\delta))/(2\log 2)] - 1$. Then

$$\int\limits_0^{\infty} e^{-t}(1-\operatorname{Re} f_0(t))\,dt \leq \int\limits_0^1 e^{-t}\delta\,dt + \sum_{k=1}^{N} \int\limits_{2^{k-1}}^{2^k} e^{-t}4^k\delta\,dt + \int\limits_{2^N}^{\infty} e^{-t}2dt \leq$$

$$\leq \delta\left\{1 + 3\sum_{j=0}^{\infty} 4^j e^{-2^j}\right\} + 2e^{-2^N}.$$

Taking into account (8), (9) and the definition of the parameter δ, we have

$$\int\limits_0^{\infty} e^{-t}(1-\operatorname{Re} f_0(t))\,dt \leq C_{16}\sqrt{\varepsilon}. \tag{35}$$

It follows from (35) and (32) that

$$\int\limits_{|x|>2\pi-\kappa} dF_0(x) \leq C_{17}\sqrt{\varepsilon}. \tag{36}$$

Combining (36) and (30), we obtain (29). ■

Now we can choose the parameter α.

Lemma 8. *Suppose conditions* (1),(18)-(20) *are satisfied. Then, for some* $C_{18} \geq 2C_{14}$, *if we set*

$$\alpha = C_{18}\sqrt{\varepsilon}, \tag{37}$$

condition (28) will be valid.

P r o o f. By (35), we have

$$\frac{\kappa^2}{1+\kappa^2}\int\limits_{|x|\geq\kappa} dF_0(x) \leq \int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^2}\, dF_0(x) =$$

$$= \int\limits_0^{\infty} e^{-t}(1-\operatorname{Re} f_0(t))\, dt \leq C_{16}\sqrt{\varepsilon}. \tag{38}$$

Inequalities (38) and (20) yield

$$\alpha \leq \int\limits_{|x|\geq\kappa} dF_0(x) \leq \frac{1+\kappa^2}{\kappa^2} C_{16}\sqrt{\varepsilon}. \tag{39}$$

From (39) and (37) we obtain

$$\kappa \leq \left(\frac{C_{18}}{C_{16}} - 1\right)^{-1/2}$$

We see that we can reach (28) by choosing the absolute constant C_{18} being large enough. Lemma 8 is proved. ■

Now we are ready to complete the proof of the Theorem.

Lemmas 6 and 7 yield (see (23) and (29))

$$J_2 \leq \sqrt{\alpha}\,\sqrt{2\varepsilon} + \left(C_{13}\sqrt{\frac{\varepsilon}{\alpha - C_{14}\sqrt{\varepsilon}}}\right)^3.$$

Since $C_{18} \geq 2C_{14}$, we have from (37) that

$$\sqrt{\frac{\varepsilon}{\alpha - C_{14}\sqrt{\varepsilon}}} \leq C_{19}\,\varepsilon^{1/4}.$$

Thus, we have

$$J_2 \leq C_{20}\,\varepsilon^{3/4}. \tag{40}$$

From (21) and (22) it follows

$$(1-\pi\sqrt{\alpha})\,|\,\omega_0(t)\,| \leq \pi\sqrt{2\alpha\varepsilon} + \pi J_2. \tag{41}$$

By (37), we have

$$\pi\sqrt{\alpha} = \pi\sqrt{C_{18}\,\varepsilon^{1/2}} = C_{21}\,\varepsilon^{1/4}.$$

Supposing $\varepsilon \leq C_{22}$, where C_{22} being small enough, we can provide $\pi\sqrt{\alpha} \leq 1/2$. There, (40) and (41) yield

$$|\,\omega_0(t)\,| \leq C_{23}\,\varepsilon^{3/4} \quad (|t| \leq 1).$$

This completes the proof of the Theorem.

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Одна теорема об устойчивости аргумента характеристической функции

А.И. Ильинский

Пусть $f(t)$ — характеристическая функция вероятностного распределения на прямой. Если $1 - |f(t)| \le \varepsilon$ при $|t| \le a$ и, кроме того, $\varepsilon \le C_1$, то

$$\min_{\beta \in R} \max_{|t| \le a} |\arg f(t) - \beta t| \le C_2 \varepsilon^{3/4},$$

где C_1 и C_2 — абсолютные постоянные.

Одна теорема про стійкість аргументу характеристичної функції

О. І. Ільїнський

Нехай $f(t)$ — характеристична функція ймовірнісного розподілу на прямій. Якщо $1 - |f(t)| \le \varepsilon$ при $|t| \le a$, а також $\varepsilon \le C_1$, тоді

$$\min_{\beta \in R} \max_{|t| \le a} |\arg f(t) - \beta t| \le C_2 \varepsilon^{3/4},$$

де C_1 та C_2 — абсолютні сталі.