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Eigenvalue distribution of large random matrices with correlated entries

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We study the normalized eigenvalue counting function $N_n(\lambda)$ of an ensemble of $n \times n$ symmetric random matrices with statistically dependent arbitrary distributed entries $u_n(x, y)$, $x, y = 1, \dots, n$. We prove that if the correlation function S of the entries is the same for each n and the correlation coefficient of random fields $\{ u_n(x, y) \}$ decays fast enough, then in the limit $n \to \infty$ the measure $N_n(d\lambda)$ weakly converges in probability to a nonrandom measure $N(d\lambda)$. We derive an equation for the Stieltjes transform of limiting $N(d\lambda)$ and show that the latter depends only on the limiting matrix of averages of $u_n(x, y)$ and the correlation function S.

1. Wigner ensembles and the semicircle law

Random matrices of large dimensions are of considerable and rapidly increasing interest both due to their numerous applications in different areas of theoretical and mathematical physics (statistical nuclear physics, statistical mechanics, quantum chaology, quantum field theory, etc.) and because the spectral theory of random matrices represents a somewhat unusual but natural part of modern operator theory. In this field a large number of results are related to eigenvalue distribution of Hermitian (or symmetric) random matrices in the limit of their infinitely increasing dimension. The principal fact here was established by E. Wigner [14]. This statement is known as the semicircle (or Wigner) law and can be formulated as follows.

Let us introduce the symmetric $n \times n$ matrix $U^{(n)}$ whose entries $u_n(x, y)$ are independent (except the symmetry condition) identically distributed random variables satisfying the conditions

$$\mathrm{E}\{ u_n(x, y) \} = 0, \quad \mathrm{E}\{ |u_n(x, y)|^2 \} = v^2, \; x, \; y = \overline{1, n}, \tag{1.1}$$

(symbol $\mathrm{E}\{ \cdot \}$ denotes the mathematical expectation) and consider the matrix

$$W^{(n)} = \frac{1}{\sqrt{n}} U^{(n)}. \tag{1.2}$$

Let the normalized eigenvalue counting function (NCF) $N_n(\lambda)$ of a symmetric $n \times n$ matrix $A^{(n)}$ be defined by the formula

$$N(\lambda; A^{(n)}) = n^{-1} \sum_{i=1}^{n} \mathbf{I}_{(-\infty, \lambda]} (\lambda_i^{(n)}),$$

where $\mathbf{I}_B(x)$ is the characteristic function of a set B and $\lambda_i^{(n)}$ are the eigenvalues of $A^{(n)}$. Then the sequence $N_n(\lambda) \equiv N(\lambda; W^{(n)})$ converges with probability 1 to a function $N_w(\lambda)$

$$\lim_{n \to \infty} N_n(\lambda) = N_w(\lambda) \tag{1.3a}$$

with the density of the semicircle form

$$N'_w(\lambda) = (2\pi v^2)^{-1}(4v^2 - \lambda^2)^{-1/2}\mathbf{I}_{[-2v, 2v]}(\lambda). \tag{1.3b}$$

The limit of NCF's, if it exists, is called the integrated density of states (IDS). Here and below the convergence of nondecreasing functions is meant to be the weak convergence of measures associated with these functions.

We will call the ensemble (1.1)-(1.2) the Wigner ensemble. For the case of the Gaussian $u_n(x, y)$ it is more convenient to consider the variant of (1.2)

$$\mathbf{E}\{u_n(x, y)u_n(s, t)\} = v^2(\delta_{xs}\delta_{yt} + \delta_{xt}\delta_{ys}). \tag{1.4}$$

Then the density of the probability distribution of $W^{(n)}$ has the form

$$\Pr\{W^{(n)} \in dW^{(n)}\} = Z_n^{-1}\exp\{-n(4v^2)^{-1}\operatorname{Tr}F(W^{(n)})\}dW^{(n)},$$

$$F(x) = x^2, \tag{1.5}$$

which is invariant with respect to orthogonal transformations of $\mathbf{R}^n$. This ensemble is known as the Gaussian Orthogonal Ensemble (GOE).

It should be noted that the spectral properties of the random matrices (1.2) with independent entries are extensively studied (see e.g. [10, 4, 6]), while the case of statistically dependent entries is not so well understood. Only ensembles of random matrices which have explicit special forms of a probabilistic distribution have been considered so far. In particular, a large number of papers is devoted to the ensembles having probability distribution (1.5) with an even $F(x)$ infinitely increasing as $|x| \to \infty$ (see, for example, the review [3] and references therein related to applications of corresponing to (1.5) matrix models in quantum field theory and other branches of theoretical physics; rigorous results on the IDS of such ensembles are obtained in [2]).

In the present paper we study the limiting eigenvalue distribution of an ensemble of random matrices having arbitrarily distributed statistically dependent random entries. Thus, in contrast to the ensembles analysed in [2] and [3], we consider the ensemble whose probability distribution is not orthogonally invariant. In other words, the ensemble we study is a generalization of the Wigner one. It has the form

$$H^{(n)} = H_0^{(n)} + \frac{1}{\sqrt{n}}U^{(n)} \tag{1.6}$$

with nonrandom $H_0^{(n)}$ and random $U^{(n)}$ whose entries $u_n(x, y)$ are statistically dependent random variables having zero mean value and a covariance matrix $S^{(n)}$. The ensemble of the form (1.6) could be called the deformed Wigner ensemble.

We prove that if:

(i) the sixth moments of the random variables $u_n(x, y)$ are uniformly bounded;

(ii) the correlation coefficient of the random field $\{ U^{(n)} \}$ vanishes fast enough;

(iii) the matrices $H_0^{(n)}$ and $S^{(n)}$ are n-dimensional restrictions of the Toeplitz matrices H_0 and S,

then the NCF's $N(\lambda; H^{(n)})$ converge in probability to a nonrandom IDS $N(\lambda)$. We derive an explicit equation determining the Stieltjes transform of the limiting $N(\lambda)$ (see (2.11)). This equation depends only on the Fourier transforms $\widetilde{H}_0$ and $\widetilde{S}$ of H_0 and S. It generalizes the equation (1.18) for the Stieltjes transform of the semicircle distribution (1.3b). Thus, corresponding to (2.11) IDS can be called the generalized semicircle distribution. We study the properties of this limiting $N(\lambda)$ and show that it differs from (1.3b) even for the case of $H_0 = 0$.

An equation similar to (2.11) was derived first by Wegner [13] for an ensemble of random matrix operators. This ensemble can be regarded as an example of random matrices whose entries are dependent in the manner opposite to the described above. In Sect. 5 we discuss interrelations between the Wigner and Wegner ensembles and present the equations determining the IDS of their generalizations.

The deformed Wigner ensemble (1.6) with the Gaussian $U^{(n)}$ having correlated entries was considered in [8]. In this paper equation (2.12) was derived by using a version of the characteristic property of a Gaussian random variable ξ with zero average:

$$\mathrm{E}\{ \xi F(\xi) \} = \mathrm{E}\{ \xi^2 \}\mathrm{E} \{ F'(\xi) \}, \tag{1.7}$$

where a nonrandom function F is such that all the integrals in (1.7) exist. This relation allowed us to employ the well developed technique based on the derivation and asymptotic analysis in the limit $n \to \infty$ of an infinite system of relations for the moments of the resolvent of $H^{(n)}$ (see also [7, 6] for applications of this technique to various random matrix and random operator ensembles).

In our non-Gaussian case, to avoid the use of (1.7), we follow a version of the perturbation theory approach developed in [6] for random matrices with independent entries. We combine these computations with the method developed by Bernstein for sums of dependent random variables (see, e.g., [5]). Another modification of the technique used in [8] is based on the observation that, to prove the theorems presented in [8], it is not necessary to study the infinite system of relations for the resolvent. We show that sufficient estimates follows from the two first relations of this system. This fact considerably simplifies the proof. To illustrate the main points of the technique which we will use below, we present here a rather short proof of the semicircle law for the GOE (1.2), (1.5).

Usually, instead of the NCF $N(\lambda; H^{(n)})$, it is convenient to study its Stieltjes transform

$$f_n(z) \equiv \int (\lambda - z)^{-1} dN(\lambda; H^{(n)}) = n^{-1} \operatorname{Tr} G^{(n)}, \quad \operatorname{Im} z \neq 0, \tag{1.8}$$

where $G^{(n)} = (H^{(n)} - z)^{-1}$. It follows from the properties of the Herglotz functions [1] and the Helly theorem [9], that to prove the existence of the IDS of the ensemble $H^{(n)}$, it is sufficient to prove the convergence (in probability or with probability 1)

$$\lim_{n \to \infty} f_n(z) = f(z) \tag{1.9}$$

in a region

$$D_0 = \{ z : | \operatorname{Im} z | \geq \eta_0 \}. \tag{1.10}$$

The limit function $f(z)$ can be extended into the region $C_{\pm} = \{ z \in \mathrm{C},\ \mathrm{Im}\, z \neq 0 \}$; one can obtain corresponding $N(\lambda)$ by using the inversion formula [1]

$$N(b) - N(a) = \pi^{-1} \lim_{\eta \downarrow 0} \int_a^b \mathrm{Im}\, f(\lambda + i\eta)\, d\lambda,$$

where b and a are the continuity points of $N(\lambda)$.

Let us write the resolvent identity

$$G - G' = -G(H - H')G', \quad G = (H - z)^{-1}, \quad G' = (H' - z)^{-1} \tag{1.11}$$

for $H(x, y) = W^{(n)}(x, y) = n^{-1/2} u(x, y)$, $x, y = \overline{1, n}$, $H' = 0$ in the form †

$$\mathrm{E}\{ g_n \} = \zeta - \zeta n^{-3/2} \sum_{x, y = 1}^{n} \mathrm{E}\{ G(x, y)\, u(y, x) \}, \tag{1.12}$$

where $g_n = n^{-1} \mathrm{Tr}\, G$ and $\zeta = -z^{-1}$. To calculate the last average in (1.12), we use the generalization of equality (1.7) to the case of the distribution (1.5) and obtain the relation

$$\mathrm{E}\{ G(x, y)\, u(y, x) \} = -v^2 n^{-1/2} \mathrm{E}\{ G(x, x)\, G(y, y) + G(x, y)\, G(x, y) \}. \tag{1.13}$$

Here we take into account the equality

$$\frac{\partial G(x, y)}{\partial u(y, x)} = -n^{-1/2} (1 + \delta_{xy})^{-1} [G(x, x)\, G(y, y) + G(x, y)\, G(x, y)], \tag{1.14}$$

which follows from (1.11). Let us denote $\tilde{g}_n = g_n - \mathrm{E}\{ g_n \}$, substitute (1.13) into (1.12) and obtain our first main relation

$$\mathrm{E}\{ g_n \} = \zeta + \zeta v^2 \mathrm{E}\{ g_n \}^2 + \zeta v^2 \mathrm{E}\{ \tilde{g}_n g_n \} + v^2 \zeta n^{-2} \mathrm{E}\{ \mathrm{Tr}\, G^2 \}, \tag{1.15}$$

we use the fact that the resolvent of a symmetric matrix is also symmetric).

We derive the second main relation for the variance of g_n

$$C_n = \mathrm{E}\{ \tilde{g}'_n \tilde{g}_n \} \equiv \mathrm{E}\{ \tilde{g}'_n g_n \}, \quad g'_n \equiv g_n(\bar{z}) = \overline{g_n(z)}.$$

Applying to the factor g_n in the last average the identity (1.11) and using (1.7) and (1.14), we obtain the equality

$$\mathrm{E}\{ \tilde{g}'_n g_n \} =$$

$$= \zeta v^2 \mathrm{E}\{ \tilde{g}'_n g_n^2 \} + \zeta v^2 n^{-2} \mathrm{E}\{ \tilde{g}'_n \mathrm{Tr}\, G^2 \} + 2 \zeta v^2 n^{-3} \mathrm{E}\{ \mathrm{Tr}\, [G']^2 G \}. \tag{1.16}$$

In fact, (1.14) and (1.16) prove (1.3). Indeed, since for $z \in D_0$ (1.10) the estimates

$$| g_n | \leq \eta_0^{-1}, \quad | n^{-1} \mathrm{Tr}\, G^2 | \leq \eta_0^{-2} \tag{1.17}$$

hold with probability 1, then the convergence in the mean value

$$\lim_{n \to \infty} \mathrm{E}\{ g_n \} = f_w, \quad z \in D_0,$$

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† Here and below we omit the superscripts n when no confusion can arise.

with f_w satisfying, the equation

$$f_w(z) = (-z - v^2 f_w(z))^{-1}, \tag{1.18}$$

follows from (1.15) as soon as the estimate stating the self-averageness property of g_n

$$\mathbf{E}\{|\tilde{g}_n(z)|^2\} = C_n, \quad \sup_{z \in D_0} C_n = o(1), \quad n \to \infty \tag{1.19}$$

is estabished. Note that (1.19) implies the convergence (1.3a) in probability (see Lemma 1, Sect. 4) and, if $\Sigma C_n < \infty$, then (1.9) and, hence, (1.3a) hold with probability 1.

Observing that

$$\mathbf{E}\{\tilde{g}'_n g_n^2\} = \mathbf{E}\{\tilde{g}'_n g_n\}\mathbf{E}\{g_n\} + \mathbf{E}\{\tilde{g}'_n \tilde{g}_n g_n\} \tag{1.20}$$

and taking into account estimates (1.17), we derive from (1.16) the following

$$C_n \le 2v^2 \eta^{-1} C_n + v^2 \eta^{-3} n^{-1} \sqrt{C_n} + 2v^2 \eta^{-4} n^{-2}, \tag{1.21}$$

where $\eta = \operatorname{Im} z$. Since $C_n < \eta^{-2}$, then for $z \in D_0$ with $\eta_0 > 2v^2 + 1$ inequality (1.21) implies the estimate $\sup_{z \in D_0} C_n = O(n^{-2})$.

Using the inversion formula, it is easy to derive (1.3b) from (1.18). The simicircle law is proved.

2. Main results and discussion

We study the ensemble of symmetric random matrices $H^{(n)} = H_0^{(n)} + W^{(n)}$ with entries

$$H^{(n)}(x, y) = H_0(x - y) + n^{-1/2} u_n(x, y), \quad x, y = \overline{1, n}, \tag{2.1}$$

where a nonrandom function H_0 satisfies the conditions

$$H_0(x) = \overline{H_0(-x)}, \quad \sum_{x \in \mathbf{Z}} |H_0(x)| < \infty. \tag{2.2}$$

Let random variables $u_n(x, y)$ be normalized by the relations

$$\mathbf{E}\{u_n(x, y)\} = 0, \tag{2.3}$$

$$\mathbf{E}\{u_n(x, y) u_n(s, t)\} = S(x - s, y - t) + S(x - t, y - s), \tag{2.4}$$

where $\mathbf{E}\{\cdot\}$ denotes the mathematical expectation and $S(x, y)$ is such that $S(x, -y) = S(x, y) = S(y, x)$ and

$$\sum_{x, y} |S(x, y)| = S_0 < \infty. \tag{2.5}$$

Assume also that each of the random fields $\{u_n(x, y) = u_n(y, x),\ x \le y,\ x, y = \overline{1, n}\}$, $n \in \mathbf{N}$ possesces the following property: there exists a sequence of monotonically decreasing functions $\psi_n(x)$, $x > 0$, such that for arbitrary measurable functions Φ_1 and Φ_2, with all the integrals below convergent,

$$\left| \mathrm{E}\{ \Phi_1 (W_1^{(n)}) \Phi_2 (W_2^{(n)}) \} - \mathrm{E}\{ \Phi_1 (W_1^{(n)}) \} \mathrm{E}\{ \Phi_2 (W_2^{(n)}) \} \right| \le$$
$$\le \prod_{i=1,2} \left(\mathrm{E}\{ | \Phi_i (W_i^{(n)}) |^2 \} \right)^{1/2} \psi_n (d(A_1, A_2)), \tag{2.6}$$

where $W_i^{(n)}) = \{ u_n (x_i, y_i) : (x_i, y_i) \in A_i \}$, $i = 1, 2$, A_i being some sets of pairs (x, y), $x, y = \overline{1, n}$, and the distance $d(A_1, A_2)$ being given by

$$d(A_1, A_2) = \min \{ \inf_{x_i, y_i \in A_i} (| x_1 - x_2 | + | y_1 - y_2 |);$$
$$\inf_{x_i, y_i \in A_i} (| x_1 - y_2 | + | y_1 - x_2 |) \}.$$

Theorem 2.1. *Let*

$$\sup_n \sup_{x, y} \mathrm{E}\{ | u_n (x, y) |^6 \} = S_6 < \infty \tag{2.7}$$

and

$$\lim_{n \to \infty} n^{1/2} \psi_n (n^{1/6}) = 0. \tag{2.8}$$

Then

(a) *the convergence in probability*

$$p - \lim_{n \to \infty} N(\lambda; H^{(n)}) = N(\lambda) \tag{2.9}$$

holds;

(b) *the IDS $N(\lambda)$ is nonrandom and its Stieltjes transform $f(z) = \int (\lambda - z)^{-1} dN(\lambda)$ can be found from the relation*

$$f(z) = \int_0^1 \tilde{g}(p; z)\, dp, \tag{2.10}$$

where $\tilde{g}(p; z)$ is the unique solution of the equation

$$\tilde{g}(p; z) = \left(\tilde{H}_0 (p) - z - \int_0^1 \Gamma(p, q)\, \tilde{g}(q; z)\, dq \right)^{-1}, \tag{2.11}$$

in which

$$\tilde{H}_0 (p) = \sum_{x \in \mathbf{Z}} H_0 (x) \exp \{ 2\pi i x \}, \tag{2.12}$$

$$\Gamma(p, q) = \sum_{x, y \in \mathbf{Z}} S(x, y) \exp \{ 2\pi i (xp + yq) \}; \tag{2.13}$$

(c) *equation* (2.11) *is uniquely solvable in the class* G *of the functions* $f(p; z)$ *bounded in* $p \in [0, 1]$ *for each fixed* z, $\mathrm{Im}\, z \neq 0$, *analytic in* z, $\mathrm{Im}\, z \neq 0$ *for each fixed* $p \in [0, 1]$ *and such that*

$$\mathrm{Im}\, f(p; z)\, \mathrm{Im}\, z > 0, \quad \text{for } \mathrm{Im}\, z \neq 0.$$

Remarks.

1) Let us consider the ensemble (2.1) with (2.4) given by

$$S(x, y) = v^2 \delta_{x,0} \delta_{y,0} \tag{2.14}$$

(this is the case of uncorrelated random entries). Then relations (2.10) and (2.11) can be rewritten in the form (cf. (1.18))

$$f(z) = f_0(z + v^2 f(z)), \tag{2.15}$$

where

$$N_0(\lambda) = \lim_{n \to \infty} N(\lambda; H_0^{(n)}).$$

Equation (2.15) is a generalization of the equation for the Stieltjes transform of the semicircle distribution. It was obtained first in [11] for the deformed Wigner ensemble (1.7) with the diagonal $H_0^{(n)}$ having limiting IDS $N_0(\lambda)$ and jointly independent random entries $u_n(x, y)$ normalized by (1.1) and satisfying the well-known Lindeberg condition. Thus, it follows from Theorem 2.1 that the condition for $u_n(x, y)$ to be independent is not necessary for the semicircle law to be valid.

2) Conditions (2.3) and (2.4) represent a natural generalization of conditions (1.4) (see also Sect. 5). To be more close to the definition of the Wigner ensemble, whose matrices have entries with equal variance (1.1), one should consider the "triangle" random fields $\{ u_n(x, y),\ x \le y,\ x, y = \overline{1, n} \}$ and define the covariance matrix as

$$\mathbf{E}\{ u_n(x, y)\, u_n(s, t) \} = S(x - s, y - t), \ \text{for } x \le y,\ s \le t. \tag{2.16}$$

In this case Theorem 2.1 is also true, but its proof requires more cumbersome computations than the proof of the case of (2.4).

3) The property (2.6) with decaying $\psi(x)$ is known as the Bernstein condition which is sufficient for the central limiting theorem for sums of dependent random variables to be valid (see, for example, [5]). This condition is a consequence of the strong mixing condition [5] which characterizes random fields with weak dependence (such fields can also be referred to as locally dependent).

Note that Theorem 2.1 will also be true if we consider a nonmonotone function ψ instead of ψ_n. In this case the condition (2.8) is to be replaced by

$$\lim_{n \to \infty} n^{1/2} \sup_{x \ge n^{1/6}} \psi(x) = 0. \tag{2.17}$$

4) Equation (2.11) was derived first by F. Wegner [13] in a study of the n-component generalization of the discrete random Schrödinger operator. Infinite matrices of this model are of the block form with n being the dimension of the block. In this case relation (2.10) defines the IDS of the ensemble in the limit $n \to \infty$ (see [7] for rigorous results).

Random matrices of this random matrix ensemble (we call it the Wegner ensemble), as well as matrices of the Wigner ensemble we deal with, also have dependent random entries. But this dependence is opposite in some sense to the dependence considered in Theorem 2.1. Corresponding random fields do not satisfy the condition (2.17) and could be called asymptotically nonlocally dependent (see Sect. 5 for the definition of the Wegner ensemble, results for its generalizations and discussion).

Now we shall list some properties of the IDS $N(\lambda)$ which follows from (2.10) and (2.11) [8]:

(i) let α_1 and α_2 be the left and the right endpoints of the support of the measure † corresponding to the IDS N_0; then the support of $N(\lambda)$ is a subset of the interval $(\alpha_1 - S_0 - 1; \alpha_2 + S_0 + 1)$;

(ii) if there exists $v_0 > 0$ such that

$$\inf_{p, q \in [0,1]} \tilde{S}(p, q) \geq v_0 ,$$

then $N(\lambda)$ is absolutely continuous with a bounded derivative:

$$dN(\lambda) = \rho(\lambda)\, d\lambda, \quad \rho(\lambda) \leq \pi^{-1} v_0^{-1} ;$$

(iii) let H_0 be the finite-difference operator with the matrix

$$H_0(x - y) = \delta_{x+1,y} + \delta_{x-1,y}, \tag{2.18}$$

$S(x, y)$ be given by (2.15) (in this case the condition of (ii) is obviously satisfied), and α be an endpoint of the support of the corresponding $\rho(\lambda)$; then

$$\rho(\lambda) = C_0 \mid \lambda - \alpha \mid^{1/2} (1 + o(1)), \ \lambda \to \alpha, \tag{2.19}$$

where a constant C_0 depends on the properties of the Stieltjes transform of the IDS of (2.18). In fact, the asymptotics (2.19) holds for a wide class of H_0 and therefore reflects the generic situation.

It is widely known that the limiting distribution of normalized sums of dependent random variables is Gaussian, i.e. coincides with that obtained for independent variables. In contrast to this, the IDS defined by (2.10) and (2.11) in the general case differs from the semicircle distribution (1.3b) even for $\tilde{H}_0(p) \equiv 0$. Indeed, considering two first nontrivial coefficients in the expansion

$$f(z) = (-z)^{-1} + a^{(1)}(-z)^{-3} + a^{(2)}(-z)^{-5} + o(|z|^{-5}), \ z \to \infty,$$

one can derive from (2.10), (2.11), and (2.13) that

$$a^{(1)} = S(0, 0), \quad a^{(2)} = 2S^2(0, 0).$$

The latter relations contradict (except the special case of (2.14)) to those obtained from the similar expansion for $f_w(z)$ given by (1.18): $a_w^{(2)} = 2[a_w^{(1)}]^2 = 2v^4$.

† We define such a support as a set of those points λ for which $N_0(\lambda + 0) \neq N_0(\lambda - 0)$ or $N_0'(\lambda) > 0$.

3. Main relations and proof of theorem 2.1

The main goal of this section is to derive and analyse relations (3.13) and (3.16) which are analogs of relations (1.15) and (1.16). To do so, we use a version of the technique developed in [6] for the deformed Wigner ensemble. The essence of the argument is that in the last average of (1.12) the fixed entry $n^{-1/2} u_n(x, y)$ can be regarded as a vanishing perturbation for the resolvent $G^{(n)}(x, y)$. Using twice the resolvent identity (1.11), one can replace $G^{(n)}$ by $\hat{G} = (\hat{H} - z)^{-1}$, where $\hat{H} = H^{(n)}\big|_{u(x,y)=0}$. The term $\hat{G}$ does not depend on $u_n(x, y)$ and this makes possible an exact computation of the average. After this, using (1.11) again, one can pass back from $\hat{G}$ to $G^{(n)}$ in the relations obtained.

In the case of correlated entries, $\hat{G}$ strongly depends on the element $u_n(x, y)$. Thus, we have to modify the argument and we proceed with the classical Bernstein method developed for sums of dependent random variables (see, e.g., [5]). Following this method, we introduce an "approximating" ensemble of random matrices $W_1^{(n)}$ of the block form, where a fixed block workes as a vanishing perturbation of the resolvent under consideration, each block being asymptotically independent of other blocks in the limit $n \to \infty$.

With these conditions in mind, let us consider the ensemble

$$H_1^{(n)} = H_0^{(n)} + W_1^{(n)}, \tag{3.1a}$$

where $H_0^{(n)}$ is the same as in (2.1) and

$$W_1^{(n)}(x, y) = \begin{cases} W^{(n)}(x, y), & \text{if } (x, y) \in A_n \\ 0, & \text{if } (x, y) \in B_n \end{cases} \tag{3.1b}$$

for each realization of $W^{(n)}$. We define the sets of pairs A_n and B_n as follows: introduce numbers a_n and b_n such that the ratio $p_n = \frac{n}{\Delta_n}$ be integer, where $\Delta_n = 2a_n + b_n + 1$; then

$$A_n = \{ (x, y) : \exists (j, k),\ j, k = \overline{1, p_n}\ |x - j\Delta_n| \le a_n,\ |y - k\Delta_n| \le a_n \},$$

$B_n = \{ (x, y) : x, y = \overline{1, n} \} \setminus A_n$. Thus, the matrix $W_1^{(n)}$ consists of p_n^2 square arrays (blocks) each of them having $(2a_n + 1) \times (2a_n + 1)$ entries; these blocks are separated by corridors of a width b_n.

The following proposition states the approximation property of the ensemble $H_1^{(n)}$.

Proposition 1. *Let us denote* $g_n = n^{-1} \operatorname{Tr} G^{(n)} = n^{-1} \operatorname{Tr}(H^{(n)} - z)^{-1}$, $r_n = n^{-1} \operatorname{Tr} R^{(n)} = n^{-1} \operatorname{Tr}(H_1^{(n)} - z)^{-1}$. *If* $b_n = o(a_n)$, *then*

$$\lim_{n \to \infty} \mathbf{E}\{ |r_n - g_n|^2 \} = 0. \tag{3.2}$$

It follows from (3.2) that Theorem 2.1 can be proved for the ensemble (3.1) and due to Lemma 1 of Sect. 4 it will also be true for $H^{(n)}$ (2.1).

P r o o f. The resolvent identity (1.7) yields the equality†

$$r_n - g_n = n^{-1} \sum_{x,s,t=1}^{n} G(x,s)[W - W_1](s,t)R(t,x) =$$
$$= n^{-3/2} \sum_{s,t} [RG](s,t) u_n(s,t) \mathbf{I}_B(s,t),$$

where $\mathbf{I}_B$ is the characteristic function of the set B_n. This equality implies that

$$\mathbf{E}\{|r_n - g_n|^2\} \le$$
$$\le \mathbf{E}\{n^{-1}(\sum_{s,t=1}^{n} |[RG](s,t)\mathbf{I}_B(s,t)|^2)^{1/2} n^{-2}(\sum_{s,t=1}^{n} |u_n(s,t)\mathbf{I}_B(s,t)|^2)^{1/2}\} \le$$
$$\le \eta^{-4}\mathbf{E}\{n^{-2} \sum_{(s,t)\in B_n} |u_n(s,t)|^2\} \le \eta^{-4} b_n (2a_n + b_n + 1)^{-1}(V(0,0) + V_0),$$
$$\eta = \mathrm{Im}\, z. \tag{3.3}$$

When deriving these inequalities, we used the Schwartz inequality and the estimate

$$\sup_x \sum_{y=1}^{n} |G^{(n)}(x,y)|^2 \equiv \sup_x \|G^{(n)} e_x\| \le \|G^{(n)}\|^2 \le \eta^{-2}, \tag{3.4}$$

where $e_x(s) = \delta_{xs}$ is a unit vector. Proposition is proved.

In Sect. 4 we show that it is sufficient to choose

$$b_n = n^{1/6} + c_n, \quad a_n = n^{1/6} \ln n, \tag{3.5}$$

where $c = o(n^{1/6})$ is used to make p_n an integer.

D e r i v a t i o n o f m a i n r e l a t i o n s. In this subsection we deal with the resolvent $R^{(n)}$ of $H_1^{(n)}$. It is easy to derive from (1.7) the relation

$$\mathbf{E}\{R(x,y)\} = G_0(x,y) -$$
$$- n^{-1/2} \sum_{j,k=1}^{p} \sum_{|\alpha|,|\beta|\le a} \mathbf{E}\{R(x, j\Delta + \alpha) u(j\Delta + \alpha, k\Delta + \beta)\} G_0(k\Delta + \beta, y). \tag{3.6}$$

For fixed j and k, we denote $u_{jk}(\alpha,\beta) \equiv u(j\Delta + \alpha, k\Delta + \beta)$, $R_{jk}(\gamma,\delta) \equiv R(j\Delta + \gamma, k\Delta + \delta)$, $R_{xj}(\gamma) \equiv R(x, j\Delta + \gamma)$, and $R_{ky}(\delta) \equiv R(k\Delta + \delta, y)$ and

† See the footnote on page 4.

define the matrix $\widehat{R} = R|_{u_{jk} = u_{kj} = 0}$, where u_{jk} stands for the corresponding "block". Applying (1.7) to the pair R and $\widehat{R}$, we obtain †

$$R(x, y) = \widehat{R}(x, y) - n^{-1/2} \sum_{\gamma, \delta} [R_{xj}(\gamma) u_{jk}(\gamma, \delta) \widehat{R}_{ky}(\delta) + R_{xk}(\gamma) u_{kj}(\gamma, \delta) \widehat{R}_{jy}(\delta)]. \quad (3.7)$$

Using (3.7) twice, we derive for the last term of (3.6) the relation

$$\sum_{j,k} \sum_{\alpha, \beta} \mathrm{E}\{ R_{xj}(\alpha) u_{jk}(\alpha, \beta) \} G_0(k\Delta + \beta, y) =$$

$$= n^{-1} \sum_{j,k} \sum_{\alpha, \beta, \gamma, \delta} \mathrm{E}\{ \widehat{R}_{xk} \widehat{R}_{jj}(\delta, \alpha) u_{jk}(\alpha, \beta) u_{kj}(\gamma, \delta) \} G_0(k\Delta + \beta, y) +$$

$$+ \sum_{i = 1, 2, 3} \sum_{k, \beta} \Phi_i^{(n)}(x, k\Delta + \beta) G_0(k\Delta + \beta, y), \quad (3.8)$$

where

$$\Phi_1^{(n)}(x, k\Delta + \beta) = - n^{-1/2} \sum_{j, \alpha} \widehat{R}_{xj}(\alpha) u_{jk}(\alpha, \beta),$$

$$\Phi_2^{(n)}(x, k\Delta + \beta) = - n^{-1/2} \sum_{j, \alpha, \gamma, \delta} \widehat{R}_{xj}(\gamma) \widehat{R}_{kj}(\delta, \alpha) u_{jk}(\alpha, \beta) u_{jk}(\gamma, \delta),$$

$$\Phi_3^{(n)}(x, k\Delta + \beta) = n^{-1} \sum_{j, \alpha, \gamma, \delta, \mu, \nu} [R_{xj}(\mu) \widehat{R}_{kj}(\nu, \gamma) \widehat{R}_{kj}(\delta, \alpha) +$$

$$+ R_{xj}(\nu) \widehat{R}_{jj}(\mu, \gamma) \widehat{R}_{kj}(\delta, \alpha) + R_{xj}(\mu) \widehat{R}_{kk}(\nu, \delta) \widehat{R}_{jj}(\gamma, \alpha) +$$

$$+ R_{xk}(\nu) \widehat{R}_{jk}(\mu, \delta) \widehat{R}_{jj}(\gamma, \alpha)] u_{jk}(\mu, \nu) u_{jk}(\gamma, \delta) u_{jk}(\alpha, \beta).$$

Let us denote also

$$\Phi_4^{(n)}(x, k\Delta + \beta) = n^{-1} \sum_{j, \alpha, \gamma, \delta} \widehat{R}_{xk}(\gamma) \widehat{R}_{jj}(\delta, \alpha) (u_{jk}(\alpha, \beta) u_{kj}(\gamma, \delta) -$$

$$- \mathrm{E}\{ u_{jk}(\alpha, \beta) u_{kj}(\gamma, \delta) \}). \quad (3.10)$$

Using (3.8) and (2.4), we rewrite (3.7) as

$$\mathrm{E}\{ R(x, y) \} = G_0(x, y) +$$

$$+ n^{-1} \sum_{j,k} \sum_{\alpha, \beta; \gamma, \delta} \mathrm{E}\{ \widehat{R}_{xk}(\gamma) \widehat{R}_{jj}(\delta, \alpha) \} V(\alpha - \delta, \gamma - \beta) G_0(k\Delta + \beta, y) +$$

$$+ \sum_{i=1}^{5} \sum_{k, \beta} \mathrm{E}\{ \Phi_i^{(n)}(x, k\Delta + \beta) \} G_0(k\Delta + \beta, y), \quad (3.11)$$

where

$$\Phi_5^{(n)}(x, k\Delta + \beta) =$$

† Here and below we omit the limits of summation when no confusion can arise; j and k run from 1 to p_n and the Greek variables always belong to the interval $[-a_n, a_n]$.

$$= n^{-1} \sum_{j,k} \sum_{\alpha,\gamma,\delta} \hat{R}_{xk}(\gamma)\, \hat{R}_{jj}(\delta,\alpha)\, V((i-k)\Delta + a\gamma, (k-j)\Delta + \beta - \delta). \quad (3.12)$$

To obtain the main relation for E$\{ R(x,y) \}$, we pass from $\hat{R}$ to R in the right-hand side of (3.11), using again (3.8). Denoting $r_n(x,y) = R^{(n)}(x,y)$ and $\widetilde{R}^{(n)} = R^{(n)} - \mathrm{E}\{ R^{(n)} \}$, we finally obtain (cf. (1.15))

$$r_n(x,y) = G_0(x,y) +$$

$$+ n^{-1} \sum_{j,k} \sum_{\alpha,\beta,\gamma,\delta} r_n(x, k\Delta + \gamma)\, r_n(j\Delta + \delta, j\Delta + \alpha)\, V(\alpha - \delta, \gamma - \beta)\, G_0(k\Delta + \beta, y) +$$

$$+ \sum_{k,\beta} \Phi^{(n)}(x, k\Delta + \beta)\, G_0(k\Delta + \beta, y), \quad (3.13)$$

where $\Phi^{(n)}(x,y) = \sum_{i=1}^{7} \Phi_i^{(n)}(x,y)$ with

$$\Phi_6^{(n)}(x, k\Delta + \beta) =$$

$$= n^{-3/2} \sum_{j} \sum_{\alpha,\gamma,\delta,\mu,\nu} \hat{R}_{xk}(\gamma)\, [\hat{R}_{kk}(\delta,\mu)\, R_{jj}(\nu,\alpha) + \hat{R}_{kj}(\delta,\nu)\, R_{kj}(\mu,\alpha) +$$

$$+ R_{jj}(\gamma,\delta) \left(\hat{R}_{xj}(\nu)\, R_{kk}(\mu,\gamma) + \hat{R}_{kj}(\delta,\gamma)\, R_{jk}(\nu,\gamma) \right) u_{kj}(\mu,\nu)$$

and

$$\Phi_7^{(n)}(x, k\Delta + \beta) = n^{-1} \sum_{j,\alpha,\gamma,\delta} \mathrm{E}\{ R_{xk}(\gamma)\, \widetilde{R}_{jj}(\delta,\alpha) \}\, V(\alpha - \delta, \gamma - \beta). \quad (3.14)$$

Our second main relation is related to the variance of the generalized "trace" .

$$R_S(x) \equiv n^{-1} \sum_{l} \sum_{\mu,\nu} R(l\Delta + \mu, l\Delta + \nu)\, V(\mu - \nu, x), \quad x \in \mathbb{Z}. \quad (3.15)$$

Let us take E$\{ \widetilde{R}'_S(s)\, R(x,y) \}$ and apply to $R(x,y)$ the identity (3.7). Repeating the procedure described above, it is not hard to obtain the relation (cf. (1.16))

$$\mathrm{E}\{ \widetilde{R}'_S(s)\, R(x,y) \} =$$

$$= \sum_{k} \sum_{\beta,\gamma} \mathrm{E}\{ \widetilde{R}'_S(s)\, R(x, k\Delta + \gamma) \}\, \mathrm{E}\{ R_S(\gamma,\beta) \}\, G_0(k\Delta + \beta, y) +$$

$$+ \sum_{k} \sum_{\gamma,\beta} \mathrm{E}\{ R'_S(s)\, R(x, k\Delta + \gamma)\, \widetilde{R}_S(\gamma - \beta) \}\, G_0(k\Delta + \beta, y) +$$

$$+ \sum_{i=1}^{8} \mathrm{E}\{ \Psi_i^{(n)}(x, k\Delta + \beta) \}\, G_0(k\Delta + \beta, y), \quad (3.16)$$

where

$$\Psi_i^{(n)}(x,y) = \widetilde{R}'_S\, \Phi_i^{(n)}(x,y), \quad i = 1, 2, 3, 4, 5,$$

$$\Psi_6^{(n)} = -n^{-3/2}\sum_{l=1}^{p}\sum_{\mu,\nu,\sigma,\tau}[R'_{lj}(\mu,\sigma)\widehat{R}'_{kl}(\tau,\nu) + R'_{lk}(\mu,\tau)\widehat{R}'_{jl}(\sigma,\nu)]\, u_{jk}(\sigma,\tau)\times$$
$$\times V(\mu - -\nu, s)\widehat{R}(x,y),$$
$$\Psi_7^{(n)}(x, k\Delta + \beta) =$$
$$= n^{-5/2}\sum_{j,l}\sum_{\alpha,\gamma,\delta}\sum_{\mu,\nu,\sigma,\tau}[R_{lk}(\mu,\sigma)\widehat{R}_{jl}(\tau,\nu) + R_{lk}(\mu,\tau)\widehat{R}_{jl}(\sigma,\nu)]\times$$
$$\times \widehat{R}_{xk}(\gamma)\widehat{R}_{jj}(\gamma,\delta)\, V(\mu-\nu,s)\, u_{jk}(\sigma,\tau)\, u_{jk}(\alpha,\beta)\, u_{kj}(\gamma,\delta),$$
$$\Psi_8^{(n)}(x, k\Delta + \beta) =$$
$$= n^{-3/2}\sum_{l,j}\sum_{\alpha,\gamma,\delta}\sum_{\mu,\nu,\sigma,\tau}[R_{lj}(\mu,\sigma)\widehat{R}_{kl}(\tau,\nu) + R_{lk}(\mu,\tau)\widehat{R}_{jl}(\sigma,\nu)]\, u_{jk}(\sigma,\tau)\times$$
$$\times R_{xk}(\gamma)\, R_{jj}(\alpha,\delta)\, V(\mu-\nu,s)\, V(\alpha-\delta,\gamma-\beta).$$

P r o o f o f T h e o r e m 2.1. We start with proving that

$$p-\lim_{n\to\infty} n^{-1}\sum_{x=1}^{n} R^{(n)}(x,x) = \int_0^1 g(p;z)\,dp \ \ \forall\ z\in D_0,\quad \eta_0 \geq 2S_0 + 1. \tag{3.17}$$

First we show that

$$\lim_{n\to\infty} n^{-1}\sum_{x=1}^{n} r_n(x,x) - \int_0^1 g(p;z)\,dp = 0. \tag{3.18}$$

Let us consider the equation

$$f_n(x,y) = G_0^{(n)}(x,y) + n^{-1}\sum_{p,r,s,t=1}^{n} f_n(x,p) f_n(r,s)\, S(r-s, p-t)\, G_0^{(n)}(t,y),$$
$$x,\ y = \overline{1,n}, \tag{3.19}$$

which is uniquely solvable for $z\in D_0$ (see Lemma 4.4.). We prove in Lemma 4.5 that the relations

$$\lim_{n\to\infty}\sup_{x,y}|\,\mathrm{E}\{\Phi_i^{(n)}(x,y)\}\,| = 0,\ i = 1,\dots,7,\ z\in D_0, \tag{3.20}$$

imply the convergence

$$\lim_{n\to\infty} n^{-1}\sum_{x=1}^{n}[r_n(x,x) - f_n(x,x)] = 0.$$

This relation, together with Lemma 4.4, proves (3.18).

Estimates (3.20) for the cases of $i = 1, \dots, 6$ are proved in Lemma 4.7. The estimate for $\Phi_7^{(n)}$ reflects the selfaveraging property of the generalized "trace" R_S (3.15). In Lemma 4.9 we show that this estimate follows from the relation

$$\sum_{t} \mathrm{E}\{ | \widetilde{R}_S(t) |^2 \} = o(1), \ n \to \infty, \tag{3.21}$$

which is a consequence of (3.16) and the relations

$$\lim_{n \to \infty} \sup_{x, y} | \mathrm{E} \{ \Psi_i^{(n)} (x, y) \} | = 0, \ i = 1, \dots, 8. \tag{3.22}$$

It should be noted that the estimate (3.21) can be derived for R_S with an arbitrary function S satisfying (2.5). Taking S of the form given in (2.14), we obtain the relation

$$\lim_{n \to \infty} \mathrm{E}\{ | n^{-1} \operatorname{Tr} \widetilde{R} |^2 \} = 0,$$

which, together with (3.18), proves (3.17).

Now, combining (3.17) and (3.2) with Lemma 4.1, we complete the proof of items (a) and (b) of Theorem 2.1. Item (c) follows from Lemma 4.4.

4. Auxiliary facts and estimates

In Lemmas 4.1-4.4 we list facts that were proved in [8].

Lemma 4.1. *Let $\{ N_n(\lambda, \omega) \}$ be a sequence of random nondecreasing and bounded functions, and let $\{ f_n(z, \omega) \}$ be the sequence of their Stieltjes transforms, where ω is a point (realization) of the respective probability space Ω_n. Suppose that there exists a nonrandom function $f(z)$ which is analytic for* $\operatorname{Im} z \neq 0$, *satisfies (2.10) and (2.11) and such that*

$$\lim_{n \to \infty} \sup_{|\operatorname{Im} z| \geq \eta_0} \mathrm{E}\{ | f_n(z) - f(z) |^2 \} = 0$$

for some $\eta_0 > 0$. If $N(\lambda)$ is a nondecreasing function which corresponds to $f(z)$ and satisfies conditions $N(-\infty) = 0$, $N(+\infty) < \infty$, then at each continuity point of $N(\lambda)$,

$$p - \lim_{n \to \infty} N_n(\lambda, \omega) = N(\lambda),$$

or, in other words, the measures $N_n(d\lambda, \omega)$ corresponding to $N_n(\lambda, \omega)$ converge (weakly) in probability to $N(d\lambda)$.

Lemma 4.2. *Let $H_0^{(n)}$ be a nonrandom matrix with entries $[H_0^{(n)}](x,y) = H_0(x, y)$, where H_0 satisfies (2.2) and $G_0^{(n)} = (H_0^{(n)} - z)^{-1}$. Then,*

$$\sup_{y} \sum_{x} | G_0^{(n)} (x, y; z) | \leq 2\eta_0^{-1} \ \text{for} \ z \in D_0. \tag{4.1}$$

Lemma 4.3. *Let us consider $n \times n$ matrices K and L satisfying the conditions*

$$\| K \| \leq K_0, \ \| L \| \leq L_0, \ \sup_{x, y} | K(x, y) | \leq K_1, \ \textit{and} \ \sup_{x} \sum_{y} | L(x, y) | \leq L_1$$

with the norm $\|\vec{e}\|^2 = \sum_{x=1}^{n} | e(x) |^2$ *and define the matrices*

$$A(V)(x, y) = n^{-1} \sum_{p, r, s, t = 1}^{n} V(x, p) K(r, s) S(r - s, p - t) L(t, y),$$

$$B(V)(x, y) = n^{-1} \sum_{p, r, s, t = 1}^{n} K(x, p) V(r, s) S(r - s, p - t) L(t, y),$$

and

$$C = \sum_{r, s = 1}^{n} S(x - s, y - r) K(s, r).$$

Then,

$$\|A\| \le K_1 L_0 S_0 \|V\|, \quad \sup_y \sum_x | A(x, y) | \le K_1 L_1 S_0 \sup_y \sum_x | V(x, y) |, \tag{4.2}$$

$$\|B\| \le K_0 L_0 S_0, \quad \sup_{x, y} | B(x, y) | \le K_0 L_0 S_0 \sup_{x, y} | V(x, y) |, \tag{4.3}$$

and

$$\|C\| \le V_0 K_0. \tag{4.4}$$

Lemma 4.4. *Equations* (3.19) *and the equation*

$$g(x) = G_0(x) + \sum_{p, r, t \in \mathbf{Z}} g(x - p) g(r) S(r, p - t) G_0(t), \quad x \in \mathbf{Z}, \tag{4.5}$$

where $G_0(x) = \int (\widetilde{H}_0(p) - z)^{-1} \exp\{-2\pi i px\} dp$, *have unique solutions for* $z \in D_0$. *The estimate*

$$\sup_x \sum_y | f^{(n)}(x, y; z) | \le 2\eta_0^{-1} \tag{4.6}$$

holds for each n and

$$\sup_{z \in D_0} \lim_{n \to \infty} \left| n^{-1} \sum_{x=1}^{n} f^{(n)}(x, x) - g(0) \right| = 0. \tag{4.7}$$

Equation (4.5) *is equavalent to* (2.11)-(2.12) *with* $\widetilde{g}(p; z) = \sum_x g(x) \exp\{2\pi i xp\}$.

Lemma 4.5. *Let us consider relation* (3.13). *If estimates* (3.20) *hold, then,*

$$\lim_{n \to \infty} \left| n^{-1} \sum_{x=1}^{n} [r^{(n)}(x, x) - f^{(n)}(x, x)] \right| = 0. \tag{4.8}$$

P r o o f. Denoting $d_n(x, y) = r_n(x, y) - f_n(x, y)$ and subtracting (3.19) from (3.13), we obtain the following matrix relation

$$d_n = d_n PS_{PrP} PG_0 + f_n PS_{PdP} PG_0 + (\Phi + \Xi_1 + \Xi_2) G_0, \tag{4.9}$$

where $P = \sum_{k=1}^{n} P_k$ with orthoprojectors P_k defined by the formula

$$[P_{k+1} \vec{e}\,](x) = \begin{cases} e(x), & \text{if } x = k\Delta + \alpha, \ |\alpha| \le a; \\ 0, & \text{if } |x - k\Delta| > a, \end{cases}$$

and

$$S_T(x, y) \equiv n^{-1} \sum_{p,r} T(p, r)\, S(p - r, x - y),$$

$$\Xi_1(x, y) = \sum_p f(x, p)[S_f - PS_f P](p, y), \quad \Xi_2(x, y) = \sum_p f(x, p)[PS_{(f - PfP)} P](p, y).$$

Since $|r_n(x, y)| \le 1/\eta$, it follows from (4.1) and (4.2) that $\| PS_{PrP} PG_0 \| \le 2S_0/\eta_0$, $z \in D_0$. This inequality and (4.6) imply the following estimates for the matrix $A' = (H_0 - z - PS_{PrP} P)^{-1}$:

$$\| A' \| \le \eta_0^{-1} (\eta_0^2 - S_0)^{-1} = \alpha_1 < 1, \ \sup_y \sum_x | A'(x, y) | \le 4S_0 \eta_0^{-3}, \ z \in D_0. \tag{4.10}$$

Rewrite (4.12) as

$$d_n = f_n PS_{PdP} PA' + (\Phi + \Xi_1 + \Xi_2) A'.$$

It is easy to check that the mapping B defined as in Lemma 4.3 with $K = fP$ and $L = PA'$ satisfies the inequality $\| B \| \le \alpha_2 < 1$ for $z \in D_0$. Thus, to prove (4.8), it suffices to estimate the term $n^{-1} \operatorname{Tr} B^k (|\Phi + \Xi_1 + \Xi_2|)$. It follows from (4.3) that

$$\sup_{x,y} | B(T)(x, y) | \le \alpha_1 \eta^{-1} \sup_{x,y} | T(x, y) |.$$

This relation together with (4.10) shows that estimates (3.20) and

$$\sup_{x,y} | \Xi_i^{(n)}(x, y) | = o(1), \ n \to \infty, \ i = 1, 2 \tag{4.11}$$

imply (4.8). We prove (3.20) for $i = 1, \dots, 6$ in Lemma 4.8 and for $i = 7$ in Lemma 4.6. Estimates (4.11) follow from Lemma 4.7.

Lemma 4.6. *Let estimates* (3. 22) *hold. This implies that relations* (3.21) *and* (3. 20) *for the case of* $i = 7$ *are true.*

P r o o f. Let us rewrite relation (3. 16) as

$$\sum_{t \in \mathbb{Z}} \mathrm{E}\{ \tilde{R}'_S(t) R_S(t) \} = n^{-1} \sum_{x,y=1}^{n} \mathrm{E}\{ \tilde{R}'_S(t) [RP\tilde{R}_S PA'](x, y) \} S(x - y, t) +$$

$$+ \sum_{i=1}^{8} \mathrm{E}\{ [\Psi_i^{(n)} PG_0 A'](x, y) \} S(x - y, t);$$

where we use the notations of Lemma 4. 5. The first term of the right-hand side of this relation can be estimated by the expression

$$\mathrm{E}\{\sum_t |\tilde{R}'_S(t)|^2\}^{1/2}\,\mathrm{E}\{\sum_t [\sup_{x,y} |RP\tilde{R}_S PA'|]^2 [n^{-1}\sum_{x,y} |S(x-y,t)|]^2\}^{1/2} \le$$
$$\le S_0^2\eta_0^{-2}(\eta_0^2 - S_0)^{-1}\sum_t \mathrm{E}\{|\tilde{R}_S|^2\}.$$

Thus, taking into account the relation $R' = \overline{R}$ and inequalities (4. 10), we obtain

$$\sum_t \mathrm{E}\{|\tilde{R}_S(t)|^2\} \le \sum_{i=1}^{8} n^{-1}\sum_t\sum_{x,y} |[\Psi_i^{(n)}A'](x,y)S(x-y,t)| \le$$
$$\le S_0\eta_0^{-1}\sum_{i=1}^{8}\sup_{x,y}|\Psi_i^{(n)}(x,y)|,$$

for $z \in D_0$ with $\eta_0 > 2S_0 + 1$.

Now let us show that (3. 21) implies (3. 20) for $i = 7$. To do that it sufficies to observe that

$$\sup_{x,y}|\Phi_7^{(n)}(x,y)| \le$$
$$\le \sup_{x,y}\sup_{k=1,p_n}\Big[\mathrm{E}\{\sum_t |[RP_k](x,t)|^2\}\Big]^{1/2}\Big[\mathrm{E}\{\sum_t |\tilde{R}_S(t-y)|^2\}\Big]^{1/2}.$$

Lemma is proved.

Lemma 4. 7. *Let $n \times n$ matrices K and L be such as in Lemma 4. 3. Then,*

$$\lim_{n\to\infty} n^{-1}\sum_{x,y}[K - PKP](x,y)L(y,x) = 0.$$

P r o o f. It is easy to see that

$$\Big| n^{-1}\sum_{|x-y|>m}[K-PKP](x,y)L(y,x)\Big| \le 2K_1 n^{-1}\sum_{x=1}^{n}\sum_{y=0}^{\infty}L(|m|+|y|).$$

Observing that in the region $|x-y| \le m$ the matrix $K - PKP$ has no more than $4b(2m+1)n/\Delta$ nonzero entries and taking $m = \log n$, one can easily obtain (4. 11).

Lemma 4. 8. *Estimates* (3. 20) *are true.*

P r o o f. Using (2.6) and the obvious inequality $|\hat{R}(x,y)| \le \eta^{-1}$, we derive the following estimate for $\mathrm{E}\{\Phi_1^{(n)}\}$:

$$n^{-1/2}\Big|\sum_{j=1}^{p}\sum_{|\alpha|\le a}\mathrm{E}\{\hat{R}_{xj}(\alpha)\,u_{jk}(\alpha,\beta)\}\Big| \le n^{-1/2}\eta^{-1}v^2\psi_n(b)2ap.$$

Taking into account (3. 5) and (2. 8), we obtain (3. 20) for $i = 1$.

Now let us consider $E\{\Phi_3^{(n)}\}$. It contains four terms which can be estimated by the same expression. Thus, we have

$$| E\{\Phi_3^{(n)}(x, k\Delta+\beta)\} | \le$$

$$\le 4n^{-3/2}\sum_{j=1}^{n}\sum_{\alpha,\gamma,\delta,\mu,\nu}[E\{ | R_{xl}(\mu)\hat{R}_{ms}(\nu,\gamma)\hat{R}_{tj}(\delta,\alpha)|^2\}]^{1/2}\cdot[E\{|u_n(x,y)|^6\}]^{1/2}, \tag{4.12}$$

where l, m, s, t are equal to j or k. It follows from (3.4) that

$$(2a+1)^{-1/2}\sum_{\mu}[E\{|R_{xl}(\mu)|^2\}]^{1/2}\le\eta^{-1},$$

$$(2a+1)^{-3/2}\sum_{\gamma,\nu}[E\{|\hat{R}_{ms}(\nu,\gamma)|^2\}]^{1/2}\le\eta^{-1}. \tag{4.13}$$

Hence, the r.h.s of (4. 12) can be estimated by the value $C(z)\,S_6^{1/2}\,n^{-3/2}\,p(2a+1)^{7/2}$ that vanishes due to (3.5)

This estimate with S_6 changed by $\sup_n\sup_{x,y} E\{|u_n(x,y)|^2\}$ is valid also for $E\{\Phi_6^{(n)}\}$.

Applying (2. 8) to $E\{\Phi_2^{(n)}\}$ and using (2. 4), we obtain that

$$E\{\Phi_2^{(n)}(x,k\Delta+\beta)\} = n^{-1}\sum_{j=1}^{n}\sum_{\alpha,\gamma,\delta}E\{\hat{R}_{xj}(\gamma)\hat{R}_{kj}(\delta,\alpha)\}\times$$
$$\times[S(\gamma-\alpha,\delta-\beta)+S((j-k)\Delta+\gamma-\beta,(k-j)\Delta+\delta-\alpha)]+\Xi_{xk}(\beta), \tag{4.14}$$

where the last term vanishes as $n\to\infty$:

$$S_4^{-1/2}|\Xi_{xk}(\beta)|\le$$

$$\le n^{-1}\sum_{j=1}^{n}\sum_{\alpha,\gamma,\delta}[E\{|\hat{R}_{xj}(\gamma)\hat{R}_{kj}(\delta,\alpha)|^2\}]^{1/2}\psi_n(b)\le\eta^{-2}(2a+1)^2\psi_n(b).$$

Here we denote $S_4=\sup_n\sup_{x,y}E\{|u_n(x,y)|^4\}$ and use inequalities (4. 13).

Let us note that the proof of vanishing of $E\{\Phi_4^{(n)}\}$ is literally the same as for $\Xi_{xk}(\beta)$.

It follows from Lemma 4. 3 that $\|\sum_{\alpha,\delta}\hat{R}_{kj}(\delta,\alpha)\,S(\cdot-\alpha,\delta-\cdot)\|\le S_0\eta^{-1}$. Since the vector $\hat{R}_{xj}(\cdot)$ has the norm bounded by η^{-1}, then the corresponding term of (4. 14) can be estimated by $S_0\eta^{-2}p/n$.

Due to the property (2. 5), one can write the inequality

$$\sum_{j=1}^{n}\sum_{\alpha,\gamma,\delta}|S(j\Delta+\gamma-\beta,j\Delta+\delta-\alpha)|\le(2a+1)^3n^{-1}S_0, \tag{4.15}$$

which completes the proof of (3. 20) for the case of $i=2$.

Note that (4. 15) also provides the estimate for $E\{\Phi_5^{(n)}\}$. Lemma is proved.

Lemma 4.9. *Estimates* (3. 22) *are true.*

P r o o f. All the estimates are based on (4. 13) and mostly repeat the calculations used in the proof of Lemma 4. 8. Thus, we explain here how to treat the terms that were not present in Lemma 4. 8.

For the cases of $i = 2, \dots, 5$, estimates (3. 22) follow immediately from (3. 20) for $E\{\Phi_i^{(n)}\}$, $i = 2, \dots, 5$. Since we use (2. 6) in the proof of (3. 20) for $\Phi_1^{(n)}$ and $\Phi_6^{(n)}$, then to estimate terms $\Psi_1^{(n)}$ and $\Psi_6^{(n)}$, we have to pass from $R'_S(x, y)$ to

$$H_0^{(n)} R'_S(x, y) = n^{-1} \sum_{i=1}^{n} \sum_{\sigma,\tau} \hat{R}'(i\Delta + \sigma, i\Delta + \tau)\, S(\sigma - \tau, x - y).$$

The following inequality proves vanishing, as $n \to \infty$, of the additional terms that arise when this transition is done:

$$n^{-2} \Big| \sum_{i,j=1}^{p} \sum_{\alpha,\mu,\nu,\sigma,\tau} E\{ R'_{il}(\mu, \sigma)\, u_{lm}(\sigma, \tau)\, \hat{R}_{mi}(\tau, \nu)\, S(\mu - \nu, s)\, \hat{R}_{xj}(\alpha)\, u_{jk}(\alpha, \beta)\} \Big| \le$$

$$\le S_4^{1/2} n^{-2} \sum_{j=1}^{n} \sum_{\alpha,\mu,\nu,\sigma,\tau} |E\{ \sum_i | R'_{il}(\mu, \sigma)|^2 \times$$

$$\times \sum_i | \hat{R}'_{mi}(\tau, \nu)|^2 \, | \hat{R}_{xj}(\alpha)|^2 \}]^{1/2} \, | S(\mu - \nu, s)| \le$$

$$\le S_4^{1/2} \eta^{-3} S_0 (2a + 1)^{7/2} p n^{-2}.$$

Combining this inequality with (4.13), one can also estimate the averages $E\{\Phi_i^{(n)}\}$, $i = 7, 8$ by $\eta^{-4} S_0 S_6^{1/2} (2a + 1)^6 p n^{-5/2}$ and $\eta^{-4} S_0^2 (2a + 1)^3 p n^{-3/2}$, respectively. Lemma is proved.

5. Other ensembles of matrices with correlated entries

To study the eigenvalue distribution of the matrices $H^{(n)}$ (2.1), we introduced in Sect. 3 the "approximating" ensemble of random matrices $H_1^{(n)}$ having a block form. We construct $H_1^{(n)}$ in a special way so that in the limit $n \to \infty$ (more precisely, in the limit of infinite a_n and b_n (3. 5)) these blocks become, roughly speaking, independent. In other words, the dependence between the entries of $H_1^{(n)}$ in this limit concentrates inside each block.

In this section we discuss ensembles of block random matrices that represent dependence between entries of a type somewhat different from the above type. One of such ensembles was introduced and studied at the theoretical physics level of rigour by F. Wegner [13]. Corresponding infinite matrices $H^{(a)}$ have the entries

$$H_{\alpha\beta}^{(a)}(x, y) = H_0(x - y) + a^{-1/2} V_{\alpha\beta}(x, y), \quad x, y \in \mathbf{Z}, \quad \alpha, \beta = \overline{1, a}, \qquad (5.1)$$

where H_0 is the same as in Sect. 2 and $V_{\alpha\beta}(x, y)$ are random variables with zero mean value and the covariance matrix

$$\mathbf{E}\{ V_{\alpha\beta}(x, y)\, V_{\mu\nu}(s, t) \} = T(x, y, s, t)(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}) \tag{5.2}$$

satisfying the symmetry condition

$$T(x, y, s, t) = T(y, x, s, t) = T(y, x, t, s) = T(s, t, x, y) \tag{5.3}$$

and

$$T(x + k, y + k, s + k, t + k) = T(x, y, s, t) \text{ for all } k \in \mathbf{Z}. \tag{5.4}$$

Thus, $H^{(a)}$ can be regarded as an infinite symmetric matrix $\widetilde{H}^{(a)}$ consisting of $a \times a$ squares (or blocks) labelled by pairs (x, y).

Let us consider the case of the Gaussian $V_{\alpha\beta}(x, y)$'s. Then (5. 2) implies that all entries inside a block labelled by the pair (x, y) are jointly independent, while the entries with the coinciding Greek subscripts but with different (x, y)'s have nonzero correlation. This means that random fields corresponding to $\widetilde{H}^{(a)}$ do not satisfy the condition (2. 17), no matter what function $\psi(x)$ may be chosen. Thus, the ensemble $H^{(a)}$ (5. 1)-(5. 2) can be regarded as giving an example of matrices with asymptotically non-local dependence between entries. In this connection, the following result seems somewhat unexpected.

It was shown in [13] (see [7] for the rigorous results concerning this ensemble) that if the random variables $V_{\alpha\beta}(x, y)$ have all their moments finite and

$$\sum_{y, s, t} | T(0, y, s, t) | \le T_0 < \infty \tag{5.5}$$

then the NCF $N(\lambda, H^{(a)})$ converges, as $a \to \infty$, to a nonrandom function whose Stieltjes transform can be found from relations (2.10)-(2.12) with

$$\Gamma(p, q) = \sum_{y, s, t} T(0, y, s, t) \exp\{ 2\pi i [(y - s)q + tp] \}. \tag{5.6}$$

Thus, the ensembles (2. 1) and (5. 1), while being rather differently defined, have the same equation for the Stieltjes transform of the IDS.

In order to understand how the local and non-local dependence enters into the limiting equation, it is natural to consider the ensemble with the mixed type of dependence which can be called the Wegner-Wigner ensemble. Using the method described in sections 1 and 2, one can prove the following statement.

Theorem 5.1. *Consider the ensemble* (5.1) *with* $H_0 = 0$. *Let the Gaussian random variables* $V_{\alpha\beta}(x, y)$ *have zero averages and*

$$\mathbf{E}\{ V_{\alpha\beta}(x, y)\, V_{\mu\nu}(s, t) \} = T(x, y, s, t)\, [S(\alpha - \mu, \beta - \nu) + S(\alpha - \nu, \beta - \mu)],$$

where T satisfies (5.3) *and* (5.4) *and S is the same as in Theorem 2.1. Then there exists the limiting IDS of* $H^{(a)}$, $a \to \infty$ *whose Stieltjes transform* $f(z)$ *can be found as*

$$f(z) = \int\!\!\int_{[0,1]} \widetilde{g}(p, q; z)\, dp\, dq, \tag{5.7}$$

where $\widetilde{g}(p, q; z)$ *is the unique solution of the equation*

$$\widetilde{g}(p, q; z) = \left[- z - \int\!\!\int_{[0,1]} \Gamma_1(p, p_1)\, \Gamma_2(q, q_1)\, \widetilde{g}(p_1, q_1)\, dp_1\, dq_1 \right]^{-1} \tag{5.8}$$

with Γ_j, $j = 1, 2$ *given by* (2. 13) *and* (5.5), *respectively.*

It is worth noting that the condition (5. 4) for the covariance matrix implies that the joint distribution of the Gaussian random field $V_{\alpha\beta}(x, y)$ is invariant with respect to one-parameter group of translations. More precisely, for every fixed a, α, β, there exists a one-parameter group of measure preserving automorphisms Π_k, $k \in \mathbf{Z}$ defined on the probability space (Ω, F, μ) such that

$$V_{\alpha\beta}(x, y; \Pi_k \omega) = V_{\alpha\beta}(x + k, y + k; \omega), \quad \omega \in \Omega.$$

This fact combined with (5. 5) implies that for each fixed a, the random operator $H^{(a)}$ belongs to the class of metrically transitive operators (see [12] for the definitions and the axiomatic theory of selfadjoint metrically transitive operators).

On the other hand, the joint distribution of the Gaussian random field satisfying (2. 3) and (2. 16) is invariant with respect to the two-parameter semi-group of translations. This could be explained by the fact that the matrices we have considered represent a generalization of the Wigner ensemble whose matrices have the entries of the equal "order" (1. 1). That is why to obtain finite and nontrivial answers for the IDS's, the factor $n^{-1/2}$ is to be placed in front of the matrices U_n in (1.2) and (2.1). Taking into account this reasoning, the ensemble (2.1)-(2.4) can be regarded as the Wigner matrix analogue of the class of metrically transitive operators.

In conclusion, we present the similar to Theorem 5. 1 statement on the limiting IDS of an ensemble which can be called the correlated Wegner-Wigner ensemble.

Theorem 5.2. *Consider the ensemble of random matrices* $H^{(a, n)}$ *with entier*

$$H^{(a, n)}_{\alpha\beta}(x, y) = (an)^{-1/2} V_{\alpha\beta}(x, y), \quad x, \ y = \overline{1, n}, \ \alpha, \ \beta = \overline{1, a}.$$

Let the Gaussian random variables $V_{\alpha\beta}(x, y)$ *have zero averages and*

$$\mathrm{E}\{ V_{\alpha\beta}(x, y) V_{\mu\nu}(s, t) \} =$$

$$= [S_1(x - s, y - t) + S_1(x - t, y - s)] [S_2(\alpha - \mu, \beta - \nu) + S_2(\alpha - \nu, \beta - \mu)],$$

where S_j, $j = 1, 2$ *are the same as S in Theorem 2.1. Then there exists the limiting IDS of* $H^{(a, n)}$, $a, n \to \infty$ *whose Stieltjes transform* $f(z)$ *can be found from relations* (5.7) *and* (5.8) *with* Γ_j, $j = 1, 2$ *defined by* (2.13) *for* S_j, $j = 1, 2$ *respectively.*

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Распределение собственных значений случайных матриц большой размерности с коррелированными элементами

А. Хорунжий

Исследуется нормированная функция распределения собственных значений $N_n(\lambda)$ ансамбля $n \times n$ симметрических случайных матриц, элементы которых $u_n(x, y)$, $x, y = 1, \dots, n$ есть статистически зависимые произвольно распределенные случайные величины. Доказано, что если корреляционная функция элементов S одинакова для всех n и коэффициент корреляции случайного поля $\{ u_n(x, y) \}$ убывает достаточно быстро, то мера $N_n(d\lambda)$ при $n \to \infty$ слабо сходится по вероятности к неслучайной мере $N(d\lambda)$. Для преобразования Стильтьеса предельной $N(d\lambda)$ выводим уравнение, которое зависит только от предельной матрицы математических ожиданий $u_n(x, y)$ и корреляционной функции S.

Розподіл властивих значень багатовимірних випадкових матриць з коррельованими елементами

О. Хорунжій

Вивчається нормована функція розподілу властивих значень $N_n(\lambda)$ ансамблю $n \times n$ симетричних випадкових матриць з елементами $u_n(x, y)$, $x, y = 1, \dots, n$, що є статистично залежні та мають довільний розподіл. Доведено, що у разі, коли корреляційна функція S елементів є єдина для кожного n та якщо коефіцієнт корреляції випадкового поля $\{ u_n(x, y) \}$ спадає досить швидко, міра $N_n(d\lambda)$ слабко збігається за ймовірністю при $n \to \infty$ до невипадкової міри $N(d\lambda)$. Ми виводимо рівняння для перетворення Стільтьеса граничної $N(d\lambda)$ та доводимо, що вона залежить тільки від граничної матриці очікувань елементів $u_n(x, y)$ та корреляційної функції S.