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On entire functions of n variables being quasipolynomials in one the variables[1)]

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A general form is found for entire functions $f(z_1, 'z)$, $z_1 \in \mathbf{C}$, $'z \in \mathbf{C}^{n-1}$, of a finite order ρ that are M-quasipolynomials in z_1 for every $'z$ from a non-pluripolar set $E \in \mathbf{C}^{n-1}$, i.e. $f(z_1, 'z) = \sum_{j=1}^{m} a_j(z_1) e^{\lambda_j z_1}$, $'z \in E$. Here m, λ_j and $a_j(z_1)$ depend on $'z$ *a priori* arbitrarily and $a_j(z_1)$ belong to the class M of entire functions of the type 0 with respect to the order 1.

An entire function is called a C-quasipolynomial or a quasipolynomial with constant coefficients[2)] of $w \in \mathbf{C}$ if it is of the form

$$f(w) = \sum_{j=1}^{\omega} a_j e^{\lambda_j w}, \tag{1}$$

where $\omega < \infty$, a_j, λ_j are constants and $a_j \neq 0$, $\forall j$, $\lambda_j \neq \lambda_i$, $\forall j \neq i$. The numbers a_j are called the coefficients of the quasipolynomial f, and the set Λ of all exponents $\lambda_1, \dots, \lambda_\omega$ is referred to as spectrum.

A P-quasipolynomial or a quasipolynomial with polynomial coefficients[3)] of $w \in \mathbf{C}$ is defined as an entire function of the form

$$f(w) = \sum_{j=1}^{\omega} a_j(w) e^{\lambda_j w}, \tag{2}$$

where, as in the case of a C-quasipolynomial, $\lambda_j \in \mathbf{C}$, $\lambda_j \neq \lambda_i$, $\forall j \neq i$ and $a_j(w) \not\equiv 0$ are polynomials. Similarly we define M-quasipolynomials as quasipolynomials whose coefficients are entire functions of degree zero.[4)]

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2) C-quasipolynomials are called also exponential sums.

3) P-quasipolynomials are called also exponential quasipolynomials.

4) An entire function $f(z)$, $z \in \mathbf{C}^n$, is called an entire function of degree zero, if $\overline{\lim_{z \to \infty}} \frac{\ln |f(z)|}{|z|} = 0$.

The value is called the degree of a P-quasipolynomial

$$\deg f = \sum_{j=1}^{\omega} (1 + \deg a_j),$$

where $\deg a_j$ is the degree of the polynomial a_j from (2).

Set

$$I_m(w; f) = \begin{vmatrix} f & f' & \dots & f^{(m)} \\ f' & f'' & \dots & f^{(m+1)} \\ \dots & \dots & \dots & \dots \\ f^{(m)} & \dots & \dots & f^{(2m)} \end{vmatrix}.$$

It is known (see, for example, [1]) that an entire function $f(z)$ is a P-quasipolynomial of degree N if and only if $I_N(w; f) \equiv 0$ and $I_{N-1}(w; f) \not\equiv 0$.

The common form of a function of n variables being P-quasipolynomial of C-quasipolynomial in every variable was found in [2, 3]. In [4] the entire functions $F(z)$, $z \in \mathbb{C}^n$, of order $\rho_F = 1$,[1)] that are M-quasipolynomials in z_1 for fixed $'z = (z_2, \dots, z_n) \in E$, were considered where E is a nonpluripolar set. It was established that every such function is of the form

$$F(z) = \sum_{j=1}^{\omega} a_j(z_1, 'z)\, e^{\lambda_j z_1}, \tag{3}$$

where ω and λ_j, $j = 1, \dots, \omega$, are independent of $'z$, and the coefficients $a_j(z_1, 'z)$ are entire functions in $\mathbb{C}^n$ of degree zero with respect to z_1. In [4] also an example was given showing that the representation (3) does not take place without the assumption $\rho_F = 1$.[2)]

In this article (Theorems 1, 1′, 1″, and 2) the problem of the common form of a function $f(z)$ being a quasipolynomial in z_1 with restriction $\rho_f < \infty$ is solved. It turns out that the above-mentioned example is in some sense universal.

Theorem 1. *Let E be a nonpluripolar set in $\mathbb{C}^{n-1}$ and $f(z)$, $z \in \mathbb{C}^n$, be an entire function of finite order $\rho_f = \rho < \infty$. Let also f be an M-quasipolynomial of z_1 for any fixed $'z = (z_2, \dots, z_n) \in E$ (with the number of terms, coefficients and exponents in general dependent on $'z$). Then $f(z)$ can be represented in the form*

$$f(z) = \sum_{j=1}^{\omega} a_j(z_1, 'z)\, e^{\lambda_j('z) z_1}, \quad z \in \mathbb{C}^n, \tag{4}$$

where:

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1) Recall that the order ρ_F is defined by the equality $\rho_F = \overline{\lim\limits_{z \to \infty}} \dfrac{\ln \ln |F(z)|}{\ln |z|}$.

2) For $\rho_F = 3/2$ and $n = 2$ the function $\cos(z_1 \sqrt{z_2})$ can be cited as an example.

a) $\omega < \infty$ *is independent of* $'z$;
b) $\lambda_1('z), \ldots, \lambda_\omega('z)$ *are arbitrarily numerated zeros of the pseudopolynomial*

$$h(z_1, 'z) = z_1^\omega + h_1('z)\, z_1^{\omega-1} + \ldots + h_\omega('z)$$

whose coefficients h_j *are polynomials in* $'z$ *of degree* $\le j(\rho - 1)$ *and whose discriminant* $D_h('z) \not\equiv 0$;
c) *the coefficients* $a_j(z_1, 'z)$ *are entire functions of degree zero in* z_1 *and are local holomorphic functions in* $'z$ *in* $\Omega_h = \{ 'z : D_h('z) \ne 0 \}$ *when the exponents* $\lambda_j('z)$ *are properly numerated.*[1)]

P r o o f. In accordance with the condition of the Theorem, for any fixed $'z \in E$ the function f is of the form

$$f(z_1, 'z) = \sum_{j=1}^{\omega('z)} b_j(z_1, 'z)\, e^{\mu_j('z) z_1}, \tag{5}$$

where $b_j(z_1, 'z)$ are entire functions of degree zero with respect to z_1. Denote by $\sigma_f('z)$ the type of the function $f(z_1, 'z)$ of order 1 with respect to z_1.[2)] Since the functions $b_j(z_1, 'z)$ are of degree zero in z_1,

$$\sigma_f('z) = \max_{1 \le j \le \omega('z)} | \mu_j('z) |, \ \forall\ 'z \in E,$$

and hence in this situation $\sigma_f('z) < \infty, \forall\, 'z \in E$. Taking into account that E is nonpluripolar and that f is of finite order, we conclude (see [5, 6]) that $\sigma_f('z) < \infty, \forall\, 'z \in \mathbb{C}^{n-1}$, and, moreover, there exist such constants $\kappa_1 > 0$ and $\kappa_2 > 0$ that

$$\sigma_f('z) \le \kappa_1\, |\, 'z\, |^{\rho - 1} + \kappa_2, \ \forall\ 'z \in \mathbb{C}^{n-1}. \tag{6}$$

Now we consider a function $F(z)$ Borel associated to the function f with respect to z_1 (see, for example [5, 7]). This function is constructed from f as follows:

$$F(z) = \sum_{m=0}^{\infty} \frac{1}{z_1^{m+1}} \frac{\partial^m f}{\partial z_1^m} \Big|_{z_1 = 0}. \tag{7}$$

In this case, e.g. when $\sigma_f('z) \le \kappa_1\, |\, 'z\, |^{\rho-1} + \kappa_2$ the series in (7) converges uniformly on every compact set in $G_f = \{ z = (z_1, 'z) : |\, z_1 \,| > \kappa_1 |\, 'z\, |^{\rho - 1} + \kappa_2 \}$. Therefore $F(z)$ is holomorphic in G_f. Furthermore, it follows from (5) that for any fixed $'z \in E$

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1) In a small enough neighbourhood of every point $'z^0 \in \Omega_h$ the zeros of pseudopolynomial h can be numerated so that the corresponding functions $\lambda_j('z)$ are holomorphic.

2) Recall that $\sigma_f('z) = \overline{\lim_{z_1 \to \infty}} \dfrac{\ln |f(z_1, 'z)|}{|z_1|}, 'z \in \mathbb{C}^{n-1}$.

$$F(z_1, 'z) = \sum_{j=1}^{\omega('z)} B_j(z_1 - \mu_j('z), 'z),$$

where $B_j(z_1, 'z)$ is a function Borel associated to $b_j(z_1, 'z)$ with respect to z_1, $j = 1, \dots, \omega('z)$. Since $b_j(z_1, 'z)$ is of degree zero in z_1, $B_j(z_1, 'z)$ is holomorphic in z_1 on $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Hence $F(z_1, 'z)$ can be holomorphically extended as a function of z_1 from G_f to the whole C except a finite set $\Lambda('z)$ of points $\mu_1('z), \dots, \mu_{\omega('z)}('z)$. If follows (see [8], also [9, 10]) that $F(z)$ can be holomorphically extended as a function of $z_1, \dots, z_n$ to $\Omega = \mathbf{C}^n \setminus \chi$, where χ is an analytic set in $\mathbf{C}^n$. Since analytic sets of dimension $\le n - 2$ are sets of removable singularity, it can be assumed without loss of generality that χ is a set of pure dimension $n - 1$ and therefore there exists such an entire function $\Phi(z)$ that $\chi = \{ z \in \mathbf{C}^n : \Phi(z) = 0 \}$ and the multiplicity of zero of $\Phi(z)$ is equal to 1 at every regular point of χ. It is obvious that the set

$$\chi('z) = \{ z_1 : (z_1, 'z) \in \chi \}$$

consists of a finite number of points for any $'z \in E$. It follows (see [11-13]) that $\Phi(z) = e^{g(z)} h(z)$, where $g(z)$ is an entire function in $\mathbf{C}^n$ and

$$h(z) = h_0('z) z_1^{\omega} + \dots + h_{\omega}('z)$$

is a pseudopolynomial whose coefficients $h_j('z)$ are entire functions in $\mathbf{C}^{n-1}$. In view of the above assumption on the multiplicity of zeros of $\Phi(z)$ the discriminant of the pseudopolynomial $h(z)$ is not identically zero. Furthermore, (16) implies the boundedness of $\chi('z)$ and therefore without loss of generality we can assume $h_0('z) = 1$. Denote by $\lambda_1('z), \dots, \lambda_{\omega}('z)$ the zeros of the pseudopolynomial. Their numeration is arbitrary. It is clear that $|\lambda_j('z)| \le \kappa_1 |'z|^{\rho - 1} + \kappa_2$. Therefore $|h_j('z)| \le \text{const} \cdot (\kappa_1 |'z|^{\rho-1} + \kappa_2)^j$ and hence $h_j('z)$ is a polynomial of degree $\deg h_j \le j(\rho - 1)$, $j = 1, \dots, \omega$. Respectively $h(z)$ is a polynomial in z of degree

$$\deg h \le \max_{1 \le j \le \omega} \{ \omega + j(\rho - 2) \} = \max \{ \omega, \omega(\rho - 1) \}.$$

Now let us return to the initial function $f(z)$. Taking into account the above established properties of $F(z)$ and the known (see, for example, [5, 7]) correlation between entire and associated functions, we conclude that the representation (4) takes place, where the coefficients are entire functions of z_1 of degree zero. Now let us consider a point $'z^0$ not belonging to the discriminant set of the pseudopolynomial. As it follows from the known properties of pseudopolynomial, for any small enough $\delta > 0$ there is an $\varepsilon > 0$ such that $B_{\varepsilon}('z^0) \subset \Omega_h = \{ 'z : D_h('z) \ne 0 \}$, and by the proper numeration of zeros $\lambda_1, \dots, \lambda_{\omega}$, the corresponding functions $\lambda_1 = \lambda_1('z; 'z^0)$, . . . , $\lambda_{\omega}('z) = \lambda_{\omega}('z; 'z^0)$ will be holomorphic in $B_{\varepsilon}('z^0) = \{ 'z : |'z - 'z^0| < \varepsilon \}$, and $|\lambda_j('z) - \lambda_i('z^0)| > 2\delta$, $\forall j \ne i$, and $|\lambda_j('z) - \lambda_j('z^0)| < \delta$, $\forall j$, $'z \in B_{\varepsilon}('z^0)$. In this situation the terms $a_j(z_1, 'z) \exp\{\lambda_j('z) z_1\}$ in (4) are defined as follows:

$$e^{\lambda_j('z)z_1} a_j(z_1, 'z) = \frac{1}{2\pi i} \int\limits_{|\zeta - \lambda_j('z^0)| = \delta} F(\zeta, 'z)\, e^{\zeta z_1}\, d\zeta. \qquad (8)$$

It is obvious that the set $\{ z : | z_1 - \lambda_j('z^0 | = \delta,\ 'z \in B_\varepsilon('z^0) \}$ does not intersect the zero set of the pseudopolynomial h that coincides with the singularity set of F. Therefore it follows from (8) that this term and hence also $a_j(z_1, 'z)$ are holomorphic in $B_\varepsilon('z^0)$. The proof is complete.

From the criterion, when a function belongs to the P-quasipolynomial class cited at the beginning of the note, it follows that set of the points $'z \in \mathbf{C}^{n-1}$ such that an entire function $f(z_1, 'z)$ is P-quasipolynomial in z_1 either coincides with $\mathbf{C}^{n-1}$ or is a union of a countable family of analytic sets $\{ 'z \in \mathbf{C}^{n-1} : I_k(z_1; f) = 0,\ \forall\ z_1 \in \mathbf{C} \} \neq \mathbf{C}^{n-1}$. Therefore the following version of Theorem 1 is valid:

Theorem 1′. *Let a set $E \subset \mathbf{C}^{n-1}$ be not representable as a countable union of analytic sets in $\mathbf{C}^{n-1}$. Further, let $f(z)$ be an entire function of finite order $\rho_f = \rho$ being P-quasipolynomial in z_1 for any fixed $'z \in E$ (with coefficients, spectrum and number of terms in general dependent on $'z$). Then $f(z_1, 'z)$ is P-quasipolynomial in z_1 for any fixed $'z$. For any $'z$ the spectrum $\Lambda_f('z)$ of the P-quasipolynomial coincides with the corresponding zero set of the pseudopolynomial $h(z_1, 'z)$ satisfying the conditions of Theorem 1. Under a proper numeration the coefficients f, that are polynomial of z_1, are locally holomorphic in $'z$ on $\Omega_h = \{ 'z : D_h('z) \neq 0 \}$.*

Now let us consider the case when a set E satisfies the conditions of Theorem 1′ and an entire function f is C-quasipolynomial for every $'z \in E$. Then all the statements of Theorem 1′ are true for f. However, concerning the coefficients of (4) we can state more than that they are locally holomorphic.

Theorem 1″. *Let a set $E \subset \mathbf{C}^{n-1}$ be not representable as a countable union of analytic sets in $\mathbf{C}^{n-1}$. Further, let $f(z)$ be an entire function of order $\rho_f = \rho < \infty$ and let it be a C-quasipolynomial in z_1 for any $'z \in E$. Then it is a P-quasipolynomial in z_1 for any $'z \in \mathbf{C}^{n-1}$. The spectrum of it coincides for the same $'z$ with the set of zeros of a pseudopolynomial $h(z_1, 'z)$ satisfying the conditions pointed in Theorem 1. Then locally with respect to $'z$ in Ω_h the representation*

$$f(z_1, 'z) = \sum_{j=1}^{\omega} a_j('z)\, e^{\lambda_j('z) z_1} \qquad (9)$$

takes place where $\lambda_j('z)$ and $a_j('z)$ are holomorphic. Furthermore, $a_1, \dots, a_\omega$ are zeros of a pseudopolynomial $g(z_1, 'z)$ with meromorphic coefficients, whose polar sets are contained in the discriminant set of the pseudopolynomial h.

P r o o f. First of all note, that in view of Theorem 1 and above connection between the degree of a P-quasipolynomial and the determinant I_m, only the last statement of Theorem 1'' should be proved.

Let us consider the representation (9) in a small enough ball $B_\varepsilon('z^0) \in \Omega_h$. Set for brevity

$$f_m = f_m('z) = \frac{\partial^m f}{\partial z_1^m}\Big|_{z_1 = z_1^0}. \tag{10}$$

It follows from (9) that the functions a_j are the solutions of the system

$$\begin{cases} a_1 + \dots + a_\omega = f_0 \\ a_1\lambda_1 + \dots + a_\omega \lambda_\omega = f_1 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_1\lambda_1^{\omega-1} + \dots + a_\omega\lambda_\omega^{\omega-1} = f_{\omega-1} \end{cases}$$

and hence

$$a_j = \frac{V_j}{V}, \quad j = 1, \dots, \omega, \tag{11}$$

where $V_j = V_j('z\,;'z^0)$ and $V = V('z\,;'z^0)$ are determinants constructed from $\lambda_1, \dots, \lambda_\omega$ and $f_1, \dots, f_\omega$ according to the Kramer rule. In particular, V is equal to the Wandermond determinant $W(\lambda_1, \dots, \lambda_\omega)$ of values $\lambda_1, \dots, \lambda_\omega$.

It follows from (11) that an elementary symmetric function $\Psi_k(a_1, \dots, a_\omega)$ of a rang k in $a_1, \dots, a_\omega$ is a quotient of $\Psi_k(V_1, \dots, V_\omega)$ and V^k. It is obvious that a change of numeration of $\lambda_1, \dots, \lambda_\omega$ changes correspondingly only the numeration of the coefficients $a_1, \dots, a_\omega$. Therefore the function $\Psi_k(a_1, \dots, a_\omega)$ is correctly defined in Ω_h: its values neither depend on the choice of $'z^0$ nor the way of numeration of λ_j. Set $\tilde{\Psi}_k('z) = \Psi_k(a_1('z), \dots, a_\omega('z))$. Since $a_j('z\,;'z^0)$ are holomorphic on $B_\varepsilon('z^0)$, $\tilde{\Psi}_k('z)$ is holomorphic on Ω_h. Let us show that $\tilde{\Psi}_k('z)$ is meromorphic in $\mathbb{C}^{n-1}$. We shall use that

$$V = W(\lambda_1, \dots, \lambda_\omega) = (-1)^{\frac{\omega(\omega-1)}{2}} \prod_{j<k} (\lambda_j - \lambda_k)$$

and

$$D_h = \prod_{j<k} (\lambda_j - \lambda_k)^2.$$

Therefore

$$\tilde{\Psi}_k('z) = \frac{\Psi_k(V_1, \dots, V_\omega)}{V^k} = \frac{V^k \Psi_k(V_1, \dots, V_\omega)}{D_h^k}$$

and hence

$$V^k \Psi_k(V_1, \dots, V_\omega) = \tilde{\Psi}_k('z) D_h('z).$$

From the last equality it follows that $\Phi_k = V^k \Psi_k(V_1, \dots, V_\omega)$ as a function of $'z$ for any k is uniquely defined and holomorphic in Ω_h. Further, it follows from the definitions of V, V_j and the inequality $\max_j |\lambda_j('z)| \le \kappa_1 |'z|^{\rho - 1} + \kappa_2$ that V and V_j are bounded on every bounded subset of Ω_h. Therefore every function $\Phi_k('z)$ is also bounded on bounded subsets of Ω_h and hence it can be holomorphically extended from Ω_h on to the whole $\mathbf{C}^{n-1}$. Thus

$$\tilde{\Psi}_k('z) = \frac{V^k \Psi_k(V_1, \dots, V_\omega)}{D_h^k} = \frac{\Phi_k('z)}{D_h^k}$$

and $\tilde{\Psi}_k('z)$ is meromorphic in $\mathbf{C}^{n-1}$ as a quotient of two entire functions. According to the form of $\tilde{\Psi}_k('z)$ the coefficients a_j are the solutions of the equation

$$a^\omega - \tilde{\Psi}_1('z) a^{\omega - 1} + \dots + (-1)^\omega \tilde{\Psi}_\omega('z) = 0.$$

The Theorem is proved.

The above reasoning implies also that the entire function $f(z_1, 'z)$ from Theorem 1'' is uniquely defined by the pseudopolynomial h and the functions $f_0, \dots, f_{\omega - 1}$ (see (11)). The following Theorem is a more powerful and in some sense a converse statement.

Theorem 2. *Let* $h(z_1, 'z) = z_1^\omega + h_1('z) z_1^{\omega - 1} + \dots + h_\omega('z)$ *be a pseudopolynomial in* $\mathbf{C}^n$ *with discriminant* $D_h \not\equiv 0$ *and let* $f_0, \dots, f_{\omega - 1}$ *be arbitrary entire functions of* $'z$. *Then there exists an unique entire function* $f(z_1, 'z)$ *such that*

$$\frac{\partial^k f}{\partial z_1^k}\Big|_{z_1 = 0} = f_k('z), \ k = 0, \dots, \omega - 1,$$

and for any fixed $'z \in \Omega_h$, *it is a quasipolynomial of* z_1 *with spectrum* $\Lambda('z) = \{ z_1 : h(z_1, 'z) = 0 \}$. *Furthermore, if* $f_0, \dots, f_{\omega - 1}$ *are of finite order* $\le \rho_1$ *and* $h_1, \dots, h_\omega$ *are polynomials of degree* $\le \rho_2$, *then* $f(z)$ *is an entire function of an order* $\rho_f \le 1 + \max(\rho_1, \rho_2)$.

Proof. If such a function exists then locally

$$\sum_{j=1}^{\omega} a_j \lambda_j^m = \frac{\partial^m f}{\partial z_1^m}\Big|_{z_1 = 0}, \quad m = 0, 1, \dots, \tag{12}$$

Here λ_j are the exponents of the quasipolynomial (they are also solutions of the equation $h(z_1, 'z) = 0$) and a_j are its coefficients. Therefore it is natural to look for a function f in the form

$$f = \sum_{m=0}^{\infty} \frac{f_m}{m!} z_1^m ,$$

where $f_0, \dots, f_{\omega-1}$ are given and $f_\omega, f_{\omega+1}, \dots$ are defined by the equalities

$$f_m = \sum_{j=1}^{\omega} a_j \lambda_j^m$$

with functions a_j obtained from the same equalities considered for $m = 0, 1, \dots, \omega - 1$ as equations with respect to a_j. There arise problems whether f_m are correctly defined, about their holomorphic property and estimates.

Let us fix any point $'z^0 \in \Omega_h$ and consider functions $\lambda_1, \dots, \lambda_\omega$ holomorphic in a ball $B_\varepsilon('z^0) \subset \Omega_h$ which are the solutions of $h(z_1, 'z) = 0$ with respect to z_1. We define functions a_j in $B_\varepsilon('z^0)$ as solutions of the system of equations

$$\begin{cases} a_1 + \dots + a_\omega = f_0 , \\ a_1 \lambda_1 + \dots + a_\omega \lambda_\omega = f_1 , \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_1 \lambda_1^{\omega-1} + \dots + a_\omega \lambda_\omega^{\omega-1} = f_{\omega-1} . \end{cases} \tag{13}$$

Since $B_\varepsilon('z^0) \subset \Omega_h$, the determinant of this system $W(\lambda_1, \dots, \lambda_\omega)$ is not equal to zero on $B_\varepsilon('z^0)$ and hence $a_j('z)$ are correctly defined and holomorphic. The functions λ_j as the solutions of the equation $h(z_1, 'z) = 0$ can be holomorphically extended along any curve L starting at $'z^0$ and belonging to Ω_h. Together with λ_j, the functions a_j can be holomorphically extended. Note that if at the end ζ of L we get holomorphic extensions $\mu_1, \dots, \mu_\omega$ of $\lambda_1, \dots, \lambda_\omega$ then by extending along any other similar curve L_1 we get the functions $\mu_1, \dots, \mu_\omega$ again but the numeration differs from the original one. This is naturally valid for the extensions of a_j too and the numeration difference of the resulting extensions is the same as for λ_j. From what was written above about extensions of λ_j and a_j it follows that for any $m \geq \omega$ $\sum_j a_j \lambda_j^\omega$ can be holomorphic extended along any curve $L \subset \Omega_h$. The result of extension depends only on the end of the curve and is independent of the curve itself. Thus a holomorphic function is correctly defined in Ω_h. We denote it by f_m. It is clear from the definition of f_m that in a small enough neighbourhood of any $\zeta \in \Omega_h$ the function $f_m('z)$ can be represented in the form

$$f_m('z) = \sum_{j=1}^{\omega} a_j('z; \zeta) \lambda_j^m('z; \zeta), \tag{14}$$

where $\lambda_1('z; \zeta), \dots, \lambda_m('z; \zeta)$ are the holomorphic solutions of the equation $h(z_1, 'z)$ and $a_j('z; \zeta)$ are the solutions of the corresponding system (13). Let us solve the system (13) and substitute the expressions obtained for a_j into (14). We get:

$$f_m('z) = \sum_{j=1}^{\omega-1} A_{j,m}('z) f_j('z),$$

where

$$A_{j,m} = \frac{1}{W(\lambda_1, \dots, \lambda_\omega)} \begin{vmatrix} 1 & \dots & 1 \\ \dots & \dots & \dots \\ \lambda_1^{j-1} & \dots & \lambda_\omega^{j-1} \\ \lambda_1^m & \dots & \lambda_\omega^m \\ \lambda_1^{j+1} & \dots & \lambda_\omega^{j+1} \\ \dots & \dots & \dots \\ \lambda_1^{\omega-1} & \dots & \lambda_\omega^{\omega-1} \end{vmatrix} = \frac{W_{j,m}}{W}. \tag{15}$$

Note that $A_{j,m}$ is a polynomial in $\lambda_1, \dots, \lambda_\omega$. This follows from the fact that $W_{j,m}(\lambda_1, \dots, \lambda_\omega)$ vanishes when $\lambda_j = \lambda_k, j \neq k$, and that

$$W(\lambda_1, \dots, \lambda_\omega) = (-1)^{\frac{\omega(\omega-1)}{2}} \prod_{j<k} (\lambda_j - \lambda_k).$$

Also it follows from (15) that $A_j(\lambda_1, \dots, \lambda_\omega)$ is a symmetric function of $\lambda_1, \dots, \lambda_\omega$ and hence by the equalities

$$\tilde{A}_{j,m}('z) = A_{j,m}(\lambda_1('z;\zeta), \dots, \lambda_\omega('z;\zeta)), \ 'z \in B_\varepsilon(\zeta) \subset \Omega_h$$

the holomorphic function $\tilde{A}_{j,m}$ is well defined in Ω_h. Since the highest coefficient of the pseudopolynomial h is equal to 1 then the solutions of the equation $h(z_1, 'z)$ are bounded on any compact set in $\mathbb{C}^{n-1}_{('z)}$. It follows that $\tilde{A}_{j,m}('z)$ are bounded on any bounded subset of Ω_h. Therefore they can be holomorphically extended to the whole $\mathbb{C}^{n-1}$. In order to estimate their growth we represent $\tilde{A}_{j,m}$ as a quotient of two entire functions, namely,

$$\tilde{A}_{j,m}('z) = \frac{W_{j,m}(\lambda_1('z;\zeta), \dots, \lambda_\omega('z;\zeta))\, W(\lambda_1('z;\zeta), \dots, \lambda_\omega('z;\zeta))}{W^2(\lambda_1('z;\zeta), \dots, \lambda_\omega('z;\zeta))} = \frac{W_{j,m} W}{D_h}. \tag{16}$$

It is obvious that

$$|\, W(\lambda_1, \dots, \lambda_\omega) \,| \leq \text{const} \cdot (\max_j |\, \lambda_j \,|)^{\frac{\omega(\omega-1)}{2}}, \tag{17}$$

$$|\, W_{j,m}(\lambda_1, \dots, \lambda_\omega) \,| \leq \text{const} \cdot (\max_j |\, \lambda_j \,|)^{\frac{\omega(\omega-1)}{2} + m - j}. \tag{18}$$

Set

$$M_\Phi(R) = \max_{|z| \leq R} |\, \Phi(z) \,|.$$

It is known (see, for example, [5], Lemmas 1.3.1, 1.3.2) that if a quotient of entire functions φ and ψ is itself an entire function then for any $k>1$ and some constant C dependent on ψ, k and n the following inequality [1)] is valid:

$$M_{\varphi/\psi}(R) \le C\left[M_{\varphi}(kR)\,M_{\psi}(kR)\right]^{\frac{k}{k-1}},\ \forall\ R>0. \tag{19}$$

Note also that λ_j as zeros of the pseudopolynomial h can be estimated through its coefficients h_j in the following standard way:

$$\max_j |\lambda_j| \le \sum_{j=1}^{\omega} |h_j|.$$

We conclude from this and (17)-(19) using (19) that

$$M_{\tilde{A}_{j,m}}(R) \le C_1\left\{\max_{|'z|\le kR}\sum_{j=1}^{\omega}|h_j('z)|\right\}^{(j+m)\frac{k}{k-1}}, \tag{20}$$

where C_1 and γ are constants dependent on ω and n only. It follows from this estimate that the series

$$\sum_{m=0}^{\infty}\left\{\frac{1}{m!}\sum_{j=0}^{\omega-1}\tilde{A}_{j,m}f_j\right\}z_1^m = f(z)$$

converges and $f(z)$ can be estimated as follows:

$$\max\{|f(z_1,'z)| : |z_1|\le r,\ |'z|\le R\} \le$$

$$\le C_1\left(\max_j M_{f_j}(R)\right)\cdot\left(\max_{|'z|\le kR}\sum_{j=1}^{\omega}|h_j('z)|\right)^{\gamma\frac{k}{k-1}}\times$$

$$\times\exp\left\{r\max_{|'z|\le kR}\sum_{j=1}^{\omega}|h_j('z)|\right\}. \tag{21}$$

Under an additional assumption that the order of functions $f_0,\dots,f_{\omega-1}$ is not larger than ρ_1 and $h_0,\dots,h_{\omega-1}$ are polynomials of degree $\le\rho_2$, taking into account that k is arbitrary, it follows from (21) that the function f is of order $\le\max(\rho_1,\rho_2)$ with respect to the totality of variables $z_2,\dots,z_n$. Hence f is of order $\le 1+\max(\rho_1,\rho_2)$ with respect to all variables $z_1,\dots,z_n$.

In order to complete the proof of the Theorem let us note that, as it follows from the construction of f, for any fixed $'z\in\Omega_h$ the function f is of the form $f(z_1,'z)=\sum_{j=1}^{\omega}a_j e^{\lambda_j z_1}$ where $a_j=a_j('z;\zeta)$ and $\lambda_j=\lambda_j('z;\zeta)$ are the same as above. The Theorem is proved.

1) In [5] the corresponding inequality is given in a somewhat different form.

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О целых функциях от n переменных, являющихся квазиполиномами по одной из переменных

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Установлен общий вид целой функции $f(z_1, 'z)$, $z_1 \in C$, $'z \in C^{n-1}$, конечного порядка ρ, которая при фиксированных $'z$ из некоторого неплюриполярного множества E как функция от z_1 является М-квазиполиномом, то есть $f(z_1, 'z) = \sum_{j=1}^{m} a_j(z_1) e^{\lambda_j z_1}$, где m, λ_j и $a_j(z_1)$ *априори* произвольно зависят от $'z \in E$ и при этом $a_j(z_1)$ принадлежат некоторому классу М целых функций от z_1 типа 0 при порядке 1.

Цілі функції від n змінних, що є квазіполіномами за одну з змінних

Л.І. Ронкін

Знайдено загальний вигляд цілої функції $f(z_1, 'z)$, $z_1 \in C$, $'z \in C^{n-1}$, скінченого порядку ρ, що за фіксованих $'z$ з деякої неплюріполярної множини E як функція від z_1 є М-квазіполіном, тобто $f(z_1, 'z) = \sum_{j=1}^{m} a_j(z_1) e^{\lambda_j z_1}$, де m, λ_j, $a_j(z_1)$ *апріорі* довільно залежать від $'z \in E$ та де $a_j(z_1)$ належать деякому класу М цілих функцій від z_1 типу 0 за порядком 1.