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On asymptotics of entire functions of finite logarithmic order

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The asymptotic behaviour of an entire function is studied whose zero counting function $n(t)$ satisfies the condition $n(t) = \Delta \ln^p t + \Delta_1 \ln^q t + o(\ln^q t)$, $t \to +\infty$, where $0 < q < p < \infty$, $0 < \Delta < \infty$, $-\infty < \Delta_1 < \infty$.

1. Introduction and principal results

Let f be a transcendental entire function such that $f(0) = 1$, and let $n(t)$ be the counting function of its zeroes. G. Valiron [1] showed that if all zeroes of the function f are negative and $n(t) \sim \Delta t^\rho$ $(t \to \infty)$, $\Delta \in (0, +\infty)$, and ρ is not an integer, then for any $\delta > 0$ uniformly on $\theta \in [-\pi + \delta, \pi - \delta]$

$$\ln f(re^{i\theta}) \sim \frac{\pi\Delta}{\sin \pi\rho} e^{i\rho\theta} r^\rho \ (r \to \infty),$$

where $\ln f(z)$ is the analitic branch of $\operatorname{Ln} f(z)$ in the angle $\{z : |\arg z| \le \pi - \delta\}$. In [1] is was also proved that if f has only negative zeros and non-entire order ρ, and

$$\ln |f(r)| \sim \frac{\pi\Delta}{\sin \pi\rho} r^\rho \ (r \to \infty), \ \Delta \in (0, +\infty),$$

then $n(t) \sim \Delta t^\rho$ $(t \to \infty)$. A simple proof of this statement is given in [2]. The connection between the growth of $n(t)$ and the behaviour of $\ln f$ in terms of the two-term asymptotics was studied by V. M. Logvinenko [3-4]. He considered the case, when

$$n(t) = \Delta t^\rho + \Delta_1 t^{\rho_1} + o(t^{\rho_1}) \ (t \to \infty)$$

and, accordingly,

$$\ln f(re^{i\theta}) = \frac{\pi\Delta}{\sin \pi\rho} e^{i\rho\theta} r^\rho + \frac{\pi\Delta_1}{\sin \pi\rho_1} e^{i\rho_1\theta} r^{\rho_1} + o(r^{\rho_1})(t \to \infty),$$

where $[\rho] < \rho_1 < \rho$, $\Delta \in [0, +\infty)$ and $\Delta_1 \in \mathbb{R}$.

Here we shall consider the case $\rho = 0$, but

$$\rho_l = \limsup_{r \to \infty} (\ln \ln M_f(r) / \ln \ln r) < \infty,$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$. The quantity ρ_l is called a logarithmic order of the function f, and it is clear that $\rho_l \ge 1$ for any entire function. Our principal results are formulated in the following theorems.

Theorem 1: *Let $2 < q < p < q + 1 < \infty$, $\Delta \in [0, +\infty)$, $\Delta_1 \in \mathbf{R}$, and φ_1 be an integrable on each finite interval on $\mathbf{R}_+$ function such that for some $m \geq 1$*

$$\int_T^{2T} |\varphi_1(x)|^m dx = o\left(T(\ln T)^{m(q-2)}\right), \quad T \to +\infty. \tag{1.1}$$

Thus if $\rho = 0$, all zeros of the function f are negative and

$$n(t) = \Delta \ln^p t + \Delta_1 \ln^q t + \varphi_1(t) \; (t \geq 1), \tag{1.2}$$

then

$$\ln f(z) = \frac{\Delta}{p+1} \ln^{p+1} r + \frac{\Delta_1}{q+1} \ln^{q+1} r + i\theta (\Delta \ln^p r + \Delta_1 \ln^q r) +$$

$$+ \frac{1}{2}\left(\frac{\pi^2}{3} - \theta^2\right)(\Delta p \ln^{p-1} r + \Delta_1 q \ln^{q-1} r) + \tilde{\varphi}_1(z), \; z = re^{i\theta}, \tag{1.3}$$

where

$$\tilde{\varphi}_1(z) = o(\ln^{q-1} r) \tag{1.4}$$

when $z \to \infty$ outside some exceptional set $E(f)$. This exceptional set $E(f)$ consists of no more than a countable union of rectangles $\{ z : x_n' < \operatorname{Re} z < x_n'', |\operatorname{Im}| < y_n \}$ such that $x_n'' < 0$, $\sum_{|x_n'| < R} (x_n'' - x_n') = o(R)$ and $\sup \{ y_n : |x_n'| < R \} = o(R)$ as $R \to \infty$. If, moreover, condition (1.1) is satisfied for some $m > 1$, then uniformly on θ

$$\int_T^{2T} |\tilde{\varphi}_1(re^{i\theta})|^m dx = o\left(T(\ln T)^{m(q-1)}\right), \quad T \to +\infty. \tag{1.5}$$

Theorem 2. *Let numbers p, q, Δ and Δ_1 be defined as in Theorem 1, let the function f has only negative zeros and*

$$\ln |f(re^{i\theta})| = \frac{\Delta}{p+1} \ln^{p+1} r + \frac{\Delta_1}{q+1} \ln^{q+1} r +$$

$$+ \frac{1}{2}\left(\frac{\pi^2}{3} - \theta^2\right)(\Delta p \ln^{p-1} r + \Delta_1 q \ln^{q-1} r) + \psi_2(re^{i\theta}) \tag{1.6}$$

for values $\theta = 0$ and $\theta = \pi$, where ψ_2 satisfies the condition

$$\int_{T < |x| < 2T} |\psi_2(x)|^m dx = o\left(T(\ln T)^{m(q-1)}\right), \quad T \to +\infty \tag{1.7}$$

for some $m \geq 1$. Then

$$n(t) = \Delta \ln^p t + \Delta_1 \ln^q t + \tilde{\varphi}_2(t) \; (t \geq 1), \tag{1.8}$$

where

$$\widetilde{\varphi}_2(t) = o(\ln^q t) \tag{1.9}$$

as $t \to \infty$ *outside some set of zero density, that is set* $E \subset [0, +\infty)$ *such that* $\mathrm{mes}\,(E \cap [0, t]) = o(t)$, $t \to +\infty$. *If condition (1.7) is fulfilled for some* $m > 1$, *then*

$$\int_T^{2T} |\widetilde{\varphi}_2(t)|^m dt = o(T \ln^{mq} T), \quad T \to +\infty. \tag{1.10}$$

We need some lemmas to prove these theorems. They will received in p. 2. In p. 3 we shall prove theorems 1 and 2 and in p. 4 we shall give some remarks connected with the cases, when p and q are not linked with conditions in Theorems 1 and 2.

2. Asymptotics of some integrals

Suppose that f has order $\rho = 0$ and only negative zeros $z_k = -a_k < 0$. Then by the Hadamard theorem this function can be represented as $f(z) = \prod_{k=1}^{\infty}(1 + z/a_k)$. Therefore,

$$\ln f(z) = \int_0^{\infty} \ln\left(1 + \frac{z}{t}\right) dn(t) = z \int_a^{\infty} \frac{n(t)}{t(t+z)} dt + O(1), \quad z \to \infty, \tag{2.1}$$

where $a = \exp(3\pi)$. Replace $n(t)$ here with expression (1.2). Then we come to necessity of finding asymptotics of corresponding integrals. For $p > 0$ and $n \in \mathbb{N}$ denote $\binom{o}{p} = 1$ and $\binom{n}{p} = \frac{1}{n!} p(p-1)\ldots(p-n+1)$, and for $p > -1$ put

$$J_p(z) = \int_a^{\infty} \frac{\ln^p x}{x(x+z)} dx, \quad z \in \mathbb{C}.$$

Lemma 1. *Let* $z = re^{i\theta}$ *and* $|\theta| < \pi$. *Then*

$$J_p(z) = \frac{1}{(p+1)z} \ln^{p+1} r + O\left(\frac{1}{r}\right), \quad -1 < p < 0, \tag{2.2}$$

$$J_p(z) = \frac{1}{(p+1)z} \ln^{p+1} r + \frac{i\theta}{z} \ln^p r + O\left(\frac{1}{r}\right), \quad 0 \le p < 1, \tag{2.3}$$

and

$$J_p(z) = \frac{(\ln r + i(\pi + \theta))^{p+1}}{(p+1)z} - \sum_{n=1}^{[p]} \binom{n}{p} \frac{(2\pi i)^n}{n+1} J_{p-n}(z) -$$

$$- \binom{[p]}{p} \frac{(2\pi i)^{[p]+1}}{([p]+1)([p]+2)z} \ln^{p-[p]} r + O\left(\frac{1}{r}\right) =$$

$$+\frac{\ln^{p+1} r}{(p+1)z}+\sum_{n=1}^{[p]}\binom{n}{p}\left\{\frac{(i(\pi+\theta))^n}{(p-n+1)z}\ln^{p-n+1} r-\frac{(2\pi i)^n}{n+1}J_{p-n}(z)\right\}+$$

$$+\left\{\binom{[p]+1}{p+1}\frac{(i(\pi+\theta))^{[p]+1}}{p+1}-\binom{[p]}{p}\frac{(2\pi i)^{[p]+1}}{([p]+1)([p]+2)}\right\}\frac{\ln^{(p-[p])} r}{z}+O\left(\frac{1}{r}\right),\quad p\geq 1, \tag{2.4}$$

as $r\to\infty$ and, moreover, these estimates are uniform on θ, $|\theta|\leq\pi-\delta$ $(\delta>0)$.

P r o o f. In the set $\{w: |w|>a, 0<\arg w<2\pi\}$ we choose the principal branch $\Phi(w)$ of a multivalent function $\frac{1}{w}\mathrm{Ln}^{p+1} w$, $p>-1$, such that $\Phi(a+i0)=0$. Let $C_R=\{w: |w|=R\}$, $R\geq a$, γ_1 and γ_2 being accordingly the upper and lower shores of the cut $\{w: \arg w=0, a<|w|<R\}$, $R>a$. We consider the curves C_R, γ_2, C_a and γ_1 oriented so that the set D bounded by them can be passed around counter-clockwise. Then

$$\left(\int_{C_R}+\int_{\gamma_2}+\int_{C_a}+\int_{\gamma_1}\right)\frac{\Phi(w)}{w+z}\,dw=2\pi i\Phi(-z) \tag{2.5}$$

for all $z\in D^*$, where D^* is the region symmetrical with D the relative to the imaginary axes. It is easy to show the first of the integrals in the LHS of (2.5) tends to 0 when $R\to+\infty$, the third one equals $O\left(\frac{1}{r}\right)$ when $|z|=r\to+\infty$, and

$$\int_{\gamma_1}\frac{\Phi(w)}{w+z}\,dw\to\int_a^{+\infty}\frac{\ln^{p+1}x}{x(x+z)}\,dx,$$

$$\int_{\gamma_2}\frac{\Phi(w)}{w+z}\,dw\to-\int_a^{+\infty}\frac{(\ln x+2\pi i)^{p+1}}{x(x+z)}\,dx$$

when $R\to+\infty$. Therefore, as $r\to+\infty$, we have from (2.5)

$$\int_a^{+\infty}\frac{(\ln x+2\pi i)^{p+1}-\ln^{p+1}x}{x(x+z)}\,dx=\frac{2\pi i}{z}(\ln r+i(\pi+\theta))^{p+1}+O\left(\frac{1}{r}\right). \tag{2.6}$$

Let $p\geq 0$. Then (with $x\geq a$)

$$(\ln x+2\pi i)^{p+1}=\ln^{p+1}x+\sum_{n=1}^{[p]+1}\binom{n}{p+1}(2\pi i)^n\ln^{p+1-n}x+\gamma_p^*(x),$$

where $\gamma_p^*(x)=0$, if $p=[p]$ and

$$\gamma_p^*(x)=\binom{[p]+2}{p+1}(2\pi i)^{[p]+2}\ln^{p-[p]-1}x+O(\ln^{p-[p]-2}x),\quad x\to+\infty,$$

if $p>[p]$. Therefore it follows from (2.6) that

$$(p+1)J_p(z) + \gamma_p^{**}(z) = \frac{1}{z}(\ln r + i(\pi+\theta))^{p+1} + O\left(\frac{1}{r}\right), \quad r \to \infty, \tag{2.7}$$

if $0 \le p < 1$, and

$$(p+1)J_p(z) + \sum_{n=2}^{[p]+1} \binom{n}{p+1} (2\pi i)^{n-1} J_{p-n+1}(z) + \gamma_p^{**}(z) =$$

$$= \frac{1}{z}(\ln r + i(\pi+\theta))^{p+1} + O\left(\frac{1}{r}\right), \quad r \to \infty,$$

if $p \ge 1$, where $\gamma_p^{**}(z) \equiv 0$, if $p = [p]$, and if $p > [p]$, then

$$\gamma_p^{**}(z) = \binom{[p]+2}{p+1} (2\pi i)^{[p]+1} J_{p-[p]-1}(z) +$$

$$+ O\left(\int_a^{+\infty} \frac{(\ln x)^{p-[p]-2}}{x|x+z|} dx\right), \quad r \to \infty. \tag{2.8}$$

Since $|x+z| \ge (x+r)\left|\sin\frac{\pi+\theta}{2}\right|$ when $|\theta| < \pi$, then, dividing the interval of integration $(a, +\infty)$ into intervals (a, r) and $(r, +\infty)$ and estimating in a suitable manner the expression $x(x+r)$, we obtain the estimate

$$\int_a^{+\infty} \frac{(\ln x)^{p-[p]-2}}{x|x+z|} dx \le K \frac{(\ln r)^{p-[p]-1}}{r\left|\sin\frac{\pi+\theta}{2}\right|},$$

where K is a positive constant. Therefore if $p > p \ge 0$, then

$$\gamma_p^{**}(z) = \binom{[p]+2}{p+1} (2\pi i)^{[p]+1} J_{p-[p]-1}(z) + O\left(\frac{1}{r}\right), \quad r \to \infty,$$

and, moreover, the estimate is uniform on θ if $|\theta| \le \pi - \delta$, $\delta > 0$. Thus from (2.7) and (2.8) we have

$$J_p(z) = \frac{(\ln r + i(\pi+\theta))^{p+1}}{(p+1)z} + \gamma_p(z), \tag{2.9}$$

if $0 \le p < 1$, and

$$J_p(z) = \frac{(\ln r + i(\pi+\theta))^{p+1}}{(p+1)z} - \sum_{n=2}^{[p]+1} \binom{n-1}{p} \frac{(2\pi i)^{n-1}}{n} J_{p+1-n}(z) + \gamma_p(z), \tag{2.10}$$

where $\gamma_p(z) = O\left(\frac{1}{r}\right)$, $r \to \infty$, if $p = [p] \ge 0$, and

$$\gamma_p(z) = -\binom{[p]+2}{p+1} \frac{(2\pi i)^{[p]+1}}{n} J_{p-[p]-1}(z) + O\left(\frac{1}{r}\right), \quad r \to \infty, \tag{2.11}$$

$p > [p] \ge 0$; moreover, these estimates are uniform on θ, $|\theta| \le \pi - \delta$ $(\delta > 0)$.

Now let $-1 < p < 0$. Then $p - 1 < -1$ and

$$(\ln x + 2\pi i)^{p+1} = \ln^{p+1} x + 2\pi i(p+1)\ln^p x + O(\ln^{p-1} x), \quad x \to +\infty.$$

Therefore, instead of (2.8) and, hence, instead of (2.9) (or (2.10)) from (2.6) we have

$$(p+1)J_p(z) - \frac{1}{z}(\ln r + i(\pi+\theta))^{p+1} = O\left(\frac{1}{r}\right), \quad r\to\infty.$$

whence (2.2) follows.

If $p > [p] \geq 0$, then $-1 < p - [p] - 1 < 0$ and from (2.11) and (2.2) imply that

$$\gamma_p(z) = -\binom{[p]}{p}\frac{(2\pi i)^{[p]+1}}{([p]+1)([p]+2)z}\ln^{p-[p]} r + O\left(\frac{1}{r}\right), \quad r\to\infty. \tag{2.12}$$

This equality is true for $p = [p] \geq 0$ too. Therefore, if $0 \leq p < 1$, then $[p] = 0$, and from (2.9) and (2.12) we have

$$J_p(z) = \frac{\ln^{p+1} r}{(p+1)z} + \frac{i(\pi+\theta)}{z}\ln^p r + O\left(\frac{\ln^{p-1} r}{r}\right) - \frac{\pi i}{z}\ln^p r + O\left(\frac{1}{r}\right), \quad r\to\infty,$$

and from here follows (2.3).

At last, if $p \geq 1$, then from (2.2) and (2.10) is easy to obtain the first of the equalities (2.4), and then

$$J_p(z) = \frac{1}{(p+1)z}\left\{\ln^{p+1} r + \sum_{n=1}^{[p]+1}\binom{n}{p+1}(i(\pi+\theta))^n \ln^{p-n+1} r + O(\ln^{p-[p]-1} r)\right\} -$$

$$-\sum_{n=1}^{[p]}\binom{n}{p}\frac{(2\pi i)^n}{n+1}J_{p-n}(z) - \binom{[p]}{p}\frac{(2\pi i)^{[p]+1}}{([p]+1)([p]+2)z}\ln^{p-[p]} r + O\left(\frac{1}{r}\right), \quad r\to\infty,$$

hence we easily obtain the second of the equalities (2.4). Lemma 1 is completely proved.

For $p > -1$ and $t > a = \exp(3\pi)$ we put

$$I_p(t) = v.p.\int_a^{+\infty}\frac{\ln^p x}{x(x-t)}\,dx.$$

Lemma 2. *When $t\to\infty$, the following relation takes place*

$$I_p(t) = -J_p(t) + O\left(\frac{1}{t}\right), \quad -1 < p < 0, \tag{2.13}$$

and

$$I_p(t) = \pi i\frac{\ln^p t}{t} - \sum_{n=0}^{[p]}\binom{n}{p}(\pi i)^n J_{p-n}(t) -$$

$$-\binom{[p]}{p}(\pi i)^{[p]+1}\frac{\ln^{p-[p]} t}{([p]+1)t} + O\left(\frac{1}{t}\right), \quad p \geq 0. \tag{2.14}$$

P r o o f. We choose the function Φ as in the proof of Lemma 1, and let $C_R^+ = \{w : |w| = R, 0 \leq \arg w \leq \pi\}$, $\gamma_1 = [-R, -a]$, $\gamma_2 = [a, R] \setminus [t-\varepsilon, t+\varepsilon]$, $a < t < R$, where $\varepsilon > 0$ is a sufficiently small number. Moreover, let

$l_\varepsilon^+ = \{ w : | w - t | = \varepsilon \}$, Im $w \ge 0$. We assume that all these curves are oriented so, that the region bounded by them can be passed around counter-clockwise. Then

$$\left(\int\limits_{C_R^+} + \int\limits_{\gamma_1} + \int\limits_{C_a^+} + \int\limits_{\gamma_2} + \int\limits_{l_\varepsilon^+} \right) \frac{\Phi(w)}{w - t}\, dw = 0. \tag{2.15}$$

The estimates of the integrals on C_a^+ and C_R^+ are the same, like the estimates of the integrals on C_a and C_R in the proof of Lemma 1, and

$$\int\limits_{\gamma_2} \frac{\Phi(w)}{w - t}\, dw \to I_{p+1}(t),$$

$$\int\limits_{l_\varepsilon^+} \frac{\Phi(w)}{w - t}\, dw \to -\frac{\pi i}{t} \ln^{p+1} t,$$

when $\varepsilon \to 0$ and $R \to +\infty$, and

$$\int\limits_{\gamma_1} \frac{\Phi(w)}{w - t}\, dw \to \int\limits_a^{+\infty} \frac{(\ln x + \pi i)^{p+1}}{x(x+t)}\, dx \ (R \to +\infty).$$

Therefore it follows from (2.15) that

$$I_p(t) = \frac{\pi i}{t} \ln^p t - \int\limits_a^{+\infty} \frac{(\ln x + \pi i)^p}{x(x+t)}\, dx + O\left(\frac{1}{t}\right), \quad t \to \infty.$$

If $-1 < p < 0$ we have from here

$$I_p(t) = O\left(\frac{1}{t}\right) - \int\limits_a^{+\infty} \frac{\ln^p x + O(\ln^{p-1} x)}{x(x+t)}\, dx = -J_p(t) + O\left(\frac{1}{t}\right), \quad t \to \infty,$$

and if $p \ge 0$, then we have, like in the proof of Lemma 1,

$$I_p(t) = \frac{\pi i}{t} \ln^p t - \int\limits_a^{+\infty} \left(\sum_{n=0}^{[p]} \binom{n}{p} (\pi i)^n \ln^{p-n} x \right) \frac{dx}{x(x+t)} - \int\limits_a^{+\infty} \frac{\gamma_p^*(x)}{x(x+t)}\, dt,$$

where $\gamma_p^*(x) = 0$ if $p = [p]$, and if $p > [p]$, then

$$\gamma_p^*(x) = \binom{[p]+1}{p} (\pi i)^{[p]+1} (\ln x)^{p-[p]-1} + O(\ln^{p-[p]-2} x), \quad x \to +\infty.$$

Hence equality (2.14) follows in the same way as in the proof of Lemma 1. Lemma 2 is proved.

We shall obtain estimates of remaining terms in equalities (1.3) and (1.8) by using the methods of Logvinenko [3-4] and the following lemmas.

Lemma 3[4]. *Let* $F \in L^p(-\infty, +\infty)$, $p > 1$, *and* $M(x) = M(x, F^*) = \sup \{ | F^*(x+iy) | : y \in \mathbf{R} \}$, where

$$F^*(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F(t)}{t+z} dt$$

if $\mathrm{Re}\, z \neq 0$, *and* $F^*(x)$ *denotes the angle limit values of* $F^*(z)$ *when* $\mathrm{Re}\, z = x \in \mathbf{R}$. *Then there exists a constant* $C_p \in (0, +\infty)$ *such that* $\| M \|_p \leq C_p \| F \|_p$, *where* $\| \cdot \|_p$ *is the norm in* L^p.

Lemma 4 [4]. *Let* $F \in L^p(-\infty, +\infty)$, $p \geq 1$ *and* $h \in (0, +\infty)$. *Then there exists a constant* $C_p \in (0, +\infty)$ *such that* $\mathrm{mes} \{ x : M(x, F^*) > h \} \leq C_p h^{-p} \| F \|_p^p$.

Lemma 5. *Let* φ_1 *be an integrable function on each finite interval from* $[a, +\infty)$ *and for some* $m \geq 1$ *and* $s > -1$ *satisfies the condition*

$$\int_T^{2T} | \varphi_1(t) |^m dt = o(T \ln^{ms} T), \quad T \to +\infty, \tag{2.16}$$

and

$$\psi_1(z) = z \int_a^{+\infty} \frac{\varphi_1(t)}{t(t+z)} dt. \tag{2.17}$$

Then

$$\psi_1(z) = o(\ln^{s+1} r), \quad z = re^{i\theta}, \tag{2.18}$$

when $z \to \infty$ *outside some exceptional set E like in theorem 1. If, besides that, condition (2.16) holds for some* $m > 1$, *then uniformly on* θ

$$\int_T^{2T} | \psi_1(re^{i\theta}) |^m dr = o(T \ln^{m(s+1)} T), \ T \to +\infty. \tag{2.19}$$

P r o o f. Write

$$\frac{\psi_1(z)}{z} = \left(\int_a^{r/4} + \int_{r/4}^{4r} + \int_{4r}^{\infty} \right) \frac{\varphi_1(t)}{t(t+z)} dt = A_1(z) + A_2(z) + A_3(z)$$

and for $r \geq 16a = 16 \exp(3\pi)$ put $k(r) = [\log_2(r/a)] - 1$ and $b(r) = \exp \{ (\log_2 r - k(r)) \ln 2 \}$. Then

$$| A_1(z) | = \int_a^{b(r)} \frac{| \varphi_1(t)|}{t| (t+z) |} dt + \sum_{k=0}^{k(r)-3} \int_{r2^{-k-3}}^{r2^{-k-2}} \frac{| \varphi_1(t)|}{t(| z | - t)} dt =$$

$$= O\left(\frac{1}{r}\right) + \sum_{k=-k(r)}^{-3} \int_{r2^k}^{r2^{k+1}} \frac{| \varphi_1(t)|}{t(r-t)} dt \le O\left(\frac{1}{r}\right) + \frac{4}{3r} \sum_{k=-k(r)}^{-3} \int_{r2^k}^{r2^{k+1}} \frac{| \varphi_1(t)|}{t} dt \le$$

$$\le \frac{4}{3r^2} \sum_{k=-k(r)}^{-3} \frac{1}{2^k} \int_{r2^k}^{r2^{k+1}} | \varphi_1(t)| \, dt + O\left(\frac{1}{r}\right), \quad r \to \infty,$$

that is, using Gelder's inequality,

$$| A_1(z) | \le \frac{4}{3r^2} \sum_{k=-k(r)}^{-3} 2^{-k} \|\varphi_1\|_{m,k} \|1\|_{m',k} + O\left(\frac{1}{r}\right), \quad r \to \infty, \tag{2.20}$$

where $\|\cdot\|_{m,k}$ is the norm in $L^m(r2^k, r2^{k+1})$, and $\frac{1}{m} + \frac{1}{m'} = 1$ and $m' = \infty$ when $m = 1$. Since, from (2.16), $\|\varphi_1\|^m_{m,k} = o(2^k r \ln^{ms}(2^k r))$ when $r \to \infty$ and $\|1\|^{m'}_{m',k} = 2^k r$, then it follows from (2.20) that

$$| A_1(z) | \le o\left(\frac{1}{r} \sum_{k=-k(r)}^{-3} (\ln r + k \ln 2)^s \right) + O\left(\frac{1}{r}\right) =$$

$$= o\left(\frac{1}{r} \int_{-k(r)}^{-3} (\ln r + x \ln 2)^s dx\right) + o\left(\frac{\ln^s r}{r}\right) + O\left(\frac{1}{r}\right) =$$

$$= o\left(\frac{\ln^{s+1} r}{r}\right) + O\left(\frac{1}{r}\right) = o\left(\frac{\ln^{s+1} r}{r}\right), \quad r \to \infty. \tag{2.21}$$

We estimate A_3. Since $\|\frac{1}{t}\|_{m',k} \le (2^k r)^{1/m' - 1}$, then, as above, we have

$$| A_3(z) | \le \int_{4r}^{\infty} \frac{| \varphi_1(t) |}{t^2 (1 - | z |/t)} dt \le \frac{4}{3} \sum_{k=2}^{\infty} \int_{r2^k}^{r2^{k+1}} \frac{| \varphi_1(t)|}{t^2} dt \le$$

$$\le \frac{4}{3} \sum_{k=2}^{\infty} \frac{1}{2^k} \|\varphi_1\|_{m,k} \left\|\frac{1}{t}\right\|_{m',k} = o\left(\frac{1}{r} \sum_{k=2}^{\infty} \frac{1}{2^k} \ln^s (r2^k)\right) =$$

$$= o\left(\frac{\ln^s r}{r} \sum_{k=2}^{\infty} \frac{(1+k)^s}{2^k}\right) = o\left(\frac{\ln^s r}{r}\right), \quad r \to \infty. \tag{2.22}$$

We estimate finally the integral A_2. If z is situated "far away" from the negative ray of the real axis, then

$$| A_2(z) | \le \int_{r/4}^{4r} \frac{| \varphi_1(t) |}{t | t+z |} dt \le \frac{4}{r} \| \varphi_1 \|_{m,r} \left\| \frac{1}{| t + re^{i\theta} |} \right\|_{m',r}, \tag{2.23}$$

where $\| \cdot \|_{m,r}$ is the norm in the space $L^m(r/4, 4r)$. Since, in view, (2.16) $\| \varphi_1 \|_{m,r} = o(r^{1/m} \ln^s r)$, $r \to \infty$, and

$$\left\| \frac{1}{| t + re^{i\theta} |} \right\|_{m',r} \le \left(\int_{r/4}^{4r} \frac{dt}{\left((t+r) \sin \frac{\pi + \theta}{2}\right)^{m'}} \right)^{1/m'} \le \frac{K}{r^{1/m} \sin \frac{\pi + \theta}{2}},$$

where K is a positive constant, it follows from (2.23) that

$$| A_2(z) | = o\left(\frac{\ln^s r}{r \sin \frac{\pi + \theta}{2}} \right), \quad r \to \infty,$$

that is, we can choose the positive function $\theta(r)$ such that $\theta(r) \downarrow 0$ $(r \to +\infty)$ and when $| \theta | \le \pi - \theta(r)$ we obtain the relation

$$| A_2(z) | = o\left(\frac{1}{r} \ln^s r \right), \quad r \to \infty. \tag{2.24}$$

In order to estimate A_2 "near" the negative ray of the real axis, we put $G = \{ z : -2r \le \operatorname{Re} z = x \le -\frac{r}{2}, 0 < | \arg z - \pi | \le 3\pi/4 \}$, and let $\chi_r(t)$ be a characteristic function of the interval $[r/4, 4r]$, $\varepsilon(r)$ being an arbitary function on the interval $[a, +\infty)$ such that $\varepsilon(r) \downarrow 0$ $(r \to \infty)$. Denote

$$E_r = \left\{ x \in [-2r, -r/2]: M(x, A_2) > \frac{\varepsilon(r)}{r} \ln^s r \right\},$$

and $g_r(t) = \chi_r(t) \varphi_1(t)/t$. Then $g_r \in L^m(-\infty, +\infty)$, $m \ge 1$, and from (2.16) $\| g_r \|_m = o(r^{1/m - 1} \ln^s r)$, $r \to \infty$, and

$$A_2(z) = \int_{-\infty}^{+\infty} \frac{g_r(t)}{t + z} dt. \tag{2.25}$$

By Lemma 4 (with $F = g_r$) we have

$$mes\, E_r \le C_m \frac{\| g_r \|_m^m}{(\varepsilon(r) \ln^s r / r)^m} = o\left(\frac{r}{\varepsilon(r)^m} \right), \quad r \to \infty,$$

so we can choose the function $\varepsilon(r)$ in order to ensure $mes\, E_r = o(r)$, $r \to \infty$. Put $E_* = \bigcup_{k=1}^{\infty} E_{r_k}$, $r_k = 2^k$. It is easy to show that the set E_* has density zero in this case.

If $\operatorname{Re} z \notin E_*$, then $M(x, A_2) < \frac{\varepsilon(r)}{r} \ln^s r = o\left(\frac{1}{r} \ln^s r\right)$, $r \to \infty$, and therefore from (2.21) and (2.22) we obtain the relation $\psi_1(z) = o(\ln^{s+1} r)$ as $r \to \infty$, $\operatorname{Re} z \notin E_*$. This result with the following from ((2.21), (2.22) and (2.24)) same estimate ψ_1 for "remote" z from the negative ray yields (2.18) when $z \to \infty$ outside the set E, which is union of rectangles, like in Theorem 1.

It remains to show that if $m > 1$ then (2.19) holds. Denote

$$\| g(z) \|_m^* = \sup_\theta \left\{ \int_T^{2T} | g(re^{i\theta}) |^m \, dr \right\}^{1/m}.$$

Then, by the Minkovski inequality, $\| \psi_1(z) \|_m^* \le \sum_{j=1}^{3} \| zA_j(z) \|_m^*$. From (2.21) and (2.22) we have $\| zA_j(z) \|_m^* = o(T^{1/m} \ln^{s+1} T)$ and $\| zA_3(z) \|_m^* = o(T \ln^s T)$ when $T \to +\infty$. In order to estimate $\| zA_2(z) \|_m^*$ denote $\Theta_1 = \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$ and $\Theta_2 = [-\pi, \pi] \backslash \Theta_1$. Then from (2.24).

$$\sup_{\theta \in \Theta_1} \left(\int_T^{2T} r^m \, | A_2(re^{i\theta}) |^m \, dr \right)^{1/m} = o(T^{1/m} \ln^s T), \quad T \to +\infty,$$

and from (2.25) and Lemma 3 it is easy to see that

$$\sup_{\theta \in \Theta_2} \left(\int_T^{2T} r^m \, | A_2(re^{i\theta}) |^m \, dr \right)^{1/m} \le 2T \sup_{\theta \in \Theta_2} \left(\int_T^{2T} M^m (r \cos\theta, A_2) \, dr \right)^{1/m} =$$

$$= 2T \sup_{\theta \in \Theta_2} \left(\frac{1}{|\cos\theta|} \left\{ \int_{T\cos\theta}^{2T\cos\theta} M^m (x, A_2) \, dx \right\} \right)^{1/m} \le$$

$$\le 2\sqrt{2}\, T \, \| M(x, A_2) \|_m \le 2\sqrt{2}\, C_m T \, \| g_r \|_m = o(T^{1/m} \ln^s T), \quad T \to +\infty,$$

since in view of the definition of $g_r(t)$ and (2.16) $\| g_r \|_m = o(r^{1/m - 1} \ln^s r)$, $T \to +\infty$. Hence $\| zA_3(z) \|_m^* = o(T^{1/m} \ln^s T)$, $T \to +\infty$, and so $\| \psi_1(z) \|_m^* = o(T^{1/m} \ln^{s+1} T)$, $T \to +\infty$. Lemma 5 is proved completely.

Lemma 6. *Let a function ψ_2 be integrable on each final interval from* **R**, $\int_0^A \frac{|\psi_2(x)|}{x} \, dx < \infty$ *and for some $m \ge 1$ and $s > -1$*

$$\int\limits_{T<|x|<2T} |\psi_2(x)|^m dx = o(T\ln^{ms} T), \quad T \to +\infty. \tag{2.26}$$

For $t > a = \exp(3\pi)$ *put*

$$\varphi_2(t) = v.\,p. \int\limits_{-\infty}^{+\infty} \frac{\psi_2(x)}{x(x+t)}\,dx. \tag{2.27}$$

Then

$$\varphi_2(t) = o\left(\frac{1}{t}\ln^{s+1} t\right) \tag{2.28}$$

when $t \to +\infty$ *outside some set of zero density. If, moreover, (2.26) holds at some* $m > 1$, *then*

$$\int\limits_{T}^{2T} |\varphi_2(t)|^m\,dt = o(\ln^{m(s+1)} T), \ T \to +\infty. \tag{2.29}$$

P r o o f. When $t \to +\infty$, we have

$$\varphi_2(t) = v.\,p. \left(\int\limits_{-\infty}^{-a} + \int\limits_{a}^{+\infty} + \int\limits_{-a}^{a}\right) \frac{\psi_2(x)}{x(x+t)}\,dx = \varphi_2^{(1)}(t) + \varphi_2^{(2)}(t) + O\left(\frac{1}{t}\right).$$

Since

$$\varphi_2^{(1)}(t) = v.\,p. \int\limits_{a}^{+\infty} \frac{\psi_2(-x)}{x(x-t)}\,dx,$$

as was shown in proof of Lemma 5, there exists set a $E_* \subset (-\infty, 0)$ of there zero density such that

$$\varphi_2^{(1)}(t) = o\left(\frac{1}{t}\ln^{s+1} t\right), \quad t \to +\infty, \quad -t \notin E_*, \tag{2.30}$$

and if (2.26) holds at $m > 1$, then

$$\int\limits_{T}^{2T} |\varphi_2^{(1)}(t)| = o(\ln^{m(s+1)} T), \quad T \to +\infty. \tag{2.31}$$

Let us estimate $\varphi_2^{(2)}$. Let $k(t)$, $b(t)$ and $\|\cdot\|_{m,k}$ be such as in the proof of Lemma 5. Then, from (2.26),

$$|\varphi_2^{(2)}(t)| \le \left(\int\limits_{a}^{b} + \sum_{k=-k(t)}^{\infty} \int\limits_{2^k t}^{2^{k+1} t}\right) \frac{|\psi_2(x)|}{x(x+t)}\,dx \le$$

$$\leq O\left(\frac{1}{t}\right) + \sum_{k=-k(t)}^{-1} \frac{1}{2^k t^2} \|\psi_2\|_{m,k} \|1\|_{m',k} + \sum_{k=0}^{\infty} \frac{1}{2^k t} \|\psi_1\|_{m,k} \left\|\frac{1}{x}\right\|_{m',k} =$$

$$= O\left(\frac{1}{t}\right) + o\left(\frac{1}{t} \sum_{k=-k(t)}^{-1} \ln^s (2^k t)\right) + o\left(\frac{1}{t} \sum_{k=0}^{\infty} \ln^s (2^k t)\right) =$$

$$= o\left(\frac{1}{t} \ln^{s+1} t\right), \quad t \to +\infty. \tag{2.32}$$

Relation (2.28) follows from (2.30) and (2.32), and relation (2.29) follows from (2.31) and (2.32). Lemma 6 is proved.

3. Proofs of the theorems

We prove Theorem 1. From (2.1) and (1.2) for $z = re^{i\theta}$, $|\theta| < \pi$, we have

$$\ln f(z) = \Delta z J_p(z) + \Delta_1 z J_q(z) + \psi_1(z) + O(1), \quad z \to +\infty, \tag{3.1}$$

where J_p and ψ_1 are the same as in Lemmas 1 and 5. Since $p > q > 2$, then, in view of (2.4),

$$zJ_p(z) = \frac{\ln^{p+1} r}{p+1} zp\left(\frac{i(\pi+\theta)}{zp} \ln^p r - \pi i J_{p-1}(z)\right) +$$

$$+ z\frac{p(p+1)}{2}\left(-\frac{(\pi+\theta)^2}{(p-1)z} \ln^{p-1} r + \frac{4\pi^2}{3} J_{p-2}(z)\right) + O\left(\ln^{p-2} r\right) =$$

$$= \frac{\ln^{p+1} r}{p+1} + i(\pi+\theta)\ln^p r - \frac{p(\pi+\theta)^2}{2}\ln^{p-1} r -$$

$$- ip\pi z J_{p-1}(z) + \frac{2\pi^2 p(p-1)}{3} zJ_{p-2}(z) + O(\ln^{p-2} r), \quad r \to +\infty.$$

Hence

$$zJ_{p-1}(z) = \frac{\ln^p r}{p} + i(\pi+\theta)\ln^{p-1} r - i(p-1)\pi z J_{p-2}(z) + O(\ln^{p-2} r), \quad r \to +\infty,$$

that is

$$zJ_p(z) = \frac{\ln^{p+1} r}{p+1} + i\theta \ln^p r + \frac{p}{2}(\pi^2 - \theta^2)\ln^{p-1} r -$$

$$- \frac{p(p+1)\pi^2}{3} zJ_{p-2}(z) + O(\ln^{p-2} r) =$$

$$= \frac{\ln^{p+1} r}{p+1} + i\theta \ln^p r - \left(\frac{\pi^2}{6} - \frac{\theta^2}{2}\right)\ln^{p-1} r + O(\ln^{p-2} r), \quad r \to +\infty. \tag{3.2}$$

Hence, from (3.1), we obtain (1.3), where $\tilde{\varphi}_1(z) = \psi_1(z) + O(\ln^{p-2} r)$, $r \to +\infty$. Since the function φ_1 satisfies (1.1) (that (2.16) with $s = q - 2 > 0$), then for the function ψ_1 relations (1.4) and (1.5) hold by Lemma 5. Simultaneously

$$\int_T^{2T} (\ln^{p-2} r)^m \, dr = O(T \ln^{m(p-2)} T), \quad T \to \infty,$$

and since $p - 2 < q - 1$, $\widetilde{\psi}_1$ satisfies conditions (1.4) and (1.5). Theorem 1 is proved.

Let us prove Theorem 2. For $x \in \mathbf{R}$ the relation

$$\frac{\ln |f(x)|}{x} = v.\,p. \int_0^{+\infty} \frac{n(t)}{t(t+x)} \, dt$$

may be interpreted in the following manner: the function $\dfrac{\ln |f(x)|}{x}$ is a Hilbert's transformation which equals 0 when $t \le t_0$, $0 \le t_0 < -a_1$, and $n(t)/t$ when $t \ge t_0$. It is clear that $\dfrac{n(t)}{t} \in L^m(0, +\infty)$ when $m > 1$, and by the theorem about the inverse Hilbert's transformation

$$\frac{n(t)}{t} = \frac{1}{\pi^2} v.\,p. \int_{-\infty}^{+\infty} \frac{\ln |f(x)|}{x(x+t)} \, dx, \quad t > t_0.$$

Therefore, in view of (1.6),

$$\frac{n(t)}{t} = \frac{1}{\pi^2} \left(v.\,p. \int_{-\infty}^{-a} + \int_a^{\infty} \right) \frac{\ln |f(x)|}{x(x+t)} \, dt + O\left(\frac{1}{t}\right) =$$

$$= \frac{1}{\pi^2} \left\{ \frac{\Delta}{p+1} (I_{p+1}(t) + J_{p+1}(t)) + \frac{\Delta_1}{q+1} (I_{q+1}(t) + J_{q+1}(t)) + \right.$$

$$+ \frac{\pi^2}{6} (\Delta p J_{p-1}(t) + \Delta_1 q J_{q-1}(t)) -$$

$$\left. - \frac{\pi^2}{3} (\Delta p I_{p-1}(t) - \Delta_1 q I_{q-1}(t)) + \varphi_2(t) \right\} + O\left(\frac{1}{t}\right), \quad t \to +\infty, \tag{3.3}$$

where J_p, I_p and φ_2 are such as in Lemmas 1,2 and 6. In view of (3.2) with $\theta = 0$, we obtain from (2.14)

$$I_{p+1}(t) = \frac{\pi i}{t} \ln^{p+1} t - J_{p+1}(t) - \pi i(p+1) J_p(t) + \frac{\pi^2 p(p+1)}{2} J_{p-1}(t) +$$

$$+ O\left(\frac{1}{t} \ln^{p-1} t\right) = - J_{p+1}(t) + \frac{\pi i}{t} \ln^{p+1} t - \pi i(p+1) \frac{1}{(p+1)t} \ln^{p+1} t +$$

$$+ \frac{\pi^2 (p+1)}{2t} \ln^p t + O\left(\frac{1}{t} \ln^{p-1} t\right) =$$

$$= - J_{p+1}(t) + \frac{\pi^2 (p+1)}{2t} \ln^p t + O\left(\frac{1}{t} \ln^{p-1} t\right), \quad t \to +\infty. \tag{3.4}$$

From (3.2) and (3.4) it follows that

$$I_{p+1}(t) = -\frac{1}{(p+2)t}\ln^{p+2}t + \frac{2\pi^2(p+1)}{3t}\ln^p t + O\left(\frac{1}{t}\ln^{p-1}t\right), \quad t \to +\infty, \quad (3.5)$$

and from (3.4) we have

$$J_{p+1}(t) + I_{p+1}(t) = \frac{\pi^2(p+1)}{2t}\ln^p t + O\left(\frac{1}{t}\ln^{p-1}t\right), \quad t \to +\infty. \quad (3.6)$$

Combining (3.2), (3.3), (3.5) and (3.6) it is easy to obtain

$$\frac{n(t)}{t} = \frac{\Delta}{t}\ln^p t + \frac{\Delta_1}{t}\ln^q t + \varphi_2(t) + O\left(\frac{1}{t}\ln^{p-1}t\right), \quad t \to +\infty,$$

that is, we have (1.8) with $\widetilde{\varphi}_2(t) = \varphi_2(t) + O(\ln^{p-2}t)$, $t \to +\infty$. Since ψ_2 satisfies (1.7) (that is condition (2.26) with $s = q - 1 > 0$), relations (2.28) and (2.29) hold by Lemma 6 for the function φ_2. Therefore, it is easy to see that the function $\widetilde{\varphi}_2$ satisfies conditions (1.9) and (1.10). Theorem 2 is proved.

4. Remarks

From the proof of Theorem 1 it is obvious that we can obtain various analogues of this theorem in the case, when the condition $2 < q < p < q + 1$ is not fulfiled. We shall state some results in this direction.

Suppose that $-1 < s < q < p < +\infty$, $\Delta \in (0, +\infty)$, $\Delta_1 \in \mathbb{R}$, and the function φ_1 is integrable on each finite interval from $\mathbb{R}_+$ and for some its $m \geq 1$

$$\int_T^{2T} |\varphi_1(x)|^m\,dx = o(T\ln^{ms}T), \quad T \to \infty. \quad (4.1)$$

Then, if an entire function f has zero order and only negative zeros, $n(t)$ has (1.2), then

$$\ln f(z) = \frac{\Delta}{p+1}\ln^{p+1}r + \frac{\Delta_1}{q+1}\ln^{q+1}r + \widetilde{\psi}_1(z), \; 0 < p < s + 1;$$

$$\ln f(z) = \frac{\Delta}{p+1}\ln^{p+1}r + \frac{\Delta_1}{q+1}\ln^{q+1}r + i\theta\Delta\ln^p r + \widetilde{\psi}_1(z), \; q < s + 1 \leq p < s + 2;$$

$$\ln f(z) = \frac{\Delta}{p+1}\ln^{p+1}r + \frac{\Delta_1}{q+1}\ln^{q+1}r + i\theta(\Delta\ln^p r + \Delta_1\ln^q r) + \widetilde{\psi}_1(z),$$
$$s + 1 \leq q < p < s + 2;$$

$$\ln f(z) = \frac{\Delta}{p+1}\ln^{p+1}r + \frac{\Delta_1}{q+1}\ln^{q+1}r + i\theta(\Delta\ln^p r + \Delta_1\ln^q r) +$$
$$+ \frac{\Delta p}{2}\left(\frac{\pi^2}{3} - \theta^2\right)\ln^{p-1}r + \widetilde{\psi}_1(z), \; s + 1 \leq q < s + 2 \leq p < s + 3;$$

$$\ln f(z) = \frac{\Delta}{p+1}\ln^{p+1}r + \frac{\Delta_1}{q+1}\ln^{q+1}r + i\theta(\Delta\ln^p r + \Delta_1\ln^q r) +$$
$$+ \frac{1}{2}\left(\frac{\pi^2}{3} - \theta^2\right)(\Delta p\ln^{p-1}r + \Delta_1 q\ln^{q-1}r) + \widetilde{\psi}_1(z), \; s + 2 \leq q < p < s + 3.$$

Here $\tilde{\psi}_1(z) = o(\ln^{s+1} r)$ when $z \to \infty$ outside some exceptional set $E(f)$ described in Theorem 1, and if (4.1) holds for some $m > 1$, then

$$\sup_{\theta} \int_T^{2T} |\psi_1(re^{i\theta})|^m \, dr = o(T \ln^{m(s+1)} T), \quad T \to +\infty.$$

As is obvious from proof of Theorem 2, there exist its analogues, corresponding to above considered cases (the conditions on p and q). We shall not give their formulation. However, we notice, that if in imposed Theorem 2 we shall restrict only by two-term asymptotics of $\ln |f|$ on the positive ray, then we cannot obtain two-term asymptotics (1.8) for $n(t)$. Actually, the following analogue of one theorem be M.M. Tyan takes place.

Theorem 3. *Let an entire function f has zero order and only negative zeros, and*

$$\ln |f(r)| = \frac{\Delta}{p+1} \ln^{p+1} r + \frac{\Delta_1}{q+1} \ln^{q+1} r + O(\ln^{p-1} r), \quad r \to \infty, \tag{4.2}$$

where $-1 < q < p$. Then the exact estimate for $n(t)$ is

$$n(t) = \Delta \ln^p t + O\left(\frac{\ln^p t}{\ln \ln t}\right), \quad t \to +\infty. \tag{4.3}$$

To prove this theorem we shall use the following lemmas.

Lemma 7. *Let $g(x)$ be a differential function on $[1, +\infty)$, and $g'(x)$ be a decreasing function. If $-1 \le q^* - 1 < q < p$ and*

$$g(x) = A \ln^{p+1} x + B \ln^{q+1} x + O(\ln^{q^*} x), \quad x \to \infty,$$

then

$$g'(x) = A(p+1) \frac{\ln^p x}{x} + B(q+1) \frac{\ln^q x}{x} + O\left(\frac{\ln^{(p+q^*)/2} x}{x}\right), \quad x \to \infty.$$

Lemma 8 [6, Theorem 3.2.1]. *Let h_1, h_2 be positive increasing on $[0, \infty)$ functions, $h_1(x) = 0$ when $x \in [0, x_1]$, $h_2(x) = 0$ when $x \in [0, x_2]$, and there exist constants c, d such that*

$$\left(\frac{v}{u}\right)^c < \frac{h_1(v)}{h_1(u)} < \left(\frac{v}{u}\right)^d, \quad x_1 < u < v, \quad 0 < c < d. \tag{4.4}$$

Let, moreover, Stieltjes untegrals

$$H_1(t) = \int_0^{+\infty} \frac{dh_1(x)}{x^\beta (x+t)^\nu},$$

$$H_2(t) = \int_0^{+\infty} \frac{dh_2(x)}{x^\beta (x+t)^\nu}$$

where $\nu > 0$, $0 \le \beta < c < d < \nu + \beta$, *the convergent when* $t > 0$. *Then be from the estimate*

$$H_2(t) = H_1(t)(1 + O(r(t))), \quad t \to \infty,$$

exact estimate follows:

$$h_2(x) = h_1(x)\left(1 + O\left(\frac{1}{\ln(1/r(x))}\right)\right), \quad x \to \infty,$$

provided that $r(x)$ *is a positive decreasing function satisfying the conditions* $r(\infty) = 0$, $r(\eta x) \le K\eta^{-\omega} r(x)$, $K > 0$, $0 < \eta < 1$, $0 \le \omega < c$, $x \to \infty$.

We will not prove Lemma 7, since it basically similar is to the proof of the Lemma [5]. Moreover, in [7] of the left inequality in (4.4) is shown to be unnecessary.

P r o o f of Theorem 3. From representation

$$f(z) = \prod_{n=1}^{\infty} (1 + z/a_k), \quad 0 < a_1 \le a_2 \ldots \le a_n \le \ldots,$$

we obtain

$$\frac{f'(r)}{f(r)} = \sum_{n=1}^{\infty} \frac{1}{r + a_n} = \int_0^{\infty} \frac{dn(t)}{r + t}, \tag{4.5}$$

which implies that $f'(r)/f(r)$ is a decreasing function. By Lemma 7 with $g(r) = \ln |f(r)|$, we have from (4.2)

$$\frac{f'(r)}{f(r)} = \Delta \frac{\ln^p r}{r} + \Delta_1 \frac{\ln^q r}{r} + O\left(\frac{\ln^{(2p-1)/2} r}{r}\right) =$$

$$= \Delta \frac{\ln^p r}{r}(1 + O(\ln^{-\delta} r)) \quad (r \to \infty), \quad \delta = \min\{\tfrac{1}{2}, p - q\}. \tag{4.6}$$

Further, it follows from (3.2)

$$\int_0^{+\infty} \frac{d(\ln^p t)^+}{x + t} = p \int_1^{+\infty} \frac{\ln^{p-1} t}{t(x + t)}\, dt = pJ_{p-1}(x) + O\left(\frac{1}{x}\right) =$$

$$= \frac{\ln^p x}{x} + \frac{\pi^2 p(p-1)}{6} \frac{\ln^{p-2} x}{x} + \gamma_0(x), \quad x \to \infty,$$

where $\gamma_0(x) = O\left(\frac{\ln^{p-3} x}{x}\right)$ when $x \to \infty$, if $p > 3$, and $\gamma_0(x) = O\left(\frac{1}{x}\right)$ when $x \to \infty$, if $p \le 3$. Therefore, from (4.5) and (4.6) we obtain

$$\int_0^{+\infty} \frac{dn(t)}{t + x} = \Delta \int_0^{+\infty} \frac{d(\ln^p t)^+}{t + x}(1 + O(\ln^{-\alpha} x)), \quad x \to \infty,$$

where $\alpha = \min\{p - q, \frac{1}{2}p\}$. From Lemma 8 (with $\nu = 1$, $\beta = 0$, $d = \frac{1}{2}$) and the last relation, we obtain (4.3). Theorem 3 is proved.

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Об асимптотике целых функций конечного логарифмического порядка

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Изучается асимптотическое поведение целой функции, считающая функция $n(t)$ нулей которой удовлетворяет условию $n(t) = \Delta \ln^p t + \Delta_1 \ln^q t + o(\ln^q t)$, $t \to +\infty$, где $0 < q < p < \infty$, $0 < \Delta < \infty$, $-\infty < \Delta_1 < \infty$.

Про асимптотику цілих функцій скінченого логарифмічного порядку

М.М. Шеремета, Р.І. Тарасюк, Н.В. Заболоцький

Вивчається асимптотичне поводження цілої функції, рахуюча функція $n(t)$ нулів якої задовольняє умові $n(t) = \Delta \ln^p t + \Delta_1 \ln^q t + o(\ln^q t)$, $t \to +\infty$, де $0 < q < p < \infty$, $0 < \Delta < \infty$, $-\infty < \Delta_1 < \infty$.