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A note on the Hall-Mergelyan theme

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Let μ be a measure supported by the points e^k, $k = 1, 2, \dots$, with the weights $\mu_k = e^{-sk^2/2}$ where $s > 1$ is a parameter. Then the polynomials are dense in the space $\mathcal{L}^p(\mu)$ for $p < s$ and are not dense in the space $\mathcal{L}^p(\mu)$ for $p > s$. This answers the question posed by Christian Berg and Jens Peter Reus Christensen.

Let μ be a measure supported by the points e^k, $k = 1, 2, \dots$, with the weights $\mu_k = e^{-sk^2/2}$ where $s > 1$ is a parameter. We show that *the polynomials are dense in the space $\mathcal{L}^p(\mu)$ for $p < s$ and are not dense in the space $\mathcal{L}^p(\mu)$ for $p > s$*. This answers the question posed in [BC2] (see also [Berg]). In fact, using results pertaining to the Hamburger moment problem, Berg and Christensen produced (in a somewhat implicit way) a measure with the same property but only for an arbitrary $s \le 2$ [BC1, BC2].

Verifying the aforementioned property of the measure μ, we rely on the well-known Mergelyan-type criterion of density of polynomials in terms of the boundedness of the point evaluation (see [Koo] or [Lev, Theorem 1.1]). Let

$$M_p(z) = \sup |Q(z)|$$

where Q ranges over the set of all polynomials satisfying

$$\left\| \frac{Q(t)}{1+|t|} \right\|_{\mathcal{L}^p(\mu)} \le 1.$$

The function $M_p(z)$ is called a Hall-Mergelyan majorant.

Theorem M. *Let μ be a measure on the real line having moments of all orders and let $S(\mu)$ be its support. In order that the system of polynomials be dense in the space $\mathcal{L}^p(\mu)$ it is sufficient that $M_p(z) = +\infty$ for some $z \notin S(\mu)$ and necessary that $M_p(z) = +\infty$ for all $z \notin S(\mu)$.*

1. Let $p < s$. Consider a sequence of polynomials

$$Q_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{e^k}\right).$$

We show that

$$\left\| \frac{Q_n(t)}{1+|t|} \right\|_{\mathcal{L}^p(\mu)} \to 0 \text{ as } n \to \infty.$$

Since $Q_n(0) = 1$, Theorem M says that the polynomials are dense in $\mathcal{L}^p(\mu)$.

For $m > n$ we have

$$|Q_n(e^m)| \le \frac{e^{nm}}{e^{1+2+\ldots+n}} = \exp\left\{n\left(m - \frac{n+1}{2}\right)\right\}.$$

Therefore,

$$\left\| \frac{Q_n(t)}{1+|t|} \right\|^p_{\mathcal{L}^p(\mu)} =$$

$$= \sum_{m=n+1}^{\infty} \frac{|Q_n(e^m)|^p e^{-sm^2/2}}{(1+e^m)^p} \le \sum_{m=n+1}^{\infty} e^{pn[m-(n+1)/2] - sm^2/2} \le$$

$$\le e^{(p-s)(n+1)^2/2} \sum_{l=0}^{\infty} e^{-sl^2/2} \to 0 \text{ for } n \to \infty \text{ (since } p - s < 0).$$

2. Assume now that $p > s$ and show that

$$\sup\{|P(0)|\} < \infty \tag{1}$$

where P ranges over the set of all polynomials satisfying

$$\left\| \frac{P(t)}{1+|t|} \right\|^p_{\mathcal{L}^p(\mu)} = \sum_{k=1}^{\infty} \frac{|P(e^k)|^p e^{-sk^2/2}}{(1+e^k)^p} \le 1. \tag{2}$$

To verify (1) we use the Lagrange interpolation formula. If the degree of $P(z)$ is less than n, then

$$P(z) = \sum_{k=1}^{n} \frac{P(e^k)}{T'_n(e^k)} \frac{T_n(z)}{z - e^k} \tag{3}$$

where

$$T_n(z) = (z-e)(z-e^2)\cdot\ldots\cdot(z-e^n) = (-1)^n e^{n(n+1)/2} Q_n(z).$$

We use (3) with $z = 0$:

$$P(0) = -\sum_{k=1}^{n} \frac{P(e^k)}{Q'_n(e^k)\, e^k}, \tag{4}$$

and the only thing we need is a lower estimate for $|Q'_n(e^k)|$. We have

$$|Q'_n(e^k)| = \frac{1}{e^k}\prod_{m=1}^{k-1}(e^{k-m}-1)\prod_{m=k+1}^{n}(1-e^{k-m}) \geq$$

$$\geq \frac{e^{k(k-1)}}{e^k e^{k(k-1)/2}}\prod_{m=1}^{\infty}(1-e^{-m})^2 = \text{const}\, e^{k^2/2-3k/2}.$$

Substituting this into (4), we obtain

$$|P(0)| \leq \text{const}\sum_{k=1}^{n}|P(e^k)|\, e^{-k^2/2+3k/2}. \tag{5}$$

Now, it follows from (2) that

$$|P(e^k)| \leq (1+e^k)\exp\left[\frac{s}{p}\frac{k^2}{2}\right].$$

Inserting this estimate into (5), we obtain finally

$$|P(0)| \leq \text{const}\sum_{k=1}^{n} e^{(s/p-1)k^2/2+5k/2} \leq \text{const}\sum_{k=1}^{\infty} e^{(s/p-1)k^2/2+5k/2} < \infty,$$

since $s/p < 1$. We are done.

R e m a r k 1. Our estimates show that the polynomials are also dense in $\mathcal{L}^s(\mu)$. One can show that the corresponding Hamburger moment problem is determinate for $s \geq 2$ and indeterminate for $s < 2$. In the latter case, the measure μ does not give a canonical solution. If one changes slightly the weights and sets up $\mu_k = e^{-sk^2/2+3sk/2}$, then it is possible to show that the corresponding Hamburger moment problem is determinate for $s > 2$, for $s = 2$ the moment problem is indeterminate but the measure μ is still N-extremal, and for $s < 2$ the moment problem is indeterminate and the measure μ is not N-extremal. That choice of weights is equivalent to $\mu_k = [q'(e^k)]^{-s}$ where $q(z)$ is a canonical product with simple zeros at e^k. Another minor change of weights μ_k provides a measure μ such that the polynomials are dense in $\mathcal{L}^p(\mu)$ for $p < s$ and are not dense for $p \geq s$.

R e m a r k 2. As it follows from the well-known theorem which is due essentially to Akhiezer (see [Koo] or [Lev]), *if the polynomials are not dense in the space* $\mathcal{L}^p(\mu)$, *then each function in their closure is a restriction of an entire function of the Cartwright class and zero exponential type*. Thus, it would be essentially more interesting to construct measures with the same property as above whose support is a set of uniqueness for this class of entire functions (say, measures supported by the set of integers or even by the whole real axis). It is not clear to the writer of this note whether such measures exist at all.

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Заметка на тему Холла–Мергеляна

М. Содин

Пусть μ - мера, сосредоточенная в точках e^k, $k = 1, 2, \ldots$, с весами $\mu_k = e^{-sk^2/2}$, где $s > 1$ – параметр. Тогда полиномы плотны в пространстве $\mathcal{L}^p(\mu)$ при $p < s$ и не плотны в пространстве $\mathcal{L}^p(\mu)$ при $p > s$. Этот факт дает ответ на вопрос, поставленный Кристианом Бергом и Иенсом П.Р. Кристиансеном.

Нотаток до теми Холла–Мергеляна

М. Содін

Нехай μ - міра, що зосереджена у точках e^k, $k = 1, 2, \dots$, з вагою $\mu_k = e^{-sk^2/2}$, де $s > 1$ – параметр . Тоді поліноми щільні у просторі $\mathcal{L}^p(\mu)$, де $p < s$, і не щільні у просторі $\mathcal{L}^p(\mu)$, де $p > s$. Цей факт дає відповідь на питання, що було поставлене Крістіаном Бергом та Йенсом П.Р. Крістіансеном.