Matematicheskaya fizika, analiz, geometriya
1996, v. 3, No 1/2, pp. 169–213

Wave operators of Deift-Simon type for a class of Schrödinger evolutions. I

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Received November 9, 1994

We are interested in questions of the scattering theory concerning the asymptotic behaviour of some Schrödinger evolutions. More precisely we present some results of the asymptotic completeness obtained by the method of Deift–Simon wave operators recently developed in the theory of N-body systems. We consider here only the 2-body case, treating a class of general time-dependent hamiltonians, e.g. $H(t) = H_0 + V(t, x)$ with H_0 being a second order differential operator with constant coefficients and $V(t, x)$ decaying suitably when $|x| \to \infty$.

1. Introduction

The idea of Deift and Simon [DS] to modify the wave operators with suitable cut-off functions in the configuration space is motivated by the analysis of Schrödinger operators

$$H = H_0 + V(x), \tag{1.1}$$

with $H_0 = -\Delta$ and $V(x)$ being the operator of multiplication by the function $V(x_1, \dots, x_N) = \sum_{1 \le j < k \le N} V_{jk}(x_j - x_k)$, describing the potential of the N-body problem, where the nature of interactions is clearly different in different sectors of the configuration space $\mathbf{R}^{Nd}$. The use of some microlocal cut-off functions was the main tool of the first proof of the asymptotic completeness for short-range N-body systems given by Sigal and Soffer [SS1]. Further developments of the approach are due to Graf [Gr] (a new proof of the result of [SS1]) and Derezinski [De] (the proof of the asymptoptic completeness for long-range N-body systems, cf. also [SS2], [Z2]).

The aim of this paper is to present an approach based on Deift-Simon wave operators in the case of 2-body scattering, i.e. the interaction potential $V(x)$ decays when $|x| \to \infty$.

Let us note first that under a suitable assumption concerning the decay of $V(x)$ when $|x| \to \infty$, it is well known (cf. e.g. [Hö], [DKY], [Sim], [Um], [M2], [Ar]) that the

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asymptotic completeness holds for $H_0 = p(D)$ being a differential (or pseudodifferential) operator satisfying some conditions on critical points of its symbol.

From this point of view the 2-body case is totally different from the general N-body case, where the hypothesis $H_0 = -\Delta$ is essential (there are no general results in scattering if H_0 is not a Laplace operator). Similarly, by now the method of Deift–Simon operators has been used only in the case $H_0 = -\Delta$ and below [the points A) and B)] we first describe known results and ideas in this case (following [Gr] and [De]). In the point C) below we formulate our result of the asymptotic completeness and in D) the analogical theory is described for time-dependent hamiltonians $H(t) = H_0 + V(t, x)$ for $H_0 = p(D)$ being a suitable differential (or pseudodifferential) operator and $V(t, x)$ decaying suitably when $|x| \to \infty$.

We would like to note here that the ideas presented in the point D) should not be considered as a trial of developing a more general abstract theory, but as the ideas having applications in practical scattering problems. In fact, they have turned out to be a fundamental tool of an analysis of some time-dependent many body quantum models studied in a subsequent paper [Z3] (cf. also [Zl]).

A) Deift–Simon wave operators and the asymptotic velocity

Let H be the Schrödinger operator on $L^2(\mathbf{R}^d)$ of the form (1.1) with $H_0 = -\Delta$ and assume that the potential V can be decomposed in a sum of the short and long range part,

$$V = V^s + V^\ell, \tag{1.2}$$

where V^s, V^ℓ are bounded measurable functions on $\mathbf{R}^d$ going to 0 when $|x| \to \infty$ and

$$\exists C,\ \varepsilon > 0 \text{ such that } |V^s(x)| + |\nabla V^\ell(x)| \le C(1 + |x|)^{-1-\varepsilon}. \tag{1.2'}$$

Such a decay hypothesis allows to obtain easily the basic spectral properties of H via Mourre estimates, e.g. the fact that the only possible accumulation point of eigenvalues is 0 ([ABG], [BG]). Further on for an open set $\mathcal{U} \subset \mathbf{R}^d$, $C_0^\infty(\mathcal{U})$ denotes the set of C^∞ functions with compact support in $\mathcal{U}$. Following [Gr] we may affirm that for $J \in C_0^\infty(\mathbf{R}^d)$ the limit

$$\Gamma_H(J)\varphi = \lim_{t \to \infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH}\varphi, \tag{1.3}$$

exists (in the norm of $L^2(\mathbf{R}^d)$) for every $\varphi \in L^2(\mathbf{R}^d)$. The formula (1.3) defines a linear operator $\Gamma_H(J): L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ which will be called the Deift–Simon wave operator.

Note that if J is a continuous functions on $\mathbf{R}^d$ going to 0 when $|x| \to \infty$, then using a sequence of functions from $C_0^\infty(\mathbf{R}^d)$ converging to J uniformly on $\mathbf{R}^d$, we check easily that the limit (1.3) still exists. Since the limit (1.3) exists trivially if J is a constant function, we may denote by $\mathcal{A}$ the algebra of continuous functions on $\mathbf{R}^d$ having a limit when $|x| \to \infty$ and we may define a family $\{\Gamma_H(J)\}_{J \in \mathcal{A}}$ being a subalgebra of $B(L^2(\mathbf{R}^d))$ [the algebra of linear bounded operators on $L^2(\mathbf{R}^d)$], because $\Gamma_H(J_1 + J_2) = \Gamma_H(J_1) + \Gamma_H(J_2)$ and $\Gamma_H(J_1 J_2) = \Gamma_H(J_1)\Gamma_H(J_2)$ due to the chain rule

$$\Gamma_H(J_1)\Gamma_H(J_2)\varphi =$$

$$= \lim_{t\to\infty} e^{itH} J_1\left(\frac{x}{t}\right) e^{-itH} \lim_{t\to\infty} e^{itH} J_2\left(\frac{x}{t}\right) e^{-itH} \varphi = \Gamma_H(J_1 J_2)\varphi.$$

It is shown in [Gr] that the following union of linear subspaces of $L^2(\mathbf{R}^d)$,

$$\mathcal{D}_0 = \bigcup \{\operatorname{Ran} \Gamma_H(J) : J \in C_0^\infty(\mathbf{R}^d)\} \tag{1.4}$$

is dense in $L^2(\mathbf{R}^d)$. If $\varphi \in \mathcal{D}_0$ then $\varphi = \Gamma_H(J_1)\psi$ for certain $\psi \in L^2(\mathbf{R}^d)$, $J_1 \in C_0^\infty(\mathbf{R}^d)$ and for $J \in C_0^\infty(\mathbf{R}^d)$ satisfying $J_1 = JJ_1$ we get $\varphi = \Gamma_H(J_1)\psi = = \Gamma_H(JJ_1)\psi = \Gamma_H(J)\Gamma_H(J_1)\psi = \Gamma_H(J)\varphi$, i.e.

$$\mathcal{D}_0 = \{\varphi \in L^2(\mathbf{R}^d) : \exists\ J \in C_0^\infty(\mathbf{R}^d),\ \varphi = \Gamma_H(J)\varphi\}. \tag{1.4'}$$

If $\varphi \in \mathcal{D}_0$, $1 \le j \le d$ and $\varphi = \Gamma_H(J)\varphi$ then there exists the limit

$$\mathcal{V}_j \varphi = \lim_{t\to\infty} e^{itH} \frac{x_j}{t} e^{-itH} \varphi = \lim_{t\to\infty} e^{itH} \frac{x_j}{t} J\left(\frac{x}{t}\right) e^{-itH} \varphi \tag{1.5}$$

and it is easy to check that $\vec{\mathcal{V}}_H = (\mathcal{V}_1, \ldots, \mathcal{V}_d)$ is a family of commuting self-adjoint operators, called (following [De]) the asymptotic velocity. Indeed, if Z is a compact set of $\mathbf{R}^d$, then we may define $E_Z(\vec{\mathcal{V}}_H)$ as the orthogonal projection on the intersection of the family of subspaces $\{\operatorname{Ran}\Gamma_H(J) : J \in C_0^\infty(\mathbf{R}^d),\ J \ge 0,\ J(\lambda) \ge 1$ for $\lambda \in Z\}$, which will be denoted

$$E_Z(\vec{\mathcal{V}}_H) = \inf \{ \Gamma_H(J) : J \in C_0^\infty(\mathbf{R}^d),\ J \geq 0,\ J(\lambda) \geq 1 \text{ for } \lambda \in Z \} \quad (1.6)$$

and for any Borel set $B \subset \mathbf{R}^d$, define $E_B(\vec{\mathcal{V}}_H)$ as the orthogonal projection on the closed linear subspace generated by the family $\{ \operatorname{Ran} E_Z(\vec{\mathcal{V}}_H) : Z$ is compact and $Z \subset B \}$, denoting

$$E_B(\vec{\mathcal{V}}_H) = \sup \{ E_Z(\vec{\mathcal{V}}_H) : Z \text{ is compact and } Z \subset B \}. \quad (1.6')$$

Then $E_{\mathbf{R}^d}(\vec{\mathcal{V}}_H) = I$, $B \to (E_B(\vec{\mathcal{V}}_H)\varphi, \psi)$ is a complex Borel measure of $\mathbf{R}^d$ for every $\varphi, \psi \in L^2(\mathbf{R}^d)$, and $\Gamma_H(J) = J(\vec{\mathcal{V}}_H)$ holds for $J \in \mathcal{A}$, where

$$(J(\vec{\mathcal{V}}_H)\varphi, \psi) = \int_{\mathbf{R}^d} J(\lambda_1, \ldots, \lambda_d)\, d(E_{\lambda_1, \ldots, \lambda_d}(\vec{\mathcal{V}}_H)\varphi, \psi).$$

Setting $E_{B_j}(\mathcal{V}_j) = E_{\{(\lambda_1, \ldots, \lambda_d) : \lambda_j \in B_j\}}(\vec{\mathcal{V}}_H)$ for Borel sets $B_j \subset \mathbf{R}$, $1 \leq j \leq d$, we have

$$E_{B_1 \times \ldots \times B_d}(\vec{\mathcal{V}}_H) = E_{B_1}(\mathcal{V}_1) \ldots E_{B_d}(\mathcal{V}_d)$$

and $\mathcal{V}_j$ may be defined as the self-adjoint operator given by the integral representation

$$(J(\mathcal{V}_j)\varphi, \psi) = \int_{\mathbf{R}} J(\lambda)\, d(E_\lambda(\mathcal{V}_j)\varphi, \psi)$$

for a Borel function J on $\mathbf{R}$, $\psi \in L^2(\mathbf{R}^d)$ and φ in the domain of $J(\mathcal{V}_j)$.

Finally let us note that if $\varphi \in L^2(\mathbf{R}^d)$ is an eigenvector of H associated with the eigenvalue λ, i.e. $H\varphi = \lambda\varphi$, and $\bar{J} \in \mathcal{A}$ is such that $\bar{J}(0) = 0$, then for $t \to \infty$ we have

$$\| e^{itH} \bar{J}(\tfrac{x}{t}) e^{-itH} \varphi \| = \| e^{itH} \bar{J}(\tfrac{x}{t}) e^{-it\lambda} \varphi \| = \| \bar{J}(\tfrac{x}{t}) \varphi \| \to 0$$

due to the Lebesgue dominated convergence theorem (the family of operators $\bar{J}(x/t)$ converges strongly to 0 when $t \to \infty$), i.e. $\Gamma_H(\bar{J})\varphi = 0$. This implies $E_{\mathbf{R}^d \setminus \{0\}}(\vec{\mathcal{V}}_H)\varphi = 0$ if φ is an eigenvector of H or more generally if $\varphi \in \mathcal{H}_p(H)$, where $\mathcal{H}_p(H)$ denotes the closed subspace generated by all eigenvectors. Therefore $\mathcal{H}_p(H) \subset \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}_H)$ and following [Gr] we may affirm that the last inclusion appears to be the equality and we have

$$\mathcal{H}_p(H) = \text{Ran}\, E_{\{0\}}(\vec{\mathcal{V}}_H),\ \mathcal{H}_c(H) = \text{Ran}\, E_{\mathbf{R}^d\setminus\{0\}}(\vec{\mathcal{V}}_H), \tag{1,7}$$

where $\mathcal{H}_c(H)$ is the continuous subspace of H, i.e. the orthogonal complement of $\mathcal{H}_p(H)$. In other words, the eigenvectors may be characterized as the states with zero asymptotic velocity and the vectors from the continuous subspace of H may be characterized as the states with non-zero asymptotic velocity.

B) Deift-Simon wave operators and the asymptotic completeness

We are interested in the formulation of the asymptotic completeness in the spirit of [BG], [ABG], comparing evolutions e^{-itH_1} and e^{-itH_2}, where (H_1, H_2) is a pair of Schrödinger operators of the type considered in A):

$$H_k = H_0 + V_k^s + V_k^\ell \ \text{for}\ k = 1, 2, \tag{1.1'}$$

with $H_0 = -\Delta$, V_k^s, V_k^ℓ bounded measurable, going to 0 when $|x| \to \infty$ and satisfying

$$|V_k^s(x)| + |\nabla V_k^\ell(x)| \le C(1 + |x|)^{-1-\varepsilon} \ \text{with}\ \varepsilon > 0 \ \text{for}\ k = 1, 2. \tag{1.2'}$$

If the long range parts of considered Schrödinger operators are equal, i.e. $V_1^\ell = V_2^\ell$, then (cf. [BG], [ABG] or [Gr]) there exist the limits

$$\Omega_{H_2, H_1}\varphi = \lim_{t\to\infty} e^{itH_2} e^{-itH_1}\varphi \ \text{if}\ \varphi \in \mathcal{H}_c(H_1), \tag{1.8}$$

$$\Omega_{H_1, H_2}\psi = \lim_{t\to\infty} e^{itH_1} e^{-itH_2}\psi \ \text{if}\ \psi \in \mathcal{H}_c(H_2) \tag{1.8'}$$

and $\Omega_{H_2, H_1}: \mathcal{H}_c(H_1) \to \mathcal{H}_c(H_2)$ is an isometric bijection with $\Omega^{-1}_{H_2, H_1} = \Omega_{H_1, H_2}$.

The last statement will be called the asymptotic completeness and Ω_{H_2, H_1}, will be called the wave operator associated with the pair (H_1, H_2). In other words, if the asymptotic completeness holds then for every $\psi \in \mathcal{H}_c(H_2)$ there is $\varphi \in \mathcal{H}_c(H_1)$ satisfying $\psi = \Omega_{H_2, H_1}\varphi$, which implies $\| e^{-itH_2}\psi - e^{-itH_1}\varphi \| \to 0$ when $t \to \infty$.

Let us describe how the asymptotic completeness can be proved if we know that for every $\varphi, \psi \in L^2(\mathbf{R}^d)$ and $\mathcal{J} \in \mathcal{A}$ such that $\mathcal{J}(0) = 0$, there exist the limits

$$\Omega_{H_2, H_1}(\mathcal{J})\varphi = \lim_{t\to\infty} e^{itH_2}\, \mathcal{J}\left(\frac{x}{t}\right) e^{-itH_1}\varphi, \tag{1.9}$$

$$\Omega_{H_1, H_2}(\bar{J})\psi = \lim_{t\to\infty} e^{itH_1}\, \bar{J}\,(\tfrac{x}{t})\, e^{-itH_2}\psi. \qquad (1.9')$$

Following [Gr], the operators $\Omega_{H_2, H_1}(\bar{J})$, $\Omega_{H_1, H_2}(\bar{J}) \in B(L^2(\mathbf{R}^d))$ defined by (1.9), (1.9′) will be called the Deift–Simon wave operators.

In fact, if $\varphi = \Gamma_{H_1}(\bar{J})\,\tilde{\varphi}$ with $\tilde{\varphi} \in L^2(\mathbf{R}^d)$, $\bar{J} \in \mathcal{A}$ and $\bar{J}(0) = 0$, then the existence of the limit

$$\Omega_{H_2, H_1}\varphi = \lim_{t\to\infty} e^{itH_2} e^{-itH_1} \lim_{t\to\infty} e^{itH_1} \bar{J}(\tfrac{x}{t})\, e^{-itH_1}\tilde{\varphi} = \Omega_{H_2, H_1}(\bar{J})\,\tilde{\varphi}$$

is a consequence of the existence of the corresponding Deift–Simon wave operator. But the set of vectors $\varphi \in \operatorname{Ran}\Gamma_{H_1}(\bar{J})$ with $\bar{J} \in \mathcal{A}$ and $\bar{J}(0) = 0$, is dense in $\operatorname{Ran} E_{\mathbf{R}^d \setminus \{0\}}(\vec{\mathcal{V}}_{H_1}) = \mathcal{H}_c(H_1)$, hence the limit (1.8) exists for every $\varphi \in \mathcal{H}_c(H_1)$. Replacing H_1 by H_2 in the above reasoning we get the existence of (1.8′) for every $\psi \in \mathcal{H}_c(H_2)$.

C) Formulation of the result for time independent hamiltonians

We want to compare evolutions e^{-itH_k} for $k = 1, 2$, where H_k are self-adjoint operators with the domains $D(H_1) = D(H_2) = \mathcal{D}$, being Schrödinger operators of the form (1.1′), where $H_0 = p(D)$ is e.g. a second order differential operator which is not necessarily elliptic, but having its principal part non-degenerated.

More generally we assume that $H_0 = p(D)$ is a self-adjoint operator with the domain $D(H_0) = \mathcal{D}$, where $D = (-i\partial_{x_1}, \dots, -i\partial_{x_d})$, $p \in C^\infty(\mathbf{R}^d)$ is real, satisfying

$$\sup_{\xi \in \mathbf{R}^d} |\partial^\alpha p(\xi)| < \infty \text{ for } \alpha \in \mathbf{N}^d \text{ such that } |\alpha| \geq 2 \qquad (\mathbf{H}_1)$$

and for $k = 1, 2$, $H_k = H_0 + V_k$, where $V_k = V_k^s + V_k^\ell$ with the short range part V_k^s being an H_0-bounded operator defined on $\mathcal{D}$ (not necessarily a multiplication operator) satisfying

$$\| \mathbf{1}_{\{|x| \geq t\}}(x)\, V_k^s E_{[-\bar{r}; \bar{r}]}(H_k) \| \in L^1([1; \infty); dt), \text{ for every } \bar{r} > 0, \qquad (\mathbf{H}_2 a)$$

the long range part V_k^ℓ being a multiplication operator by a real function $V_k^\ell(x) \in L^\infty(\mathbf{R}^d)$ such that $[i V_k^\ell(x), \nabla_j p(D)]$ defines a bounded operator satisfying

$$\| \mathbf{1}_{\{|x| \geq t\}}(x)\,[i\,V_k^{\ell}(x), \nabla_j p(D)]\,\| \in L^1([1;\infty); dt), \text{ for } j = 1, \dots, d, \quad (\mathrm{H}_2 b)$$

where $F(t) \in L^1([1;\infty); dt)$ means $\int_1^{\infty} |F(t)|\,dt < \infty$, $\mathbf{1}_{\Theta}(\cdot)$ denotes the characteristic function of $\Theta \subset \mathbf{R}^d$ and for symmetric operators A, B with the intersection of domains $\mathcal{D}(A) \cap \mathcal{D}(B)$ dense in $L^2(\mathbf{R}^d)$, $[iA, B]$ is considered as the quadratic form given by

$$\langle [iA, B]\,\varphi, \psi \rangle = i(B\varphi, A\psi) - i(A\varphi, B\psi), \text{ for } \varphi, \psi \in \mathcal{D}(A) \cap \mathcal{D}(B), \quad (1.10)$$

$(\cdot, \cdot)$ being the scalar product of $L^2(\mathbf{R}^d)$. Clearly it defines a bounded operator if and only if

$$\sup_{\varphi, \psi \in \mathcal{D}(A) \cap \mathcal{D}(B),\ \|\varphi\| \leq 1,\ \|\psi\| \leq 1} |\langle [iA, B]\,\varphi, \psi \rangle| < \infty .$$

Finally, we consider for $k = 1, 2$ and every $h \in C_0^{\infty}(\mathbf{R})$ the following compactness hypotheses

$$h(H_k) - h(H_0) \text{ is a compact operator,} \quad (\mathrm{H}_3 a)$$

$$(1 + |x|)\, V_k^s\, h(H_k) \text{ is a compact operator,} \quad (\mathrm{H}_3 b)$$

the quadratic form $[ix_j V_k^{\ell}(x), \nabla_m p(D)]$ defines a bounded operator and $[ix_j V_k^{\ell}(x), \nabla_m p(D)]\, h(H_0)$, $V_k^{\ell}(x)\, h(H_0)$ are compact operators for $j, m = 1, \dots, d$. $(\mathrm{H}_3 c)$

We have

Theorem 1.1. *Assume that H_k, $k = 1, 2$, are given by (1.1′), the hypotheses ($\mathrm{H}_1 - \mathrm{H}_3$) are satisfied and $V_1^{\ell} = V_2^{\ell}$ holds. Let $\mathcal{R}$ be an open subset of* $\mathbf{R}$ *such that*

$$\lambda \in \mathcal{R} \Leftrightarrow \exists\ \delta > 0, \quad \inf_{\{\xi : \lambda - \delta < p(\xi) < \lambda + \delta\}} |\nabla p(\xi)| > 0, \quad (1.11)$$

where $\nabla p(\xi) = (\nabla_1 p(\xi), \dots, \nabla_d p(\xi)) = (\partial_{\xi_1} p(\xi), \dots, \partial_{\xi_d} p(\xi))$. Then the following limit exists

$$s - \lim_{t \to \infty} e^{itH_2} e^{-itH_1} E_c(H_1)\, E_{\mathcal{R}}(H_1). \quad (1.12)$$

R e m a r k s. 1°. If $\mathbf{R} \setminus \mathcal{R}$ is finite or countable, then $E_c(H_k)\,E_{\mathcal{R}}(H_k) = E_c(H_k)$ and Theorem 1.1 implies the existence of (1.8). Interchanging H_1 and H_2, we get the existence of (1.8′) and the asymptotic completeness holds. In the particular case when $p(D)$ is a second order differential operator and its principal part is a non-degenerated quadratic form, it is clear that the set $\mathbf{R} \setminus \mathcal{R}$ has only one element (the critical value of p).

2°. The hypothesis $(\mathbf{H}_1)$ is chosen to fit easily with the structure of the cut-off in x/t. If $|p(\xi)| \to \infty$ when $|\xi| \to \infty$, then it can be seen that energy cut-offs $E_{[-\bar{r};\,\bar{r}]}(H_k)$ allow to forget about the hypothesis $(\mathbf{H}_1)$. A more general approach avoiding the hypothesis $(\mathbf{H}_1)$, will be presented in the second part of the paper.

3°. The hypotheses $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ are formulated in the way we use in the proof and we give some comments about them in Appendix. Here we note only that it can be easily seen that $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ are satisfied if e.g. (1.2′) holds and H_0 is a second order differential operator with its principal part non-degenerated (cf. the compactness results of [DM]).

4°. We note that (1.2′) is a very weak condition on $V_k^{\ell}(x)$, because it involves only its first order derivatives. If stronger conditions are imposed on $V_k^{\ell}(x)$, then it is possible to compare e^{-itH_k} $(k = 1, 2)$ with a suitable evolution generated by a modified time-dependent pseudo-differential operator with constant coefficients (independent on x), cf. [E2], [Hö], [M1], [M3], [KY],[Co], [If], [Sig], [Pa]. According to [DG], conditions on second order derivatives of $V^{\ell}(x)$ can not be avoided in this construction.

D) Formulation of results concerning time-dependent hamiltonians

Let $\mathcal{H} = L^2(\mathbf{R}^d)$, $H_0 = p(D)$ with $p \in C^\infty(\mathbf{R}^d)$ real, satisfying $(\mathbf{H}_1)$ and let $\mathcal{D}$ denote the domain of H_0 with the graph norm $\|\varphi\|_{\mathcal{D}} = \|\varphi\| + \|H_0\varphi\|$. We shall say that the family $\{H(t)\}_{t \ge s}$ of self-adjoint operators in $\mathcal{H}$ is H_0-admissible if $H(t)$ is H_0-bounded (hence defined on $\mathcal{D}$) and there exists a family $\{U(t, s)\}_{t \ge s}$ of unitary operators in $\mathcal{H}$ such that

a) $\varphi \mapsto U(t, s)\varphi$ is continuous $[s; \infty) \to \mathcal{H}$ for every $\varphi \in \mathcal{H}$;

b) $\varphi \mapsto U(t, s)\varphi$ is continuous $[s; \infty) \to \mathcal{D}$ for every $\varphi \in \mathcal{D}$;

c) $\varphi \mapsto U(t, s)\varphi$ is continuously differentiable $[s; \infty) \to \mathcal{H}$ for every $\varphi \in \mathcal{D}$ with

$$i\frac{d}{dt}U(t, s)\varphi = H(t)\,U(t, s)\varphi, \quad U(s, s)\varphi = \varphi. \tag{1.14}$$

We want to compare evolutions $U_k(t, s)$ for $k = 1$ and 2, associated with H_0-admissible families of Schrödinger operators $\{ H_k(t) \}_{t \geq s}$ of the form

$$H_k(t) = H_0 + V_k(t) \ \text{with} \ V_k(t) = V_k^s(t) + V_k^\ell(t), \tag{1.15}$$

the short range part $V_k^s(t)$ being an H_0-bounded operator defined on $\mathcal{D}$ satisfying

$$\| \mathbf{1}_{\{|x| \geq t\}}(x) V_k^s(t) \| \in L^1([1; \infty); dt), \tag{$\mathrm{H}_2'a$}$$

the long range part $V_k^\ell(t)$ being a multiplication operator by a real function $V_k^\ell(t, x) \in L^\infty(\mathbf{R}^d)$ such that the quadratic form $[i V_k^\ell(t, x), \nabla_j p(D)]$ defines a bounded operator satisfying

$$\exists C, \varepsilon_0 > 0, \| \mathbf{1}_{\{|x| \geq t\}}(x) [\nabla_j p(D), V_k^\ell(t, x)] \| \leq C t^{-1-\varepsilon_0}, \text{ for } j = 1, \ldots, d \tag{$\mathrm{H}_2'b$}$$

We have

Theorem 1.2. *Assume that for $k = 1, 2$, $\{ H_k(t) \}_{t \geq s}$ are as above, the hypotheses (H_1), (H_2') are satisfied and $V_1^\ell(t) = V_2^\ell(t)$ holds. Let $\mathcal{J} \in \mathcal{A}$ be such that $\mathcal{J}(0) = 0$, where we have denoted $\mathcal{A} = \{ J \in C(\mathbf{R}^d) : \lim_{|x| \to \infty} J(x)$ exists $\}$ as before. Then the following limit exists*

$$\Omega_{2,1}(\mathcal{J}; s) = s - \lim_{t \to \infty} U_2(t, s)^* \mathcal{J}\left(\frac{x}{t}\right) U_1(t, s). \tag{1.16}$$

If $J \in C^\infty(\mathbf{R}^d)$ is such that $\lim_{|x| \to \infty} J(x)$ exists, then using Theorem 1.2 in the case $H_1(t) = H_2(t)$ with $\mathcal{J} = J - J(0)$ we get the existence of

$$\Gamma_k(J; s) = s - \lim_{t \to \infty} U_k(t, s)^* J\left(\frac{x}{t}\right) U_k(t, s) =$$

$$= s - \lim_{t \to \infty} U_k(t, s)^* \mathcal{J}\left(\frac{x}{t}\right) U_k(t, s) + J(0)I, \tag{1.17}$$

for $k = 1, 2$ and as before on $\mathcal{D}_k = \bigcup \{ \operatorname{Ran} \Gamma_k(J; s) : J \in C_0^\infty(\mathbf{R}^d) \}$ we may define $\vec{\mathcal{V}}_k(s)$ such that $J(\vec{\mathcal{V}}_k(s)) = \Gamma_k(J; s)$ (note that we do not have any energy estimates in general and it is possible that $\mathcal{D}_k$ is not dense in $L^2(\mathbf{R}^d)$. Following the analogy with the situation described in B) we define $\mathcal{H}_k^{\text{noscat}}(s) = \operatorname{Ran} E_{\{0\}}(\vec{\mathcal{V}}_k(s))$ and

$\mathcal{H}_k^{\mathrm{scat}}(s)$ as the orthogonal complement of $\mathcal{H}_k^{\mathrm{noscat}}(s)$ in $\mathcal{H} = L^2(\mathbf{R}^d)$, i.e. $\mathcal{H}_k^{\mathrm{noscat}}(s)$ is the intersection of the family of subspaces

$$\{ \operatorname{Ran} \Gamma_k(J\,;s) : J \in C_0^\infty(\mathbf{R}^d),\ \ J = 1 \text{ in a neighbourhood of } 0 \},$$

and

$$\mathcal{H}_k^{\mathrm{scat}}(s) \text{ is the closure of}$$

$$\bigcup \{ \operatorname{Ran} \Gamma_k(1 - J\,;s) : J \in C_0^\infty(\mathbf{R}^d),\ \ J = 1 \text{ in a neighbourhood of } 0 \}. \tag{1.18}$$

Following the analogy with B), we state the asymptotic completeness as

Theorem 1.3. *If for* $k = 1, 2$, $\{ H_k(t) \}_{t \geq s}$ *are as above, the hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}'_2)$ *are satisfied and* $V_1^{\ell}(t) = V_2^{\ell}(t)$ *holds, then for every* $\varphi \in \mathcal{H}_1^{\mathrm{scat}}(s)$, *the limit*

$$\Omega_{2,1}(s)\,\varphi = \lim_{t \to \infty} U_2(t, s)^* \, U_1(t, s)\,\varphi \tag{1.19}$$

exists and belongs to $\mathcal{H}_2^{\mathrm{scat}}(s)$. *The following application is an isometric bijection*

$$\Omega_{2,1}(s) : \mathcal{H}_1^{\mathrm{scat}}(s) \to \mathcal{H}_2^{\mathrm{scat}}(s). \tag{1.20}$$

Note that $\mathcal{H}_k^{\mathrm{scat}}(s)$ are not necessecarily orthogonal complements of bound states in a classical meaning (cf. [Ya], [Ho]) and another definitions were introduced in [KY], [It]. However all definitions coincide in the case when $H_k(t) = H_k$ does not depend on t, due to

Theorem 1.4. *Assume that for* $k = 1$ *or* 2, $H_k(t)$ *satisfies* $(\mathbf{H}_1)$, $(\mathbf{H}'_2)$ *and* $H_k(t) = H_k$ *does not depend on t. If* $(\mathbf{H}_3)$ *holds and* $\mathbf{R} \setminus \mathcal{R}$ *is finite or countable, then* $\mathcal{H}_k^{\mathrm{scat}}(s) = \mathcal{H}_c(H_k)$.

Let us also note that we have chosen not to look for the formulations of hypotheses as weak as possible. For instance without any changes of statements and proofs, $\mathcal{D}$ can be a densly embedded Frechet space with the topology not weaker then the graph topology of the domain of H_0 (and there is no need to assume V_k to be H_0-bounded). Also some hypotheses may be replaced by other ones, e.g. the hypothesis that $V_k^s(t)$ are bounded in the region $|x| \geq 1$ (or $|x| \geq r_0$), connected with the fact that we have no energy control. In fact, if one knows e.g. that $\{ U(t, s)_{t \geq s} \}$ is a bounded family of operators $\mathcal{D} \to \mathcal{D}$, then the norm $\| \cdot \|$ in ($\mathbf{H}_2'$ *a*) can be replaced by the norm of $\mathcal{D}$.

2. Basic propagation estimates

Further on we denote $\mathcal{H} = L^2(\mathbf{R}^d)$, $(\cdot\,,\cdot)$ is the scalar product and $\| \cdot \|$ is the norm of $\mathcal{H}$ or the norm of $B(\mathcal{H})$, the space of bounded operators on $\mathcal{H}$. In this section we assume that H_k satisfy ($\mathbf{H}_1$), ($\mathbf{H}_2$) for $k = 1, 2$, and we drop the index k, writing simply $H = H_0 + V$ instead of $H_k = H_0 + V_k$. For $\varphi \in \mathcal{H}$, $t \in \mathbf{R}$, we write $e^{-itH}\varphi = \varphi_t$ and set $\mathcal{K} = H^\infty(\mathbf{R}^d)$ with

$$H^\infty(\mathbf{R}^d) = \{ \varphi \in C^\infty(\mathbf{R}^d) \colon \ \forall\ \alpha \in \mathbf{N}^d,\ \partial^\alpha \varphi \in \mathcal{H} \}. \tag{2.1}$$

The subspace $\mathcal{K}$ is useful for algebraic operations on differential operators with constant coefficients (functions of D) and operators of multiplication by functions of class $C_b^\infty(\mathbf{R}^d)$ (all derivatives bounded on $\mathbf{R}^d$). Using the usual notation $|_{\mathcal{K}}$ for the restriction to $\mathcal{K}$, we may treat $f(D)|_{\mathcal{K}}$ and multiplication operators $J(x)|_{\mathcal{K}}$ with $J \in C_b^\infty(\mathbf{R}^d)$ as linear operators on $\mathcal{K}$. If A, B are linear operators on $\mathcal{K}$, the commutator $[A, B] = AB - BA$ is a linear operator on $\mathcal{K}$ as well. If A extends to a bounded operator in $\mathcal{H}$, $\| A \|$ denotes the $B(\mathcal{H})$-norm of this extension. If $\theta \in \mathbf{R}$, $A(t)$, $B(t)$ are operators on $\mathcal{K}$ for $t \geq 1$, then we write $A(t) = B(t) + O(t^{-\theta})$ when for every $t \geq 1$, $A(t) - B(t)$ extends to a bounded operator on $\mathcal{H}$ and that there exists a constant C such that $\| A(t) - B(t) \| \leq C t^{-\theta}$ for all $t \geq 1$.

If A_k, $k = 1, 2, \ldots$ are linear symmetric operators on $\mathcal{K}$ [i.e. $(A_k \varphi, \psi) = (\varphi, A_k \psi)$ for $\varphi, \psi \in \mathcal{K}$] then $A_1 A_2 + hc$ denotes the symmetric operator $(A_1 A_2 + A_2 A_1)/2$ and more generally

$$A_1 A_2 \ldots A_n + hc = (A_1 A_2 \ldots A_n + A_n A_{n-1} \ldots A_1)/2 \tag{2.2}$$

is the symmetrization of a product of n symmetric operators. Finally, the notation $A_1 \leq A_2$ means that $(A_1 \varphi, \varphi) \leq (A_2 \varphi, \varphi)$ for all $\varphi \in \mathcal{K}$.

In sections 2 and 3, we make the following additional assumption

$$| p(\xi) | \to \infty \ \text{ when } \ | \xi | \to \infty, \tag{2.3}$$

which simplifies the exposition because of Ran $E_{[-\bar{r};\bar{r}]}(H) \subset \mathcal{K}$ for every $\bar{r} > 0$, allowing to consider all the time only linear operators on $\mathcal{K}$. Later on, in the section 6, we will discribe the possibilities of extensions by means of quadratic forms, allowing to recover all results without the assumption (2.3).

For a self-adjoint operator A, $F_+(A)$ and $F_-(A)$ is the positive and the negative part of A when we define F_+, F_- to be real functions defined by

$$F_+(\lambda) = \lambda\, \mathbb{1}_{[0;\infty)}(\lambda), \;\; F_-(\lambda) = -\lambda\, \mathbb{1}_{(-\infty;0]}(\lambda). \tag{2.4}$$

The result of this section can be stated as follows

Theorem 2.1 **(Basic Propagation Estimates).** *If* $\bar{r}$, r_-, $r_+ \in \mathbf{R}$, $J_0 \in C_0^\infty((r_-, r_+) \setminus \{0\})$, *then there exists a constant* C *such that for all* $\varphi \in \operatorname{Ran} E_{[-\bar{r};\bar{r}]}(H)$,

$$\int_1^\infty \frac{dt}{t} \left\| F_\pm(\nabla_j p(D) - r_\pm)^{1/2} J_0\left(\frac{x_j}{t}\right) \varphi_t \right\|^2 \le C \|\varphi\|^2, \; j = 1, \ldots, d. \tag{2.5$_\pm$}$$

The notation $\pm$ means always two seperate statements: replacing $\pm$ by $+$ everywhere in (2.5$_\pm$), we get (2.5$_+$) and, replacing $\pm$ by $-$ everywhere in (2.5$_\pm$), we get (2.5$_-$). Setting

$$M_\pm(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right) F_\pm(\nabla_j p(D) - r_\pm) J_0\left(\frac{x_j}{t}\right)\Big|_{\mathcal{K}}, \tag{2.6$_\pm$}$$

we may reformulate (2.5$_\pm$) as follows

$$\forall\, T \ge 1, \; \int_1^T dt\, (M_\pm(t)\varphi_t, \varphi_t) \le C \|\varphi\|^2. \tag{2.5$'_\pm$}$$

Note also that since $\|\varphi + \psi\|^2 \le 2\|\varphi\|^2 + 2\|\psi\|^2$, it suffices to consider the cases $\operatorname{supp} J_0 \subset (0;\infty)$ and $\operatorname{supp} J_0 \subset (-\infty; 0)$ separately.

P r o o f o f T h e o r e m 2.1 i n t h e c a s e $\operatorname{supp} J_0 \subset (0;\infty)$. There exist $r'_- > r_-$ and $r'_+ < r_+$, such that $\operatorname{supp} J_0 \subset (r'_-, r'_+) \cap (0;\infty)$ and it is possible to find $J \in C_b^\infty(\mathbf{R})$, $J \ge 0$, $\operatorname{supp} J \subset (0;\infty)$ with the derivative J' such that $J' \ge 0$, $\operatorname{supp} J' \subset (r'_-, r'_+)$ and $2JJ' \ge J_0^2$. Then we can find g_-, $g_+ \in C_b^\infty(\mathbf{R})$ such that

$0 \le g_- \le 1$, $0 \le g_+ \le 1$, $g_+(\lambda) = 1$ for $\lambda \ge r_+$, $g_-(\lambda) = 1$ for $\lambda \le r_-$, $\operatorname{supp} g_+ \subset (r'_+, \infty)$, $\operatorname{supp} g_- \subset (-\infty, r'_-)$, and define $M_0^\pm(t)\colon \mathcal{K} \to \mathcal{K}$ by

$$M_0^\pm(t) = \pm J\left(\frac{x_j}{t}\right) g_\pm(\nabla_j p(D))\, J\left(\frac{x_j}{t}\right)\Big|_{\mathcal{K}}. \qquad (2.7_\pm)$$

For $\varphi \in \operatorname{Ran} E_{[-\bar r; \bar r]}(H)$, $t \to (M_0^\pm(t)\varphi_t, \varphi_t)$ is C^1 on $(0; \infty)$ and

$$\frac{d}{dt}(M_0^\pm(t)\varphi_t, \varphi_t) = ((M_1^\pm(t) + M_2^\pm(t))\varphi_t, \varphi_t), \qquad (2.8_\pm)$$

where

$$M_1^\pm(t) = \mathrm{D}_{H_0} M_0^\pm(t) = \frac{d}{dt} M_0^\pm(t) + [iH_0, M_0^\pm(t)]\colon \mathcal{K} \to \mathcal{K}, \qquad (2.9_\pm)$$

is the Heisenberg derivative with respect to H_0, and

$$M_2^\pm(t) = i\, V M_0^\pm(t) - i\, \overline{M_0^\pm(t)}\, V\colon \mathcal{K} \to \mathcal{H}, \qquad (2.9'_\pm)$$

where $\overline{M_0^\pm(t)}$ is the extension of $M_0^\pm(t)$ on the whole $\mathcal{H}$.

We shall obtain $(2.5'_\pm)$ as an easy consequence of the following two lemmas

Lemma 2.2. *There exists* $C > 0$ *such that* $M_\pm(t) \le CM_1^\pm(t) + Ct^{-2}$, *for all* $t \ge 1$.

Lemma 2.3. *One has* $\left\| E_{[-\bar r; \bar r]}(H)\, M_2^\pm(t)\, E_{[-\bar r; \bar r]}(H) \right\| \in L^1([1; \infty); dt)$.

Indeed, using Lemma 2.2 and $(2.8_\pm)$, we have for $\varphi \in \operatorname{Ran} E_{[-\bar r; \bar r]}(H)$,

$$\int_1^T dt\,(M_\pm(t)\varphi_t, \varphi_t) \le C\int_1^T dt\,(M_1^\pm(t)\varphi_t, \varphi_t) + C\int_1^T dt\, t^{-2}(\varphi_t, \varphi_t) =$$

$$= C\int_1^T dt \frac{d}{dt}(M_0^\pm(t)\varphi_t, \varphi_t) - C\int_1^T dt\,(M_2^\pm(t)\varphi_t, \varphi_t) + C\int_1^T dt\, t^{-2}(\varphi_t, \varphi_t) \le$$

$$\le C\,[(M_0^\pm(t)\varphi_t, \varphi_t)]_1^T +$$

$$+ C\int_1^T dt\left(\left\| E_{[-\bar r; \bar r]}(H)\, M_2^\pm(t)\, E_{[-\bar r; \bar r]}(H)\right\| + t^{-2}\right)\|\varphi\|^2, \qquad (2.10_\pm)$$

giving $(2.5'_\pm)$ due to $\| M_0^\pm(t) \| \le C$ allowing to estimate easily the first term and due to Lemma 2.3 allowing to estimate the second term of the last line. To prove Lemma 2.2 we need

Proposition 2.4. *Assume that* $J \in C_b^\infty(\mathbf{R})$ *is such that the derivative* $J' \in C_0^\infty(\mathbf{R})$ *and* $\eta \in C^\infty(\mathbf{R}^d)$ *is such that the derivatives* $\partial^\alpha \eta \in C_b^\infty(\mathbf{R}^d)$ *if* $|\alpha| \ge 1$. *Then there exists a constant C such that:* a) *we have the following estimate of the norm of the commutator*.

$$\left\| \left[\eta(D), J\left(\frac{x_j}{t}\right) \right] \right\| \le C t^{-1}, \text{ for } t \ge 1,\ j = 1, \dots, d; \tag{2.11a}$$

b) *we have the following Garding type inequality*:

$$\eta \ge 0,\ J \ge 0 \Rightarrow \eta(D) J\left(\frac{x_j}{t}\right) + hc \ge -Ct^{-1}, \text{ for } t \ge 1,\ j = 1, \dots, d; \tag{2.11b}$$

c) *if p satisfies* $(\mathbf{H}_1)$ *then*

$$\left[ip(D), J\left(\frac{x_j}{t}\right) \right] = \frac{1}{t} J'\left(\frac{x_j}{t}\right) \nabla_j p(D) + R_t, \tag{2.11c}$$

with $\| R_t \| \le Ct^{-2}$, *for* $t \ge 1$.

Note that in the case when η and p are symbols of the degree 1 and 2 respectively, i.e.

$$|\partial^\alpha \eta(\xi)| \le C_\alpha (1 + |\xi|)^{1-|\alpha|},\quad |\partial^\alpha p(\xi)| \le C_\alpha (1 + |\xi|)^{2-|\alpha|}, \text{ for all } \alpha \in \mathbf{N}^d,$$

then (2.11, a, b, c) follow easily by standard arguments of the symbolic calculus. The proof of Prop. 2.4 under the general hypotheses considered above is discribed in the Appendix.

Let us note that since $(d/dt) J(x_j/t) = -J'(x_j/t) x_j/t^2$, we may express the Heisenberg derivative of $J(x_j/t)$ with an error $O(t^{-2})$ in our notation:

$$\mathbf{D}_{H_0} J\left(\frac{x_j}{t}\right) = \frac{d}{dt} J\left(\frac{x_j}{t}\right) + \left[iH_0, J\left(\frac{x_j}{t}\right) \right] =$$

$$= \frac{1}{t} J'\left(\frac{x_j}{t}\right)\left(\nabla_j p(D) - \frac{x_j}{t} \right) + O(t^{-2}) = \frac{1}{t}\left(\nabla_j p(D) - \frac{x_j}{t} \right) J'\left(\frac{x_j}{t}\right) + O(t^{-2}). \tag{2.12}$$

P r o o f o f L e m m a 2.2. Using (2.12) we may write

$$M_1^{\pm}(t) = \pm \frac{2}{t} J'\left(\frac{x_j}{t}\right)\left(\nabla_j p(D) - \frac{x_j}{t}\right) g_{\pm}(\nabla_j p(D))\, J\left(\frac{x_j}{t}\right) + hc + O(t^{-2}). \tag{2.13$_\pm$}$$

If $\eta(\xi) = g_{\pm}(\nabla_j p(\xi))$, then $\partial^{\alpha}\eta \in C_b^{\infty}(\mathbf{R}^d)$ for all $\alpha \in \mathbf{N}^d$, hence (2.11a) implies that the commutator of $g_{\pm}(\nabla_j p(D))$ and $J\left(\frac{x_j}{t}\right)$ is $O(t^{-1})$ and using the decomposition

$$\nabla_j p(D) - \frac{x_j}{t} = \left(r'_{\pm} - \frac{x_j}{t}\right) + (\nabla_j p(D) - r'_{\pm}), \tag{2.14$_\pm$}$$

we may write

$$M_1^{\pm}(t) = M_{1,1}^{\pm}(t) + M_{1,2}^{\pm}(t) + O(t^{-2}), \tag{2.14$'_\pm$}$$

with

$$M_{1,1}^{\pm}(t) = \pm \frac{2}{t} JJ'\left(\frac{x_j}{t}\right)\left(r'_{\pm} - \frac{x_j}{t}\right) g_{\pm}(\nabla_j p(D)) + hc, \tag{2.15$_\pm$}$$

$$M_{1,2}^{\pm}(t) = \pm \frac{2}{t} JJ'\left(\frac{x_j}{t}\right)(\nabla_j p(D) - r'_{\pm})\, g_{\pm}(\nabla_j p(D)) + hc. \tag{2.16$_\pm$}$$

Using (2.11b) with $\eta(D) = g_{\pm}(\nabla_j p(D))$ and $\pm JJ'(x_j/t)(r'_{\pm} - x_j/t)$ instead of $J(x_j/t)$, we have

$$\pm JJ'\left(\frac{x_j}{t}\right)\left(r'_{\pm} - \frac{x_j}{t}\right) \geq 0,\ g_{\pm}(\nabla_j p(D)) \geq 0 \Rightarrow M_{1,1}^{\pm}(t) \geq -C_1 t^{-2},$$

for a certain constant C_1. If $\eta(\xi) = (\nabla_j p(\xi) - r'_{\pm})\, g_{\pm}(\nabla_j p(\xi))$, then $\partial^{\alpha}\eta \in C_b^{\infty}(\mathbf{R}^d)$ for all $|\alpha| \geq 1$ and using $2JJ' - J_0^2$ instead of J in (2.11b), we have

$$2JJ' - J_0^2 \geq 0,$$

$$\pm(\nabla_j p(D) - r'_{\pm})\, g_{\pm}(\nabla_j p(D)) \geq 0 \Rightarrow M_{1,2}^{\pm}(t) \geq \tilde{M}_{1,2}^{\pm}(t) - C_1 t^{-2},$$

where

$$\tilde{M}_{1,2}^{\pm}(t) = \pm \frac{1}{t} J_0^2\left(\frac{x_j}{t}\right)(\nabla_j p(D) - r'_{\pm})\, g_{\pm}(\nabla_j p(D)) + hc =$$

$$= \pm \frac{1}{t} J_0\left(\frac{x_j}{t}\right)(\nabla_j p(D) - r'_{\pm})\, g_{\pm}(\nabla_j p(D))\, J_0\left(\frac{x_j}{t}\right) + O(t^{-2}). \tag{2.17$_\pm$}$$

Since there exists $c_0 > 0$ such that $\pm(\lambda - r'_{\pm})\, g_{\pm}(\lambda) \geq c_0 F_{\pm}(\lambda - r_{\pm})$, we can see that for certain constants C_1, C_2, C_3,

$$M_1^{\pm}(t) \geq M_{1,2}^{\pm}(t) - C_1 t^{-2} \geq \tilde{M}_{1,2}^{\pm}(t) - C_2 t^{-2} \geq c_0 M_{\pm}(t) - C_3 t^{-2}. \blacksquare \quad (2.18)$$

P r o o f o f L e m m a 2.3. Clearly

$$\left\| \overline{M_0^{\pm}(t)}\, V^s E_{[-\bar{r}; \bar{r}]}(H) \right\| \leq C \left\| J\left(\frac{x_j}{t}\right) V^s E_{[-\bar{r}; \bar{r}]}(H) \right\| \in L^1([1; \infty); dt),$$

and due to the estimate Lemma A.1 of Appendix,

$$\| [V^{\ell}, M_0^{\pm}(t)] \| \leq 2 \left\| \left[g_{\pm}(\nabla_j p(D)),\, V^{\ell}(x) J\left(\frac{x_j}{t}\right) \right] \right\| \leq$$

$$\leq 2\, \bar{\omega}_0(g_{\pm}) \left\| \left[\nabla_j p(D),\, V^{\ell}(x) J\left(\frac{x_j}{t}\right) \right] \right\| \in L^1([1; \infty); dt). \blacksquare$$

P r o o f o f T h e o r e m 2.1 i n t h e c a s e $\operatorname{supp} J_0 \subset (-\infty; 0)$. Let r'_-, r'_+, g_-, g_+ be as before. Since $\operatorname{supp} J_0 \subset (r'_-, r'_+) \cap (-\infty; 0)$, it is possible to find $J \in C_b^{\infty}(\mathbf{R})$ such that $J \leq 0$, $J' \geq 0$, $\operatorname{supp} J \subset (-\infty; 0)$, $\operatorname{supp} J' \subset (r'_-; r'_+)$, and $-2JJ' \geq J_0^2$. Since now $JJ' \leq 0$, the reasoning analogous to the proof of Lemma 2.2 gives now $M_{\pm}(t) \leq -C M_1^{\pm}(t) + C t^{-2}$ for a certain constant $C > 0$ and to complete the proof it suffices to replace $M_1^{\pm}$, $M_0^{\pm}$ and $M_2^{\pm}$ by $-M_1^{\pm}$, $-M_0^{\pm}$ and $-M_2^{\pm}$ in $(2.10_{\pm})$. $\blacksquare$

3. Existence of wave operators of Deift–Simon type

In this section we make the same hypotheses on H_k as in the section 2.

If $g_m \in C_b^{\infty}(\mathbf{R}^d)$, $m = 1, \ldots, d$, then $g = \otimes g_m$ denotes the function of class $C_b^{\infty}(\mathbf{R}^d)$ given by

$$g(\lambda_1, \lambda_2, \ldots, \lambda_d) = g_1(\lambda_1)\, g_2(\lambda_2) \ldots g_d(\lambda_d). \quad (3.1)$$

Theorem 3.1. *Let* $\bar{J}, g \in C_b^{\infty}(\mathbf{R})$ *and* $g = \otimes g_m$ *with* $g_m \in C_b^{\infty}(\mathbf{R})$, $g_m \geq 0$ *and* $g'_m \in C_0^{\infty}(\mathbf{R})$ *for* $m = 1, \ldots, d$. *Fix* $r, \bar{r} > 0$, $j \in \{1, \ldots, d\}$ *and assume that* supp $g_j \cap [-r, r] = \emptyset$, $\bar{J} = 0$ *in a certain neighbourhood of* 0 *and* supp $\bar{J}' \subset (-r, r)$. *Then the following limit exists*

$$s-\lim_{t\to\infty} E_{[-\bar{r};\bar{r}]}(H_2)\, e^{itH_2}\bar{J}\left(\frac{x_j}{t}\right) g(\nabla p(D)) e^{-itH_1}. \tag{3.2}$$ ∘

P r o o f. Let $\tilde{J} \in C_b^\infty(\mathbb{R})$ be such that supp $\tilde{J}' \subset (-r, r)$, $\tilde{J} \geq 0$, $\tilde{J} = 0$ in a certain neighbourhood of 0 and $\tilde{J} = 1$ on the support of $\bar{J}$. Then setting

$$M_0(t) = \bar{J}\left(\frac{x_j}{t}\right) g(\nabla p(D))\, \tilde{J}\left(\frac{x_j}{t}\right)\Big|_{\mathcal{K}}, \tag{3.3}$$

we note that $\bar{J} = \bar{J}\tilde{J}$ and recall that $\left[g(\nabla p(D)), \tilde{J}\left(\frac{x_j}{t}\right)\right] = O(t^{-1})$ due to (2.11a), hence

$$\bar{J}\left(\frac{x_j}{t}\right)\tilde{J}\left(\frac{x_j}{t}\right) g(\nabla p(D))\Big|_{\mathcal{K}} = M_0(t) + O(t^{-1}). \tag{3.4}$$

Thus it suffices to prove that for any $\bar{r}' > 0$, $\varphi \in \operatorname{Ran} E_{[-\bar{r}';\bar{r}']}(H_1)$, $\lim_{t\to\infty}\Omega_t\varphi$ exists, where

$$\Omega_t\varphi = E_{[-\bar{r};\bar{r}]}(H_2)\, e^{itH_2} M_0(t) e^{-itH_1}\varphi. \tag{3.5}$$

We shall check the Cauchy condition estimating $\|\Omega_{t''}\varphi - \Omega_{t'}\varphi\|$ by

$$\sup_{\|\psi\|\leq 1}\int_{t'}^{t''} dt \left|\frac{d}{dt}(\Omega_t\varphi, \psi)\right| \leq \sup_{\psi\in\operatorname{Ran}E_{[-\bar{r};\bar{r}]}(H_2),\ \|\psi\|\leq 1}(\zeta_1+\zeta_2)(t', t'', \varphi, \psi), \tag{3.6}$$

$$\zeta_k(t', t'', \varphi, \psi) = \int_{t'}^{t''} dt\, |(M_k(t)\varphi_t^1, \psi_t^2)|, \tag{3.7}$$

where $\varphi_t^1 = e^{-itH_1}\varphi$, $\psi_t^2 = e^{-itH_2}\psi$ and

$$M_1(t) = \mathbf{D}_{H_0} M_0(t) = \frac{d}{dt} M_0(t) + [iH_0, M_0(t)], \tag{3.8}$$

$$M_2(t) = iV_2 M_0(t) - i\,\overline{M_0(t)}\, V_1, \tag{3.9}$$

$\overline{M_0(t)}$ being the extension of $M_0(t)$ on the whole $\mathcal{H}$.

We complete the proof of Theorem 3.1 by showing the following lemma

Lemma 3.2. *For* $k = 1, 2$, $\sup_{\psi \in \mathrm{Ran}\, E_{[-\bar{r};\bar{r}]}(H_2),\, \|\psi\| \le 1} \zeta_k(t', t'', \varphi, \psi) \to 0$ *when* $t', t'' \to \infty$.

P r o o f f o r $k = 1$. Let $g_m^{\pm} = g_m$ for $m \neq j$ and write $g_j = g_j^- + g_j^+$ with $\mathrm{supp}\, g_j^- \subset (-\infty; -r)$, $\mathrm{supp}\, g_j^+ \subset (r; \infty)$. Then $g = g_- + g_+$ where $g_{\pm} = \otimes g_m^{\pm}$ and $M_1 = M_{1,1}^+ + M_{1,1}^- + M_{1,2}^+ + M_{1,2}^-$ with

$$M_{1,1}^{\pm}(t) = \left(\mathbf{D}_{H_0} \bar{J}\left(\frac{x_j}{t}\right)\right) g_{\pm}(\nabla p(D))\, \tilde{J}\left(\frac{x_j}{t}\right), \tag{3.10$_\pm$}$$

$$M_{1,2}^{\pm}(t) = \bar{J}\left(\frac{x_j}{t}\right) g_{\pm}(\nabla p(D)) \left(\mathbf{D}_{H_0} \tilde{J}\left(\frac{x_j}{t}\right)\right). \tag{3.11$_\pm$}$$

Introducing $J_0 \in C_0^{\infty}((-r; r) \setminus \{0\})$ such that $J_0 \ge 0$ and $J_0 = 1$ on the support of $\bar{J}'$ and $\tilde{J}'$, we may write $\bar{J}' = \bar{J}' J_0^2$, $\tilde{J}' = \tilde{J}' J_0^2$ and

$$M_{1,1}^{\pm}(t) = \pm \bar{J}'\left(\frac{x_j}{t}\right) \tilde{M}_{\pm}(t)\, \tilde{J}\left(\frac{x_j}{t}\right) + O(t^{-2}), \tag{3.12$_\pm$}$$

$$M_{1,2}^{\pm}(t) = \pm \bar{J}\left(\frac{x_j}{t}\right) \tilde{M}_{\pm}(t)\, \tilde{J}'\left(\frac{x_j}{t}\right) + O(t^{-2}) \tag{3.13$_\pm$}$$

with

$$\tilde{M}_{\pm}(t) = \pm \frac{1}{t} J_0\left(\frac{x_j}{t}\right)\left(\nabla_j p(D) - \frac{x_j}{t}\right) g_{\pm}(\nabla p(D))\, J_0\left(\frac{x_j}{t}\right) + hc + C_0 t^{-2}. \tag{3.14$_\pm$}$$

Set $r_+ = r$, $r_- = -r$. Then as in the proof of Lemma 2.3, due to (2.11b), $\pm J_0^2 (x_j/t)(r_{\pm} - x_j/t) \ge 0$ and $\pm(\nabla_j p(D) - r_{\pm})\, g_{\pm}(\nabla p(D)) \ge 0$ imply that it is possible to choose C_0 in (3.14$_\pm$) such that $\tilde{M}_{\pm}(t) \ge 0$. Thus the Schwartz inequality for the quadratic form $\tilde{M}_{\pm}(t)$ gives

$$\left|\left(\tilde{M}_{\pm}(t)\, \tilde{J}\left(\frac{x_j}{t}\right) \varphi_t^1, \bar{J}'\left(\frac{x_j}{t}\right) \psi_t^2\right)\right| \le$$

$$\le \sqrt{\left(\tilde{M}_{\pm}(t)\, \tilde{J}\left(\frac{x_j}{t}\right) \varphi_t^1, \tilde{J}\left(\frac{x_j}{t}\right) \varphi_t^1\right)\left(\tilde{M}_{\pm}(t)\, \bar{J}'\left(\frac{x_j}{t}\right) \psi_t^2, \bar{J}'\left(\frac{x_j}{t}\right) \psi_t^2\right)}\,. \tag{3.15$_\pm$}$$

Replacing $\tilde{J}$ by a constant $\tilde{C}$ such that $\tilde{J} \le \tilde{C}$ [due to (2.11b) as in the section 2], using

$$\pm(\nabla_j p(D) - r_{\pm})\, g_{\pm}(\nabla p(D)) \le C_0 F_{\pm}(\nabla_j p(D) - r_{\pm}),$$

and $M_\pm$ being defined by $(2.6_\pm)$ with $r_+ = r$, $r_- = -r$, we may estimate

$$\left(\tilde{M}_\pm(t)\,\tilde{J}\left(\frac{x_j}{t}\right)\varphi_t^1, \tilde{J}\left(\frac{x_j}{t}\right)\varphi_t^1\right) \le$$

$$\le \tilde{C}\left((\tilde{M}_\pm(t) + t^{-2})\,\varphi_t^1, \varphi_t^1\right) \le C\left((M_\pm(t) + t^{-2})\,\varphi_t^1, \varphi_t^1\right). \qquad (3.16_\pm)$$

An analogous estimate holds with $\bar{J}'$ instead of $\tilde{J}$, hence by Cauchy-Schwartz inequality,

$$\int_{t'}^{t''} dt\,|(M^\pm_{1,1}(t)\varphi_t^1, \psi_t^2)| \le$$

$$\le C\sqrt{\int_{t'}^{t''} dt(((M_- + M_+)(t) + t^{-2})\,\varphi_t^1, \varphi_t^1)\int_{t'}^{t''} dt(((M_- + M_+)(t) + t^{-2})\,\psi_t^2, \psi_t^2)}$$

and clearly the same estimate holds with $M^\pm_{1,2}$ instead of $M^\pm_{1,1}$ [it suffices to replace $\tilde{J}$ and $\bar{J}'$ by $\tilde{J}'$ and $\bar{J}$ in $(3.15_\pm)$,$(3.16_\pm)$, (3.16)]. To complete the proof it remains to use the propagation estimates $(2.5_\pm)$ for $H = H_k$, $k = 1, 2$. ■

P r o o f o f L e m m a 3.2 f o r $k=2$. It suffices to note that $\|M_2(t)\| \in L^1([1;\infty); dt)$ in an analogous way as in the proof of Lemma 2.3. ■

4. Minimal velocity estimates

In this section H is H_1 or H_2 given by (1.1) and satisfying ($\mathrm{H}_1 - \mathrm{H}_3$).

Theorem 4.1. *Let* $\bar{r}$, $r > 0$. *Then for every* $0 < r' \le r/2$, *every* $J \in C_0^\infty((-r'; r'))$ *such that* $0 \le J \le 1$, $J = 1$ *in a certain neighbourhood of* 0 *and every* $\varphi \in \mathcal{H}_c(H)$ *such that* $\varphi = h(H)\varphi$ *for a certain* $h \in C_0^\infty((-\bar{r}; \bar{r}))$, *one has*

$$\limsup_{T\to\infty} \frac{1}{T}\int_1^T dt\left\| \mathbf{1}_{\mathbb{R}\setminus[-r;r]}(\nabla_j p(D))\,J\left(\frac{x_j}{t}\right)\varphi_t\right\|^2 \le C_0\frac{r'}{r}\|\varphi\|^2,$$

$$j = 1, \ldots, d. \qquad (4.1)$$

P r o o f. We consider separately $Z_- = (-\infty; -r)$ and $Z_+ = (r; \infty)$ proving

$$\limsup_{T\to\infty} \frac{1}{T}\int_1^T dt\,(\tilde{M}_\pm(t)\,\varphi_t\,,\,\varphi_t\,) \le C_0\frac{r'}{r}\,\|\,\varphi\,\|^2, \tag{4.1$'_\pm$}$$

where

$$\tilde{M}_\pm(t) = J\,(\frac{x_j}{t})\;\mathbf{1}_{Z_\pm}(\nabla_j p(D))\,J\,(\frac{x_j}{t})\,|_{\mathcal{K}}\,. \tag{4.2$_\pm$}$$

We use the observables

$$M_0^\pm(t) = \pm J\,(\frac{x_j}{t})\,x_j\,g_\pm(\nabla_j p(D))\,J\,(\frac{x_j}{t})\,|_{\mathcal{K}} + hc, \tag{4.3$_\pm$}$$

where $g_\pm \in C_b^\infty(\mathbf{R})$, $0 \le g_\pm \le 2/r$, $\operatorname{supp} g_+ \subset (r/2;\infty)$, $\operatorname{supp} g_- \subset (-\infty;-r/2)$, $g_+(\lambda) = 1/\lambda$ for $\lambda \ge r$ and $g_-(\lambda) = -1/\lambda$ for $\lambda \le -r$. Then

$$\frac{d}{dt}(M_0^\pm(t)\,\varphi_t\,,\,\varphi_t\,) = ((M_1^\pm + M_2^\pm + M_3^\pm)(t)\,\varphi_t\,,\,\varphi_t\,), \tag{4.4$_\pm$}$$

where

$$M_1^\pm(t) = \pm J\,(\frac{x_j}{t})\,[iH_0\,,\,x_j\,]\,g_\pm(\nabla_j p(D))\,J\,(\frac{x_j}{t}) + hc, \tag{4.5$_\pm$}$$

$$M_2^\pm(t) = \pm 2\left(\mathbf{D}_{H_0} J\,(\frac{x_j}{t})\right)x_j\,g_\pm(\nabla_j p(D))\,J\,(\frac{x_j}{t}) + hc, \tag{4.6$_\pm$}$$

$$M_3^\pm(t) = \pm\, i\,(V M_0^\pm(t) - \overline{M_0^\pm(t)V}\,), \tag{4.7$_\pm$}$$

$\overline{M_0^\pm(t)}$ being the extension of $M_0^\pm(t)$ on the whole $\mathcal{H}$. Hence

$$\frac{1}{T}\int_1^T dt\,(M_1^\pm(t)\,\varphi_t\,,\,\varphi_t\,) =$$

$$= \frac{1}{T}\Big[(M_0^\pm(t)\,\varphi_t\,,\,\varphi_t\,)\Big]_1^T - \frac{1}{T}\int_1^T dt\,((M_2^\pm + M_3^\pm)(t)\,\varphi_t\,,\,\varphi_t\,). \tag{4.8$_\pm$}$$

We note that

Lemma 4.2. *One has* $\tilde{M}_\pm(t) \le M_1^\pm(t)$ *and*

$$\limsup_{T\to\infty} \frac{1}{T}\Big[(M_0^\pm(t)\,\varphi_t\,,\,\varphi_t\,)\Big]_1^T \le C_0\frac{r'}{r}\,\|\,\varphi\,\|^2. \tag{4.9$_\pm$}$$

and $M_\pm$ being defined by $(2.6_\pm)$ with $r_+ = r$, $r_- = -r$, we may estimate

$$\left(\tilde{M}_\pm(t)\,\tilde{J}\left(\frac{x_j}{t}\right)\varphi_t^1, \tilde{J}\left(\frac{x_j}{t}\right)\varphi_t^1\right) \le$$

$$\le \tilde{C}\left((\tilde{M}_\pm(t) + t^{-2})\,\varphi_t^1, \varphi_t^1\right) \le C\left((M_\pm(t) + t^{-2})\,\varphi_t^1, \varphi_t^1\right). \qquad (3.16_\pm)$$

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and clearly the same estimate holds with $M^{\pm}_{1,2}$ instead of $M^{\pm}_{1,1}$ [it suffices to replace $\tilde{J}$ and $\overline{J}'$ by $\tilde{J}'$ and $\overline{J}$ in $(3.15_\pm)$,$(3.16_\pm)$, (3.16)]. To complete the proof it remains to use the propagation estimates $(2.5_\pm)$ for $H = H_k$, $k = 1, 2$. ■

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In this section H is H_1 or H_2 given by (1.1) and satisfying ($\mathbf{H}_1 - \mathbf{H}_3$).

Theorem 4.1. *Let $\overline{r}$, $r > 0$. Then for every $0 < r' \le r/2$, every $J \in C_0^\infty((-r'; r'))$ such that $0 \le J \le 1$, $J = 1$ in a certain neighbourhood of 0 and every $\varphi \in \mathcal{H}_c(H)$ such that $\varphi = h(H)\varphi$ for a certain $h \in C_0^\infty((-\overline{r}; \overline{r}))$, one has*

$$\limsup_{T\to\infty} \frac{1}{T}\int_1^T dt\left\| \mathbf{1}_{\mathbf{R}\setminus[-r;r]}(\nabla_j p(D))\,J\left(\frac{x_j}{t}\right)\varphi_t\right\|^2 \le C_0\frac{r'}{r}\|\varphi\|^2,$$

$$j = 1, \ldots, d. \qquad (4.1)$$

P r o o f. We consider separately $Z_- = (-\infty; -r)$ and $Z_+ = (r; \infty)$ proving

$$\limsup_{T\to\infty} \frac{1}{T}\int_1^T dt\, (\tilde{M}_\pm(t)\,\varphi_t\,,\,\varphi_t\,) \le C_0\frac{r'}{r}\,\|\,\varphi\,\|^2, \tag{4.1$'_\pm$}$$

where

$$\tilde{M}_\pm(t) = J\,(\frac{x_j}{t}\,)\,\mathbf{1}_{Z_\pm}\,(\,\nabla_j p(D))\,J\,(\frac{x_j}{t}\,)\,|_{\mathcal{K}}\,. \tag{4.2$_\pm$}$$

We use the observables

$$M_0^\pm(t) = \pm\,J\,(\frac{x_j}{t}\,)\,x_j\,g_\pm(\,\nabla_j p(D))\,J\,(\frac{x_j}{t}\,)\,|_{\mathcal{K}} + hc, \tag{4.3$_\pm$}$$

where $g_\pm \in C_b^\infty(\mathbf{R})$, $0 \le g_\pm \le 2/r$, $\operatorname{supp} g_+ \subset (r/2;\infty)$, $\operatorname{supp} g_- \subset (-\infty;-r/2)$, $g_+(\lambda) = 1/\lambda$ for $\lambda \ge r$ and $g_-(\lambda) = -\,1/\lambda$ for $\lambda \le -r$. Then

$$\frac{d}{dt}(M_0^\pm(t)\,\varphi_t\,,\,\varphi_t\,) = ((M_1^\pm + M_2^\pm + M_3^\pm)(t)\,\varphi_t\,,\,\varphi_t\,), \tag{4.4$_\pm$}$$

where

$$M_1^\pm(t) = \pm\,J\,(\frac{x_j}{t}\,)\,[iH_0\,,\,x_j\,]\,g_\pm(\,\nabla_j p(D))\,J\,(\frac{x_j}{t}\,) + hc, \tag{4.5$_\pm$}$$

$$M_2^\pm(t) = \pm\,2\left(\mathbf{D}_{H_0}\,J\,(\frac{x_j}{t}\,)\right)x_j\,g_\pm(\,\nabla_j p(D))\,J\,(\frac{x_j}{t}\,) + hc, \tag{4.6$_\pm$}$$

$$M_3^\pm(t) = \pm\,i\,(V\,M_0^\pm(t) - \overline{M_0^\pm(t)V}\,), \tag{4.7$_\pm$}$$

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$$= \frac{1}{T}\Big[(M_0^\pm(t)\,\varphi_t\,,\,\varphi_t\,)\Big]_1^T - \frac{1}{T}\int_1^T dt\,((M_2^\pm + M_3^\pm)(t)\,\varphi_t\,,\,\varphi_t\,). \tag{4.8$_\pm$}$$

We note that

Lemma 4.2. *One has* $\tilde{M}_\pm(t) \le M_1^\pm(t)$ *and*

$$\limsup_{T\to\infty} \frac{1}{T}\Big[(M_0^\pm(t)\,\varphi_t\,,\,\varphi_t\,)\Big]_1^T \le C_0\frac{r'}{r}\,\|\,\varphi\,\|^2. \tag{4.9$_\pm$}$$

P r o o f o f L e m m a 4.2. Indeed, $\tilde{M}_{\pm}(t) \leq M_1^{\pm}(t)$ follows from

$$\pm [iH_0, x_j]\, g_{\pm}(\nabla_j p(D)) = \pm \nabla_j p(D)\, g_{\pm}(\nabla_j p(D)) \geq \mathbf{1}_{Z_{\pm}}(\nabla_j p(D))$$

and $(4.9_{\pm})$ follows from $|x_j/T| \leq r'$ on the support of $J(\frac{x_j}{T})$, which implies

$$\frac{1}{T}(M_0^{\pm}(T)\,\varphi_T, \varphi_T) \leq$$

$$\leq \left(J(\frac{x_j}{T}) r'\, g_{\pm}(\nabla_j p(D))\, J(\frac{x_j}{T})\, \varphi_T, \varphi_T \right) + \frac{1}{T} C_{\varphi, J, g_{\pm}, r'}, \qquad (4.10_{\pm})$$

[due to (2.11b)] and the first term of the right hand side of $(4.10_{\pm})$ can be estimated by $2\frac{r'}{r}\|\varphi\|^2$ because of $0 \leq g_{\pm} \leq 2/r$ and $0 \leq J \leq 1$. ∎

T h e e n d o f t h e p r o o f o f T h e o r e m 4.1. Using $(4.8_{\pm})$ and Lemma 4.2, we can see that it remains to prove that for $k = 2, 3$, one has

$$\frac{1}{T}\int_1^T dt\,(M_k^{\pm}(t)\,\varphi_t, \varphi_t) \to 0 \ \text{ when } \ T \to \infty. \qquad (4.11_{\pm})$$

P r o o f o f $(4.11_{\pm})$ f o r $k = 2$. We have

$$\frac{1}{T} M_2^{\pm}(t) = \pm \frac{1}{t} J'(\frac{x_j}{t}) \left(\nabla_j p(D) - \frac{x_j}{t} \right) \frac{x_j}{T} g_{\pm}(\nabla_j p(D))\, J(\frac{x_j}{t}) +$$

$$+ hc + O(t^{-1} T^{-1}), \qquad (4.12_{\pm})$$

hence introducing $J_0 \in C_0^{\infty}((-r; r) \setminus \{0\})$ such that $J_0 = 1$ on the support of J', we may write $J' = J' J_0^2$ and repeating the reasoning of the proof of Theorem, 3.1, we get

$$1 \leq t \leq T \Rightarrow -CM_{\pm}(t) - Ct^{-2} \leq \frac{1}{T} M_2^{\pm}(t) \leq CM_{\pm}(t) + Ct^{-2}, \qquad (4.13_{\pm})$$

where $M_{\pm}(t)$ is given by $(2.6_{\pm})$. Then the propagation estimates $(2.5_{\pm})$ imply that for any $\varepsilon > 0$ we can find T_0 such that

$$\int_{T_0}^{T} dt\, C((M_{\pm}(t) + t^{-2})\,\varphi_t, \varphi_t) < \varepsilon \qquad (4.14_{\pm})$$

for all $T \geq T_0$ and it remains to note that $\frac{1}{T}\int_1^{T_0} dt\,(M_2^{\pm}(t)\,\varphi_t\,,\,\varphi_t) \to 0$ when $T \to \infty$. ■

P r o o f o f $(4.11_{\pm})$ f o r $k = 3$. It is based on RAGE theorem (cf. [RS], v. 3), which states that for every compact operator K, one has

$$\frac{1}{T}\int_1^T dt\,\|\,KE_c\,(H)\,\varphi_t\,\| \to 0 \quad \text{when} \quad T \to \infty. \tag{4.15}$$

Indeed, the operators $K^s = (1 + |\,x\,|)V^s\,h(H)$ and $K^\ell = V^\ell h(H)$ are compact, hence RAGE theorem allows to estimate easily

$$|\,([V^s, M_0\,(t)]\,\varphi_t\,,\,\varphi_t\,)\,| \leq C\,\|\,K^s\,E_c\,(H)\,\varphi_t\,\|\,\|\,\varphi\,\|,$$

then $[x_j\,,\,g_\pm\,(\,\nabla_j\,p(D))]$ being bounded, we have

$$\left|\left(J\,(\frac{x_j}{t})\,[x_j\,,\,g_\pm\,(\,\nabla_j\,p(D))]\,J\,(\frac{x_j}{t})\,V^\ell\,\varphi_t\,,\,\varphi_t\right)\right| \leq$$

$$\leq C\,\|\,K^\ell E_c\,(H)\,\varphi_t\,\|\,\|\,\varphi\,\| + C\,\|\,K^\ell(1 - J\,)(\frac{x_j}{t})\,h(H)\,\varphi_t\,\|\,\|\,\varphi\,\|\,,$$

hence RAGE theorem allows to estimate easily the first term of the last line, while the second term tends to 0 when $t \to \infty$, because $(1 - J\,)(x_j\,/t) \to 0$ strongly and K^ℓ compact imply the norm convergence $\|\,K^\ell(1 - J\,)(x_j\,/t)\,\| \to 0$ when $t \to \infty$. It remains to estimate

$$\left|\left([\,x_j\,V^\ell J\,(\frac{x_j}{t}\,),\,g_\pm\,(\,\nabla_j\,p(D))]\,h(H)\,\varphi_t\,,\,\varphi_t\right)\right|. \tag{4.16$_\pm$}$$

Note that

$$\left[x_j\,J\,(\frac{x_j}{t}\,),\,\nabla_j\,p(D)\right] = t\left[\frac{x_j}{t}\quad J\,(\frac{x_j}{t}\,),\,\nabla_j\,p(D)\right] = tO(t^{-1}\,) \tag{4.17}$$

is uniformly bounded, hence the same can be said about $[x_j\,V^\ell J\,(x_j\,/t),\,\nabla_j\,p(D)]$ and due to (A.2), about $[x_j\,V^\ell J\,(x_j\,/t),\,g_\pm\,(\,\nabla_j\,p(D))]$ as well. Since $h(H) - h(H_0\,)$ is compact, we may replace $h(H)$ by $h(H_0\,)$ in (4.16) and (A.1) allows to estimate

$$\left\|\left[\,x_j\,V^\ell J\,(\frac{x_j}{t}\,),\,g_\pm\,(\,\nabla_j\,p(D))\,\right]h(H_0\,)\,\varphi_t\,\right\| \leq$$

$$\leq C \left\| \left[x_j V^{\ell} J\left(\frac{x_j}{t}\right), \nabla_j p(D) \right] h(H_0)\, \varphi_t \right\|, \qquad (4.18_{\pm})$$

because $h(H_0)$ commutes with $\nabla_j p(D)$. Using (4.17) we can see that RAGE theorem allows to estimate the right hand side of (4.18), because $V^{\ell} h(H_0)$ is compact and

$$x_j [V^{\ell}, \nabla_j p(D)]\, h(H_0) = [x_j V^{\ell}, \nabla_j p(D)]\, h(H_0) - [x_j, \nabla_j p(D)]\, V^{\ell} h(H_0)$$

is compact as well. ■

5. Proof of Theorem 1.1

The idea of the proof consists of introducing a suitable cut-off $g(\nabla p(D))$ between two evolutions in order to be able to use Theorem 3.1 and 4.1.

It suffices to fix $\lambda_0 \in \mathcal{R}$ and to find $\delta_0 > 0$ such that for every $h \in C_0^{\infty}((\lambda_0 - \delta_0 ; \lambda_0 + \delta_0))$, the limit $s - \lim_{t \to \infty} \Omega_t$ exists, where

$$\Omega_t = e^{itH_2} e^{-itH_1} h^2 (H_1)\, E_c (H_1). \qquad (5.1)$$

By the definition of $\mathcal{R}$, we may assume that

$$\exists\ r > 0\ \forall \xi \in \mathbf{R}^d,\ \lambda_0 - \delta_0 \leq p(\xi) \leq \lambda_0 + \delta_0 \Rightarrow |\nabla p(\xi)| \geq 2rd. \qquad (5.2)$$

Lemma 5.1. *There exists a partition of unity* $\{ g_j \}_{j=0,1,\dots,d}$ *formed by functions* $g_j \in C_b^{\infty}(\mathbf{R}^d)$ *such that* $0 \leq g_j \leq 1$, $g_j = \otimes\, g_{j,m}$ *with* $g'_{j,m} \in C_0^{\infty}(\mathbf{R})$ *satisfying the conditions*

$$\operatorname{supp} g_{0,m} \subset (-2r; 2r) \ \text{ for } \ 1 \leq m \leq d, \qquad (5.3)$$

$$\operatorname{supp} g_{j,j} \cap [-r; r] = \emptyset \ \text{ for } \ 1 \leq j \leq d. \qquad (5.4)$$

P r o o f. Let $g \in C_0^{\infty}((-2r; 2r))$ be such that $0 \leq g \leq 1$ and $g = 1$ in a neighbourhood of $[-r; r]$. Denoting $\bar{g} = 1 - g$, we take $g_1(\lambda) = \bar{g}(\lambda_1)$, $g_2(\lambda) = g(\lambda_1)\,\bar{g}(\lambda_2)$, $g_3(\lambda) = g(\lambda_1)\, g(\lambda_2)\,\bar{g}(\lambda_3)$, $g_j(\lambda) = g(\lambda_1)\, g(\lambda_2) \dots g(\lambda_{j-1})\,\bar{g}(\lambda_j)$ for $4 \leq j \leq d$ and $g_0(\lambda) = g(\lambda_1)\, g(\lambda_2) \dots g(\lambda_{d-1})\, g(\lambda_d)$. ■

We shall check the Cauchy condition for $\Omega_t \varphi$ decomposing for $0 < r' \leq r/2$,

$$\Omega_t \varphi = \Omega_{r',t}\, \varphi + \tilde{\Omega}_{r',t}\, \varphi, \qquad (5.5)$$

where

$$\Omega_{r',t} = e^{itH_2} h(H_1) \sum_{1\le j\le d} \bar{J}_{r'}\left(\frac{x_j}{t}\right) g_j(\nabla p(D))\, h(H_1)\, e^{-itH_1} E_c(H_1), \quad (5.6)$$

$$\tilde{\Omega}_{r',t} = e^{itH_2} h(H_1) \left(I - \sum_{1\le j\le d} \bar{J}_{r'}\left(\frac{x_j}{t}\right) g_j(\nabla p(D)) \right) h(H_1)\, e^{-itH_1} E_c(H_1) \quad (5.7)$$

and $\bar{J}_{r'} = 1 - J_{r'}$ with $J_{r'} \in C_0^\infty((-r'; r'))$ such that $0 \le J_{r'} \le 1$, $J_{r'} = 1$ in a neighbourhood of 0.

Lemma 5.2. *The limit* $s - \lim_{t\to\infty} \Omega_{r',t}$ *exists.*

P r o o f. Theorem 3.1 gives the existence of limits analogue to (5.6) but with $h(H_2)$ instead of $h(H_1)$ in front of $\bar{J}_{r'}(x_j/t)$. However $h(H_1) - h(H_2)$ is compact and $\bar{J}_{r'}(x_j/t) \to 0$ strongly, hence $(h(H_1) - h(H_2))\bar{J}_{r'}(x_j/t) \to 0$ in norm when $t \to \infty$.■

Lemma 5.3. *There exists a constant* $C_1 > 0$ *such that*

$$\lim_{t\to\infty} \| \tilde{\Omega}_{r',t}\varphi \| \le C_1 \sqrt{r'/r}\, \|\varphi\|. \quad (5.8)$$

P r o o f. First we note that the limit in the left hand side of (5.8) exists. Indeed, Lemma 5.2 still holds if $H_2 = H_1$, hence the limit

$$\lim_{t\to\infty} e^{-itH_1} h(H_1) \left(I - \sum_{1\le j\le d} \bar{J}_{r'}\left(\frac{x_j}{t}\right) g_j(\nabla p(D)) \right) h(H_1)\, e^{-itH_1}\varphi =$$

$$= h^2(H_1)\varphi - \sum_{1\le j\le d} \lim_{t\to\infty} e^{-itH_1} h(H_1)\bar{J}_{r'}\left(\frac{x_j}{t}\right) g_j(\nabla p(D))\, h(H_1)\, e^{-itH_1}\varphi$$

exists. Thus it suffices to check that for $\varphi \in \mathcal{H}_c(H_1)$, one has

$$\limsup_{T\to\infty} \frac{1}{T}\int_1^T dt\, \| \tilde{\Omega}_{r',t}\varphi \|^2 \le C_0 \frac{r'}{r} \|\varphi\|^2. \quad (5.9)$$

However, $\{ g_j \}_{j=0,1,\dots,d}$ being a partition of unity,

$$I - \sum_{1 \le j \le d} \bar{J}_{r'}\left(\frac{x_j}{t}\right) g_j(\nabla p(D)) = g_0(\nabla p(D)) + \sum_{1 \le j \le d} J_{r'}\left(\frac{x_j}{t}\right) g_j(\nabla p(D)) \quad (5.10)$$

and (5.9) follows immediately from Theorem 4.1 and from RAGE Theorem applied to the compact operator $g_0(\nabla p(D))h(H_1)$. Indeed, for ξ belonging to the support of $g_0(\nabla p(\xi))$ we have $|\nabla p(\xi)| \le 2rd$, implying $h(p(\xi)) = 0$ due to (5.2). Therefore $g_0(\nabla p(D))h(H_0) = 0$ implies that $g_0(\nabla p(D))h(H_1) = g_0(\nabla p(D))(h(H_1) - h(H_0))$ is compact. ■

T h e e n d o f t h e p r o o f o f T h e o r e m 1.1. For any $\varepsilon > 0$ we choose $r' > 0$ small enough to assure $C_1\sqrt{r'/r}\,\|\varphi\| < \varepsilon/4$ in the right hand side of (5.8). Then there exists t_0 such that for $t', t'' \ge t_0$ we have $\|\tilde{\Omega}_{r',t''}\varphi - \tilde{\Omega}_{r',t'}\varphi\| \le \|\tilde{\Omega}_{r',t''}\varphi\| + \|\tilde{\Omega}_{r',t'}\varphi\| \le \varepsilon/2$ due to Lemma 5.3 and $\|\Omega_{r',t''}\varphi - \Omega_{r',t'}\varphi\| < \varepsilon/2$ due to Lemma 5.2. ■

6. Genaralisation

If X, X' are Banach spaces, then $B(X; X')$ denotes the Banach space of linear bounded operators $X \to X'$ and $B(X) = B(X; X)$.

Let X be a linear space, M a function $X \times X \to \mathbb{C}$ and denote the value of M on vectors $\varphi, \psi \in X$ by $\langle M\varphi, \psi \rangle$. Then M is a sesquilinear form on X if for all $\varphi, \varphi', \psi \in X$, $\alpha, \beta \in \mathbb{C}$,

$$\langle M(\alpha\varphi + \beta\varphi'), \psi \rangle = \alpha\langle M\varphi, \psi \rangle + \beta\langle M\varphi', \psi \rangle,$$

$$\langle M\psi, \alpha\varphi + \beta\varphi' \rangle = \bar{\alpha}\langle M\psi, \varphi \rangle + \bar{\beta}\langle M\psi, \varphi' \rangle.$$

If moreover $\langle M\psi, \varphi \rangle = \overline{\langle M\varphi, \psi \rangle}$ for all $\varphi, \psi \in X$, then M is quadratic form on X. If X is Banach space then $Q(X)$ will denote the Banach space of all bounded sesquilinear forms on X, i.e. all sesqulinear forms on X satisfying

$$\|M\|_{Q(X)} = \sup_{\varphi, \psi \in X, \|\varphi\|_X \le 1, \ \|\psi\|_X \le 1} |\langle M\varphi, \psi \rangle| < \infty. \quad (6.1)$$

Let Δ be a real interval and assume that $M(t) \in Q(X)$ for every $t \in \Delta$. Then we write $(t \to M(t)) \in w - C^k(\Delta; Q(X))$ if for every $\varphi, \psi \in X$, the function $t \to \langle M(t)\varphi, \psi \rangle$ is of class C^k. We note that for $1 \le j \le k$, there exists $(d/dt)^j M(t) \in Q(X)$ such that

$$(d/dt)^j \langle M(t)\varphi, \psi \rangle = \langle (d/dt)^j M(t)\varphi, \psi \rangle, \text{ if } \varphi, \psi \in X.$$

R e m a r k 6.1. If $t' \to \varphi_{t'}$, $t' \to \psi_{t'}$ are continuous $\tilde{\Delta} \to X$ and $(t \to M(t)) \in w - C(\Delta; Q(X))$, then the function $(t, t', t'') \to \langle M(t)\, \varphi_{t'}, \varphi_{t''} \rangle$ is continuous on $\Delta \times \tilde{\Delta}^2$.

Indeed, we may assume that $\tilde{\Delta}$, Δ are compact, hence $t' \to \| \varphi_{t'} \|_X$, $t'' \to \| \psi_{t''} \|_X$, $t \to \| M(t) \|_{Q(X)}$ are bounded (due to the Banach–Stainhaus uniform boundedness principle) and $\delta \to 0 \Rightarrow \langle M(t+\delta)\, \varphi_{t'}, \varphi_{t''} \rangle \to \langle M(t)\, \varphi_{t'}, \varphi_{t''} \rangle$ uniformly with respect to $t', t'' \in \tilde{\Delta}$.

Let the Banach space X be a continuously embedded dense subspace of a Hilbert space $\mathcal{H}$. Then every operator $M \in B(X; \mathcal{H})$ defines naturally $\langle M\cdot, \cdot \rangle \in Q(X)$ by the formula

$$\langle M\varphi, \psi \rangle = (M\varphi, \psi), \tag{6.2}$$

where $(\cdot, \cdot)$ is the scalar product of $\mathcal{H}$. In particular we can identify $B(\mathcal{H})$ and $Q(\mathcal{H})$. We denote by $\| \cdot \|$ the norm of $\mathcal{H}$ or the norm of $B(\mathcal{H}) = Q(\mathcal{H})$.

If $A, \tilde{A}, B \in B(X; \mathcal{H})$, then we define $\mathrm{ad}^Q_{A, \tilde{A}} B \in Q(X)$ setting

$$\langle \mathrm{ad}^Q_{A, \tilde{A}} B\varphi, \psi \rangle = (A\varphi, B\psi) - (B\varphi, \tilde{A}\psi) \tag{6.3}$$

and denote $\mathrm{ad}^Q_A B = \mathrm{ad}^Q_{A, \tilde{A}} B$.

Assume that $(t \to M(t)) \in w - C^1((0; \infty); B(\mathcal{H}))$ and $\{ H(t) \}_{t \geq s}$ is H_0-admissible in the sense defined in the section 1. Then setting

$$\mathrm{D}^Q_{H(t)} M(t) = i\, \mathrm{ad}^Q_{H(t)} M(t) + (d/dt) M(t), \tag{6.4}$$

we define $(t \to \mathrm{D}^Q_{H(t)} M(t)) \in w - C((0; \infty); Q(\mathcal{D}))$, where $\mathcal{D}$ is the domain of H_0 with the graph norm $\| \varphi \|_{\mathcal{D}} = \| \varphi \| + \| H_0 \varphi \|$.

Let X, X' be Banach spaces and $\mathcal{K}$ a dense subspace of X. We shall write $A \in B_{\mathcal{K}}(X; X')$ if A is a linear operator $\mathcal{K} \to X'$ such that

$$\| A \|_{B(X;\, X')} = \sup_{\varphi \in \mathcal{K},\, \| \varphi \|_X \leq 1} \| A \varphi \|_{X'} < \infty. \tag{6.5}$$

If $A \in B_{\mathcal{K}}(X; X')$ then the extension by continuity, $\overline{A} \in B(X; X')$, is uniquely determined and gives the natural correspondence between elements of $B_{\mathcal{K}}(X; X')$ and $B(X; X')$. We say that $\overline{A}$ is the closure of A in $B(X; X')$ and A is the restriction of $\overline{A}$ to $\mathcal{K}$ (i.e. $A = \overline{A}\,|_{\mathcal{K}}$ in the usual notation).

We write $M \in Q_{\mathcal{K}}(X)$ if M is a sesqulinear form on $\mathcal{K}$ such that

$$\| M \|_{Q(X)} = \sup_{\varphi, \psi \in \mathcal{K}, \|\varphi\|_X \le 1, \|\psi\|_X \le 1} |\langle M\varphi, \psi \rangle| < \infty. \qquad (6.1')$$

If $M \in Q_{\mathcal{K}}(X)$ then the extension by continuity, $\overline{M}^{Q(X)} \in Q(X)$ is uniquely determined and gives the natural correspondence between elements of $Q_{\mathcal{K}}(X)$ and $Q(X)$. We say that $\overline{M}^{Q(X)}$ is the closure of M in $Q(X)$ and M is the restriction of $\overline{M}^{Q(X)}$ to $\mathcal{K} \times \mathcal{K}$ (i.e. $M = \overline{M}^{Q(X)}|_{\mathcal{K}\times\mathcal{K}}$). For $M: \Delta \to Q_{\mathcal{K}}(X)$, we write $(t \to M(t)) \in w - C(\Delta; Q_{\mathcal{K}}(X))$ if for every φ, $\psi \in \mathcal{K}$ the function $t \to \langle M(t)\varphi, \psi \rangle$ is continuous and $t \to \| M(t) \|_{Q(X)}$ is locally bounded. Note that

$$(t \to M(t)) \in w - C(\Delta; Q_{\mathcal{K}}(X)) \Leftrightarrow (t \to \overline{M(t)}^{Q(X)}) \in w - C(\Delta; Q(X)). \qquad (6.6)$$

Indeed, if $\varphi_{t'}, \psi_{t'} \in \mathcal{K}$ for $t' > 0$ and $t' \to 0 \Rightarrow \varphi_{t'} \to \varphi_0 \in X$, $\psi_{t'} \to \psi_0 \in X$ in X, then applying the reasoning below Remark 6.1 we get $\langle \overline{M(t)}^{Q(X)} \varphi_0, \psi_0 \rangle = \lim_{t' \to 0} \langle M(t)\varphi_{t'}, \psi_{t'} \rangle$, where the limit is uniform with respect to $t \in \Delta$ for any compact Δ.

In order to prove Theorem 1.1 without the hypothesis $|p(\xi)| \to \infty$ for $|\xi| \to \infty$, it suffices to show that Theorem 3.1 remains valid. We take $\mathcal{K} = H^{\infty}(\mathbf{R}^d)$ and state first

R e m a r k 6.2. If $M_{\pm}(t)$ is given by ($2.6_{\pm}$), then

$$(t \to M_{\pm}(t)) \in w - C([1; \infty); Q_{\mathcal{K}}(\mathcal{D})).$$

P r o o f. Note first that if $J_0 \in C_0^{\infty}(\mathbf{R})$, g_0 is bounded, then

$$\left(t \to J_0\left(\frac{x_j}{t}\right) g_0(\nabla_j p(D)) J_0\left(\frac{x_j}{t}\right)\right) \in w - C([1; \infty); Q_{\mathcal{K}}(\mathcal{H}))$$
$$\subset w - C([1; \infty); Q_{\mathcal{K}}(\mathcal{D})).$$

Consider now $g \in C^{\infty}(\mathbf{R})$ such that $g'' \in C_0^{\infty}(\mathbf{R})$, $g = 0$ in a neighbourhood of 0 and $J \in C_b^{\infty}(\mathbf{R})$ such that $J' = J_0^2$. Using the boundedness of the commutator $[J(x_j/t), g(\nabla_j p(D))]$ (cf. (2.11a)) and applying (2.11c) we can see that for every $t \ge 1$ there exist $R_1(t), R_2(t) \in B_{\mathcal{K}}(\mathcal{H})$, such that

$$J_0\left(\frac{x_j}{t}\right) g(\nabla_j p(D)) J_0\left(\frac{x_j}{t}\right) = J_0^2\left(\frac{x_j}{t}\right) \nabla_j p(D)\, \tilde{g}(\nabla_j p(D)) + R_1(t) =$$

$$= \left[itH_0\,, J\left(\frac{x_j}{t}\right) \tilde{g}(\nabla_j p(D)) \right] + R_2(t),$$

where $\tilde{g}(\lambda) = g(\lambda)/\lambda$ is bounded and thus it is clear that both terms of the last line belong to $Q_{\mathcal{K}}(\mathcal{D})$. It remains to note that $t \to (J_0(x_j/t) g(\nabla_j p(D)) J_0(x_j/t)\varphi, \psi)$ is continuous for every $\varphi, \psi \in \mathcal{K}$, because $t \to J_0(x_j/t)\varphi$ is continuous from $(0;\infty)$ into the Frechet space $H^{\infty}(\mathbf{R}^d)$. ■

We reformulate propagation estimates as follows

Theorem 2.1 reformulated. *If $\bar{r}$, r_-, $r_+ \in \mathbf{R}$, $J_0 \in C_0^{\infty}((r_-, r_+) \setminus \{0\})$ and $M_{\pm}(t)$ is given by $(2.6_{\pm})$, then there exists a constant C such that for every $\varphi \in \operatorname{Ran} E_{[-\bar{r};\bar{r}]}(H)$, one has*

$$\int_1^{\infty} dt \,\langle \overline{M_{\pm}(t)}^{\,Q(\mathcal{D})} \varphi_t\,, \varphi_t \rangle \le C \,\|\varphi\|^2 \tag{$6.7_{\pm}$}$$

(note that the integrated function is continuous due to Remark 6.1 with $X = \mathcal{D}$).

P r o o f. Let $M_k^{\pm}(t)$ be given by $(2.7_{\pm})$ and $(2.9_{\pm})$ for $k = 0, 1, 2$. Then clearly $M_0^{\pm}(t) \in B_{\mathcal{K}}(\mathcal{H})$ and $M_k^{\pm}(t) \in Q_{\mathcal{K}}(\mathcal{D})$, because

$$\overline{M_1^{\pm}(t)}^{\,Q(\mathcal{D})} = \mathbf{D}_{H_0}^{Q} M_0^{\pm}(t), \quad \overline{M_2^{\pm}(t)}^{\,Q(\mathcal{D})} = \operatorname{ad}_V^Q M_0^{\pm}(t).$$

Note that the proof of Lemma 2.2 and 2.3 are valid and we may state them as follows

Lemma 2.2 reformulated. *There exists $C > 0$ such that for all $t \ge 1$,*

$$\overline{M_{\pm}(t)}^{\,Q(\mathcal{D})} \le C\, \overline{M_1^{\pm}(t)}^{\,Q(\mathcal{D})} + C t^{-2} I. \tag{$6.8_{\pm}$}$$

Lemma 2.3 reformulated. *One has*

$$\| E_{[-\bar{r};\bar{r}]}(H)\, \overline{M_2^{\pm}(t)}^{\,Q(\mathcal{D})} E_{[-\bar{r};\bar{r}]}(H) \|_{Q(\mathcal{D})} \in L^1([1;\infty); dt).$$

Then instead of $(2.8\pm)$ we may write

$$\frac{d}{dt}(\overline{M_0^{\pm}(t)}\,\varphi_t\,,\varphi_t) = \sum_{1\le k\le 2} \langle \overline{M_k^{\pm}(t)}^{\,Q(\mathcal{D})}\,\varphi_t\,,\varphi_t\rangle, \tag{6.9$_\pm$}$$

and instead of (2.10±),

$$\int_1^T dt\,\langle \overline{M_{\pm}(t)}^{\,Q(\mathcal{D})}\,\varphi_t\,,\varphi_t\rangle \le C\int_1^T dt\,\langle \overline{M_1^{\pm}(t)}^{\,Q(\mathcal{D})}\,\varphi_t\,,\varphi_t\rangle + C\int_1^T dt\,t^{-2}\,\|\varphi\|^2 \le$$

$$\le C\Big[(\overline{M_0^{\pm}(t)}\,\varphi_t\,,\varphi_t)\Big]_1^T + C\int_1^T dt\,\|\overline{M_2^{\pm}(t)}^{\,Q(\mathcal{D})}\|_{Q(\mathcal{D})}\,\|(H_0+i)\varphi\|^2 +$$

$$+ C\int_1^T dt\,t^{-2}\,\|\varphi\|^2, \tag{6.10$_\pm$}$$

which easily implies (6.7±). ■

P r o o f o f T h e o r e m 3.1 w i t h o u t t h e h y p o t h e s i s (2.3). Since

$$\frac{d}{dt}(\overline{M_0(t)}\,\varphi_t^1\,,\psi_t^2) = \sum_{1\le k\le 2} \langle \overline{M_k(t)}^{\,Q(\mathcal{D})}\,\varphi_t^1\,,\psi_t^2\rangle, \tag{6.11}$$

if φ, $\psi \in \mathcal{D}$ and M_k are defined by (3.3), (3.8), (3.9) for $k = 0, 1, 2$, we have (3.6) with

$$\zeta_k(t', t'', \varphi, \psi) = \int_{t'}^{t''} dt\,|\langle \overline{M_k(t)}^{\,Q(\mathcal{D})}\,\varphi_t^1\,,\psi_t^2\rangle|. \tag{6.12}$$

Clearly the proof of Lemma 3.3 holds and it remains to prove Lemma 3.2. Following the beginning of the proof of this statement in the section 3, we get

$$|(M_1(t)\varphi, \psi)| \le$$

$$\le C\sqrt{(((M_- + M_+)(t) + t^{-2})\varphi, \varphi)\,(((M_- + M_+)(t) + t^{-2})\psi, \psi)}\,, \tag{6.13}$$

for φ, $\psi \in \mathcal{K}$. Hence for φ, $\psi \in \mathcal{D}$, we have

$$|\langle \overline{M_1(t)}^{\,Q(\mathcal{D})}\,\varphi_t^1, \psi_t^2\rangle| \le$$

$$\leq C\sqrt{\langle((M_- + M_+)(t)}\ ^{Q(\mathcal{D})} + t^{-2})\varphi_t^1, \varphi_t^1\rangle\langle\overline{((M_- + M_+)(t)}\ ^{Q(\mathcal{D})} + t^{-2})\psi_t^2, \psi_t^2\rangle$$

and we complete the proof as in the section 2, applying Cauchy-Schwartz inequality for integrals and propagation estimates ($6.7_{\pm}$). ■

7. Propagation estimates of the second type

In this section $H(t) = H_1(t)$ or $H_2(t)$ satisfying the hypotheses of Theorem 1.2, we assume $s = 1$ and denote $\varphi_t = U(t, 1)\varphi_1$ for $\varphi_1 \in \mathcal{H}$.

We shall write $M(t) \in \mathcal{G}(H(t))$ if $(t \to M(t)) \in w - C([1; \infty); Q(\mathcal{D}))$ is such that $M(t)$ is a quadratic form on $\mathcal{D}$ and there exists a constant C such that for every $\varphi_1 \in \mathcal{D}$ and $T \geq 1$,

$$\int_1^T dt \langle M(t)\varphi_t, \varphi_t\rangle \leq C\|\varphi\|^2. \tag{7.1}$$

Note that the integrated function is continuous due to Remark 6.1 with $X = \mathcal{D}$.

Theorem 7.1 (Propagation estimates of the second type). *If $J_0 \in C_0^\infty(\mathbf{R} \setminus \{0\})$ and*

$$M(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right)\left|\frac{x_j}{t} - \nabla_j p(D)\right| J_0\left(\frac{x_j}{t}\right)|_{\mathcal{K}}, \tag{7.2}$$

then $\overline{M(t)}\ ^{Q(\mathcal{D})} \in \mathcal{G}(H(t))$.

At the beginning of the proof of Theorem 7.1, we state

R e m a r k 7.2. Let $f = g_0 + g$ with $g_0 \in C_0(\mathbf{R})$ and $g \in C^\infty(\mathbf{R})$ such that $g'' \in C_0^\infty(\mathbf{R})$ and set

$$M_f(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right) f\left(\frac{x_j}{t} - \nabla_j p(D)\right) J_0\left(\frac{x_j}{t}\right)|_{\mathcal{K}}. \tag{7.3}$$

Then $(t \to M_f(t)) \in w - C([1; \infty); Q_{\mathcal{K}}(\mathcal{D}))$.

P r o o f. It suffices to follow the proof of Remark 6.2, using g_0, g and $\tilde{g}$ with the argument $x_j/t - \nabla_j p(D)$ instead of $\nabla_j p(D)$. ■

Thus, taking $f(\lambda) = |\lambda|$ we get $(t \to M_f(t)) \in w - C([1;\infty); Q_{\mathcal{K}}(\mathcal{D}))$. Taking $f(\lambda) = \lambda$ or $f = F_+$ we know that $t \to \tilde{M}(t)$ and $t \to M_+(t)$ belong also to $w - C([1;\infty); Q_{\mathcal{K}}(\mathcal{D}))$, where

$$\tilde{M}(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right)\left(\frac{x_j}{t} - \nabla_j p(D)\right) J_0\left(\frac{x_j}{t}\right)|_{\mathcal{K}}, \tag{7.4}$$

$$M_+(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right) F_+\left(\frac{x_j}{t} - \nabla_j p(D)\right) J_0\left(\frac{x_j}{t}\right)|_{\mathcal{K}}. \tag{7.5}$$

Since $\pm \mathbf{D}_{H_0} J(x_j/t) = \pm \tilde{M}(t) + O(t^{-2})$ if $J \in C_b^\infty(\mathbf{R})$ is such that $J' = J_0^2$, it is clear that $\pm \overline{\tilde{M}(t)}^{Q(\mathcal{D})} \in \mathcal{G}(H(t))$. Since $|\lambda| = 2F_+(\lambda) - \lambda$, we have

$$\overline{M(t)}^{Q(\mathcal{D})} = 2\,\overline{M_+(t)}^{Q(\mathcal{D})} - \overline{\tilde{M}(t)}^{Q(\mathcal{D})}. \tag{7.6}$$

Hence it suffices to prove $\overline{M_+(t)}^{Q(\mathcal{D})} \in \mathcal{G}(H(t))$ instead of $\overline{M(t)}^{Q(\mathcal{D})} \in \mathcal{G}(H(t))$. Set

$$\gamma_\varepsilon(\lambda) = \gamma_1(\lambda/\varepsilon)/\varepsilon \quad \text{for } \varepsilon > 0, \tag{7.7}$$

where $\gamma_1 \in C_0^\infty((-1;1))$ is such that $\gamma_1 \ge 0$, $\int \gamma_1 = 1$, take $f_0 \in C(\mathbf{R})$ being C^∞ on $\mathbf{R} \setminus \{0\}$ such that $0 \le f_0 \le 1$, $f_0' \ge 0$, $f_0(\lambda) = F_+(\lambda)$ for $\lambda \le \frac{1}{2}$, $f_0(\lambda) = 1$ for $\lambda \ge 1$, and define

$$f_t = f_0 * \gamma_{t^{-\beta}} \quad \text{for } t \ge 1 \quad [\text{where } \gamma_{t^{-\beta}}(\lambda) = t^\beta \gamma_1(t^\beta \lambda)] \tag{7.8}$$

with $\beta > 0$ fixed small enough.

To prove $\overline{M_+(t)}^{Q(\mathcal{D})} \in \mathcal{G}(H(t))$, we may consider separately the cases $\operatorname{supp} J_0 \subset (0;\infty)$ and $\operatorname{supp} J_0 \subset (-\infty; 0)$, using the following observable

$$M_0(t) = -J\left(\frac{x_j}{t}\right) f_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) J\left(\frac{x_j}{t}\right)|_{\mathcal{K}}, \tag{7.9}$$

where in the case $\operatorname{supp} J_0 \subset (0;\infty)$ we take $J \in C_b^\infty(\mathbf{R})$ such that $J \ge 0$, $J' \ge 0$, $\operatorname{supp} J \subset (0;\infty)$, $J^2 \ge J_0^2$, $2JJ' \ge J_0^2$, and in the case $\operatorname{supp} J_0 \subset (-\infty; 0)$ we take $J \in C_b^\infty(\mathbf{R})$ such that $J \le 0$, $J' \ge 0$, $\operatorname{supp} J \subset (-\infty; 0)$, $J^2 \ge J_0^2$, $-2JJ' \ge J_0^2$. Then we have

Lemma 7.3. *There exist constants $C > 0$ and $\varepsilon > 0$, such that for all $t \ge 1$,*

$$M_+(t) \le C\,\mathbf{D}_{H_0} M_0(t) + Ct^{-1-\varepsilon}. \tag{7.10}$$

Lemma 7.4. *One has* $\| [V(t), \overline{M_0(t)}] \| \in L^1([1;\infty); dt)$.

As before, Lemma 7.3 and 7.4 allow to follow the reasoning based on formulas $(6.9_\pm)$ and $(6.10_\pm)$, proving that $\overline{M_+(t)}^{\mathcal{Q}(\mathcal{D})} \in \mathcal{G}(H(t))$. Thus it remains to prove Lemma 7.3 and 7.4.

Proof of Lemma 7.3. We have $\mathbf{D}_{H_0} M_0 = M_1 + M_2$ with

$$M_1(t) = -J\left(\frac{x_j}{t}\right)\left(\mathbf{D}_{H_0} f_t\left(\frac{x_j}{t} - \nabla_j p(D)\right)\right) J\left(\frac{x_j}{t}\right)|_{\mathcal{X}}, \tag{7.11}$$

$$M_2(t) = -2\left(\mathbf{D}_{H_0} J\left(\frac{x_j}{t}\right)\right) f_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) J\left(\frac{x_j}{t}\right)|_{\mathcal{X}} + hc. \tag{7.12}$$

Step 1°. We define

$$\tilde{f}_t(\lambda) = \lambda \frac{d}{d\lambda} f_t(\lambda), \tag{7.13}$$

$$\tilde{M}_1(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right) \tilde{f}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) J_0\left(\frac{x_j}{t}\right)|_{\mathcal{X}}, \tag{7.14}$$

and check that $\tilde{M}_1(t) \le M_1(t) + Ct^{-1-\varepsilon}$ for certain constants $C, \varepsilon > 0$.
Indeed, we remark first that $|\frac{d}{dt} f_t(\lambda)| \le C_0 t^{-1-\beta}$ [cf. (A.7c) of Appendix], hence

$$\mathbf{D}_{H_0} f_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) = \left([iH_0, \cdot\,] + \frac{d}{ds}\right) f_t\left(\frac{x_j}{s} - \nabla_j p(D)\right)|_{s=t} + O(t^{-1-\beta}) =$$

$$= \frac{d}{ds} e^{i(s-t)H_0} f_t\left(\frac{x_j}{s} - \nabla_j p(D)\right) e^{i(t-s)H_0} = |_{s=t} + O(t^{-1-\beta}). \tag{7.15}$$

But

$$e^{isH_0} f_t\left(\frac{x_j}{s} - \nabla_j p(D)\right) e^{-isH_0} = f_t\left(\frac{x_j}{s}\right) \tag{7.16}$$

and since $\frac{d}{ds} f_t(x_j/s) = -\tilde{f}_t(x_j/s)/s$, we can see that the first term of the second line of (7.15) equals

$$-\frac{1}{s} e^{-itH_0} \tilde{f}_t\left(\frac{x_j}{s}\right) e^{itH_0}|_{s=t} = -\frac{1}{t} \tilde{f}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right). \tag{7.17}$$

Thus for a certain $\varepsilon > 0$,

$$M_1(t) = \frac{1}{t} J\left(\frac{x_j}{t}\right) \tilde{f}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) J\left(\frac{x_j}{t}\right)|_{\mathcal{K}} + O(t^{-1-\varepsilon}) =$$

$$= \frac{1}{t} J^2\left(\frac{x_j}{t}\right) \tilde{f}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right)|_{\mathcal{K}} + O(t^{-1-\varepsilon}), \qquad (7.18)$$

and

$$\tilde{M}_1(t) = \frac{1}{t} J_0^2\left(\frac{x_j}{t}\right) \tilde{f}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right)|_{\mathcal{K}} + O(t^{-1-\varepsilon}) \qquad (7.19)$$

due to (A.2) allowing to estimate

$$\left\| \left[J\left(\frac{x_j}{t}\right), \tilde{f}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) \right] \right\| \le \overline{\omega}_0(\tilde{f}_t) \left\| \left[J\left(\frac{x_j}{t}\right), \nabla_j p(D) \right] \right\| \le C t^{\beta - 1},$$

where we have used

$$\overline{\omega}_k(\tilde{f}_t) = \| f_t^{(k+1)} \|_{L^1(\mathbf{R})} \le \| \hat{f}'_0 \|_{L^\infty(\mathbf{R})} \| \hat{\gamma}^{(k)}_{t^{-\beta}} \|_{L^1(\mathbf{R})} \le C_k t^{\beta(k+1)} \qquad (7.20)$$

(note that $\hat{f}'_0 \in L^\infty(\mathbf{R})$ because $f'_0 \in L^1(\mathbf{R})$). Using (A.9), (A.6′), we get

$$J^2\left(\frac{x_j}{t}\right) \ge J_0^2\left(\frac{x_j}{t}\right), \tilde{f}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) \ge 0 \Rightarrow \tilde{M}_1(t) \le M_1(t) + C t^{-1-\varepsilon}.$$

Step 2°. We define

$$\tilde{\tilde{f}}_t(\lambda) = \lambda f_t(\lambda), \qquad (7.21)$$

$$\tilde{M}_2(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right) \tilde{\tilde{f}}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) J_0\left(\frac{x_j}{t}\right)|_{\mathcal{K}}, \qquad (7.22)$$

and check that $\tilde{M}_2(t) \le M_2(t) + C t^{-1-\varepsilon}$ for certain constants $C, \varepsilon > 0$.

Indeed, using (2.12) we get

$$M_2(t) = \frac{2}{t} J'\left(\frac{x_j}{t}\right) \tilde{\tilde{f}}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) J\left(\frac{x_j}{t}\right)|_{\mathcal{K}} + hc + O(t^{-2}) =$$

$$= \frac{2}{t} JJ'\left(\frac{x_j}{t}\right) \left(\tilde{\tilde{f}}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) + C_0 t^{-\beta} \right)|_{\mathcal{K}} + hc + O(t^{-1-\varepsilon}). \qquad (7.23)$$

Due to (A.9), (A.6″) and (7.20), for $\beta > 0$ small enough and C_0 as in (A.7b) there is $\varepsilon > 0$ such that

$$2JJ'\left(\frac{x_j}{t}\right) \ge J_0^2\left(\frac{x_j^\circ}{t}\right),$$

$$\tilde{\tilde{f}}_t\left(\frac{x_j}{t} - \nabla_j p(D)\right) + C_0 t^{-\beta} \ge 0 \Rightarrow \tilde{M}_2(t) \le M_2(t) + C t^{-1-\varepsilon}.$$

Step 3°. To complete the proof of Lemma 7.3, we remark that the assertions of steps 1° and 2° give

$$\frac{1}{t} J_0 (\frac{x_j}{t}) (\tilde{f}_t + \tilde{\tilde{f}}_t) \left(\frac{x_j}{t} - \nabla_j p(D) \right) J_0 (\frac{x_j}{t}) |_{\mathcal{K}} \le M_1 (t) + M_2 (t) + C t^{-1-\varepsilon}, \quad (7.24)$$

while (A.7d) gives the existence of $c_0 > 0$ such that

$$\frac{c_0}{t} F_+ \left(\frac{x_j}{t} - \nabla_j p(D) \right) \le \frac{1}{t} (\tilde{f}_t + \tilde{\tilde{f}}_t) \left(\frac{x_j}{t} - \nabla_j p(D) \right) + C t^{-1-\beta}. \blacksquare \quad (7.25)$$

Proof of Lemma 7.4. It suffices to proceed as in the proof of Lemma 2.3. ■

8. Proofs of Theorems 1.2, 1.3, and 1.4

Proof of Theorem 1.2. We proceed in a similar way as in the proof of Theorem 1.1, but using the propagation estimate of Theorem 7.1 instead of the estimate of Theorem 2.1. It suffices to prove the statement for a class of functions $\bar{J}$ linearly dense in $\overset{\circ}{\mathcal{A}}$ with respect to the uniform convergence topology. Thus it suffices to consider $\bar{J} \in C^\infty (\mathbf{R}^d)$ such that $\bar{J} = 0$ in a neighbourhood of 0 and $\text{supp}\, \nabla_j \bar{J} \subset \{ x_j \in (-r; r) \setminus \{0\} \}$ for a certain $r > 0$. Denoting

$$\Omega_t = U_2 (t, s)^* \bar{J} (\frac{x}{t}) U_1 (t, s), \quad (8.1)$$

$\varphi_t^1 = U_1 (t, s) \varphi$, $\psi_t^2 = U_2 (t, s) \psi$ for $\varphi, \psi \in \mathcal{D}$, we may write (6.11) with

$$M_0 (t) = \bar{J} (\frac{x}{t}) |_{\mathcal{K}}, \quad (8.2)$$

$M_1 (t)$ given by (3.8) and

$$M_2 (t) = i V_2 (t) \bar{J} (\frac{x}{t}) - i \bar{J} (\frac{x}{t}) V_1 (t) |_{\mathcal{K}}. \quad (8.3)$$

Let $\varphi \in \mathcal{D}$ and check the Cauchy condition estimating $\| \Omega_{t''} \varphi - \Omega_{t'} \varphi \|$ as in (3.6) and ζ_k given by (6.12). The same reasoning as before gives $\sup_{\|\psi\| \le 1} \zeta_2 (t', t'', \varphi, \psi) \to 0$ for $t', t'' \to \infty$ and to complete the proof it remains to show that $\sup_{\psi \in \mathcal{D}, \|\psi\| \le 1} \zeta_1 (t', t'', \varphi, \psi) \to 0$, when $t', t'' \to \infty$. Now

$$- M_1 (t) = \sum_{1 \le j \le d} \frac{1}{t} \nabla_j \bar{J} (\frac{x}{t}) \left(\frac{x_j}{t} - \nabla_j p(D) \right) |_{\mathcal{K}} + O(t^{-2}) \quad (8.4)$$

and since $\operatorname{supp} \nabla_j \bar{J} \subset \{ x_j \in (-r; r) \setminus \{0\}\}$, it is possible to find $J_0 \in C_0^\infty((-r; r) \setminus \{0\})$ such that $\nabla_j \bar{J}(x) = (\nabla_j \bar{J})(x) J_0^2(x_j)$ for $j = 1, \ldots, d$. Setting

$$\tilde{M}_j^{\pm}(t) = \frac{1}{t} J_0\left(\frac{x_j}{t}\right) F_{\pm}\left(\frac{x_j}{t} - \nabla_j p(D)\right) J_0\left(\frac{x_j}{t}\right), \tag{8.5$_\pm$}$$

we get

$$\left|(\tilde{M}_j^{\pm}(t)\varphi, \nabla_j \bar{J}(\tfrac{x}{t})\psi)\right| \leq \sqrt{(\tilde{M}_j^{\pm}(t)\varphi, \varphi)\,(\tilde{M}_j^{\pm}(t)\,\nabla_j \bar{J}(\tfrac{x}{t})\psi, \nabla_j \bar{J}(\tfrac{x}{t})\psi)},$$

for $\varphi, \psi \in \mathcal{K}$, and using (A.8a) we have as before estimates

$$\left((\tilde{M}_j^{\pm}(t), \nabla_j \bar{J}(\tfrac{x}{t})\psi, \nabla_j \bar{J}(\tfrac{x}{t})\psi)\right) \leq C((\tilde{M}_j^{\pm}(t) + t^{-2})\,\psi, \psi). \tag{8.6$_\pm$}$$

Thus denoting

$$\tilde{M}_j(t) = \tilde{M}_j^{-}(t) + \tilde{M}_j^{+}(t) = J_0\left(\frac{x_j}{t}\right)\left|\frac{x_j}{t} - \nabla_j p(D)\right| J_0\left(\frac{x_j}{t}\right)|_{\mathcal{K}},$$

we get for $\varphi, \psi \in \mathcal{D}$, the inequality

$$|\langle \overline{M_1(t)}^{\,Q(\mathcal{D})}\varphi_t^1, \psi_t^2\rangle| \leq$$

$$\leq C \sum_{1 \leq j \leq d} \sqrt{\langle (\overline{\tilde{M}_j(t)}^{\,Q(\mathcal{D})} + t^{-2})\varphi_t^1, \varphi_t^1\rangle \langle (\overline{\tilde{M}_j(t)}^{\,Q(\mathcal{D})} + t^{-2})\,\psi_t^2, \psi_t^2\rangle}$$

and we complete the proof as before using Cauchy-Schwartz inequality and the propagation estimate $\overline{\tilde{M}_j(t)}^{\,Q(\mathcal{D})} + t^{-2} \in \mathcal{G}(H(t))$. ■

P r o o f o f T h e o r e m 1.3. It suffices to prove the existence of the limit (1.16) on vectors φ of the form

$$\varphi = \lim_{t \to \infty} U_1(t, s)^* \,\bar{J}\left(\frac{x}{t}\right) U_1(t, s)\,\tilde{\varphi}, \tag{8.8}$$

where $\bar{J} = 1 - J$, $J \in C_0^\infty(\mathbf{R}^d)$ such that $J = 1$ in a neighbourhood of 0. Hence

$$\lim_{t \to \infty} U_2(t, s)^* \, U_1(t, s)\varphi = \lim_{t \to \infty} U_2(t, s)^* \,\bar{J}\left(\frac{x}{t}\right) U_1(t, s)\,\tilde{\varphi} \tag{8.9}$$

exists due to Theorem 1.2. If $\tilde{J}$ is such that $\tilde{J} = 0$ in a neighbourhood of 0 and $\tilde{J} = 1$ on $\operatorname{supp} \bar{J}$, then

$$\Omega_{2,1}(s)\varphi = \lim_{t\to\infty} U_2(t,s)^* \mathcal{J}(\tfrac{x}{t})\tilde{\mathcal{J}}(\tfrac{x}{t})U_1(t,s)\tilde{\varphi} =$$

$$= \lim_{t\to\infty} U_2(t,s)^* \bar{\mathcal{J}}(\tfrac{x}{t})U_2(t,s) \lim_{t\to\infty} U_2(t,s)^* \tilde{\mathcal{J}}(\tfrac{x}{t})\mathring{U}_1(t,s)\tilde{\varphi},$$

hence $\Omega_{2,1}(s)\varphi \in \operatorname{Ran}\Gamma_2(\bar{\mathcal{J}})$. Thus $\Omega_{2,1}(s)$ is an isometric injection of $\mathcal{H}_1^{\rm scat}(s)$ into $\mathcal{H}_2^{\rm scat}(s)$ and an analogous reasoning shows that for $\psi \in \mathcal{H}_2^{\rm scat}(s)$,

$$\Omega_{1,2}(s)\psi = \lim_{t\to\infty} U_1(t,s)^* \; U_2(t,s)\psi \tag{8.10}$$

exists, defining an isometric injection of $\mathcal{H}_2^{\rm scat}(s)$ into $\mathcal{H}_1^{\rm scat}(s)$. Since $\psi = \Omega_{2,1}(s)\,\Omega_{1,2}(s)\psi$ for every $\psi \in \mathcal{H}_2^{\rm scat}(s)$, it is clear that $\Omega_{2,1}(s)$ is also surjective. ■

P r o o f o f T h e o r e m 1.4. We assume that $k = 1$ and prove first that $\mathcal{H}_1^{\rm scat} \supset \mathcal{H}_c(H_1)$. Let $\varphi \in \mathcal{H}_c(H)$ be such that $\varphi = h(H)\varphi$ with $h \in C_0^\infty((\lambda_0 - \delta_0; \lambda_0 + \delta_0))$, $\lambda_0 \in \mathcal{R}$ and δ_0 chosen such that (5.2) holds with a certain $r > 0$. If $J_{r'} \in C_0^\infty((-r'; r')^d)$, $J_{r'} = 1$ in a neighbourhood of 0 and $\bar{J}_{r'} = 1 - J_{r'}$, then

$$\varphi = \lim_{t\to\infty} e^{itH_1} \bar{J}_{r'}(\tfrac{x}{t})e^{-itH_1}\varphi + \lim_{t\to\infty} e^{itH_1} J_{r'}(\tfrac{x}{t})e^{-itH_1}\varphi. \tag{8.11}$$

Since the first term on the right hand side belongs to $\mathcal{H}_1^{\rm scat}$, it suffices to show that for any $\varepsilon > 0$ we can choose r' such that the second term on the right side of (8.11) has its norm smaller then ε. Since the limit exists, it suffices to prove the existence of a constant C_0 such that for $0 < r' \le r/2$,

$$\limsup_{T\to\infty} \frac{1}{T}\int_1^T dt\, \| J_{r'}(\tfrac{x}{t})e^{-itH_1}\varphi \|^2 \le C_0 \frac{r'}{r} \|\varphi\|^2. \tag{8.12}$$

But the decomposition of unity from Lemma 5.1 allows to apply Theorem 4.1 and the compactness of $g_0(\nabla p(D))h(H_1)$, implying (8.12) exactly as in the proof of Lemma 5.3.

To prove that $\mathcal{H}_1^{\rm scat} \cap \mathcal{H}_{pp}(H_1) = \emptyset$ denote by Π_0 the orthogonal projection on an eigenvector of H_1 and remark that

$$\Pi_0\varphi = \lim_{t\to\infty} e^{itH_1}\Pi_0 \mathcal{J}\left(\frac{x}{t}\right) e^{-itH_1}\tilde{\varphi} = 0,$$

because Π_0 is compact and $\mathcal{J}\left(\frac{x}{t}\right) \to 0$ strongly, hence $\Pi_0 \mathcal{J}\left(\frac{x}{t}\right) \to 0$ in the norm. ■

Appendix

Lemma A.1. a) *Let $\mathcal{H}$ be a Hilbert space, B, A, $\tilde{A}$ self-adjoint operators in $\mathcal{H}$ and $\mathcal{K}$ a dense subspace of $\mathcal{H}$, contained in the intersection of domains of B, A, $\tilde{A}$ i.e. $\mathcal{K} \subset \mathcal{D}(B) \cap \mathcal{D}(A) \cap \mathcal{D}(\tilde{A})$ and let $f \in C^\infty(\mathbf{R})$ be such that the Fourier transform of its derivative $\hat{f'} \in L^1(\mathbf{R})$. If $\mathrm{ad}^Q_{A,\tilde{A}} B$ extends to a bounded form on $\mathcal{H}$, then for $\varphi, \psi \in \mathcal{K}$,*

$$\langle \mathrm{ad}^Q_{f(A),f(\tilde{A})} B\varphi, \psi\rangle =$$

$$= \int \frac{d\lambda}{2\pi}\hat{f'}(\lambda) \int_0^1 d\sigma \langle (\mathrm{ad}^Q_{A,\tilde{A}} B) e^{i\sigma\lambda A}\varphi, e^{-i(1-\sigma)\lambda\tilde{A}}\psi\rangle. \tag{A.1}$$

Consequently $\mathrm{ad}^Q_{f(A),f(\tilde{A})} B$ extends to a bounded form on $\mathcal{H}$ and

$$\| \mathrm{ad}^Q_{f(A),f(\tilde{A})} B \| \le \frac{1}{2\pi} \|\hat{f'}(\lambda)\|_{L^1(\mathbf{R}, d\lambda)} \| \mathrm{ad}^Q_{A,\tilde{A}} B\|. \tag{A.2}$$

b) *Assume moreover that $\mathcal{K}$ is a Frechet space invariant with respect to B, A, $\tilde{A}$ and for $j \in \mathbf{N}\setminus\{0\}$ let $\mathrm{ad}^j_{A,\tilde{A}} B|_{\mathcal{K}}$ denote linear operators on $\mathcal{K}$ defined by induction as follows*

$$\mathrm{ad}_{A,\tilde{A}} B|_{\mathcal{K}} = \mathrm{ad}^1_{A,\tilde{A}} B|_{\mathcal{K}} = BA - \tilde{A}B|_{\mathcal{K}},$$

$$\mathrm{ad}^{j+1}_{A,\tilde{A}} B|_{\mathcal{K}} = \left(\mathrm{ad}^j_{A,\tilde{A}} B|_{\mathcal{K}}\right) A|_{\mathcal{K}} - \tilde{A}\left(\mathrm{ad}^j_{A,\tilde{A}} B|_{\mathcal{K}}\right). \tag{A.3}$$

Assume also that $\lambda \to e^{i\lambda A}\varphi$, $\lambda \to e^{i\lambda\tilde{A}}\varphi$ are continuous functions $\mathbf{R} \to \mathcal{K}$ for every $\varphi \in \mathcal{K}$. If $n \in \mathbf{N}\setminus\{0\}$ is such that $\mathrm{ad}^{n+1}_{A,\tilde{A}} B$ extends to a bounded operator on $\mathcal{H}$ and $\hat{f}^{(n+1)} \in L^1(\mathbf{R})$, then we may write

$$\mathrm{ad}^Q_{f(A),f(\tilde{A})} B|_{\mathcal{K}} = \sum_{j=1}^{n} \frac{f^{(j)}(\tilde{A})}{j!} \mathrm{ad}^j_{A,\tilde{A}} B|_{\mathcal{K}} + R_n, \tag{A.4}$$

where the remainder R_n extends to a bounded operator on $\mathcal{H}$, and

$$\| R_n \| \le \bar{\omega}_n (f) \| \operatorname{ad}^{n+1}_{A, \tilde{A}} B \| \text{ with } \bar{\omega}_n (f) = \| \hat{f}^{(n+1)} (\lambda) \|_{L^1(\mathbf{R}, d\lambda)} . \tag{A.4'}$$

P r o o f. We assume first that $\hat{f}^{(j)} \in L^1(\mathbf{R})$ for all $j \in \mathbf{N}$. Then, for $\varphi, \psi \in \mathcal{K}$,

$$(f(A)\varphi, B\psi) - (B\varphi, f(\tilde{A})\psi) = \int \frac{d\lambda}{2\pi} \hat{f}(\lambda) ((e^{i\lambda A}\varphi, B\psi) - (B\varphi, e^{-i\lambda\tilde{A}}\psi)).$$

Introducing for $\varphi, \psi \in \mathcal{K}$,

$$\upsilon (\sigma, \lambda, \varphi, \psi) = (Be^{i\sigma\lambda A} \varphi, e^{-i(1-\sigma)\lambda\tilde{A}} \psi),$$

we have

$$\frac{\partial\upsilon}{\partial\sigma} (\sigma, \lambda, \varphi, \psi) = i\lambda \langle (\operatorname{ad}^{Q}_{A, \tilde{A}} B) e^{i\sigma\lambda A} \varphi, e^{\ i(1-\sigma)\lambda\tilde{A}} \psi \rangle,$$

and (A.1) follows from

$$(e^{i\lambda A}\varphi, B\psi) - (B\varphi, e^{-i\lambda\tilde{A}}\psi) = [\upsilon (\sigma, \lambda, \varphi, \psi)]_0^1 = \int_0^1 d\sigma \frac{\partial\upsilon}{\partial\sigma} (\sigma, \lambda, \varphi, \psi).$$

Under the hypotheses of b), $\sigma \to \upsilon (\sigma, \lambda, \varphi, \psi)$ is C^{∞} and

$$\frac{\partial^j \upsilon}{\partial\sigma^j} (\sigma, \lambda, \varphi, \psi) = (i\lambda)^j (e^{i(1-\sigma)\lambda\tilde{A}} (\operatorname{ad}^{j}_{A, \tilde{A}} B) e^{i\sigma\lambda A} \varphi, \psi).$$

Using the Taylor formula of order n at 0 for $\upsilon (\cdot , \lambda, \varphi, \psi)$ to express $[\upsilon (\sigma, \lambda, \varphi, \psi)]_0^1$, we get (A.4) with

$$R_n \varphi = \int \frac{d\lambda}{2\pi} \hat{f}^{(n+1)} (\lambda) \int_0^1 \frac{\sigma^n d\sigma}{n!} e^{i(1-\sigma)\lambda\tilde{A}} (\operatorname{ad}^{n+1}_{A, \tilde{A}} B) e^{i\sigma\lambda A} \varphi,$$

for $\varphi \in \mathcal{K}$, which implies clearly the estimate (*A.4'*).

To complete the proof for a general f, it suffices to pass to the limit $\varepsilon \to 0$ in estimates for the sequence of functions $f(x) \gamma(\varepsilon x)$ with $\gamma \in C_0^{\infty} (\mathbf{R})$, $\gamma(0) = 1$. ■

Corollary A.2. *Let A, B satisfy the hypotheses of Lemma A.1b), $\tilde{A} = A$ and*

$$\operatorname{ad}^{j}_{A} B|_{\mathcal{K}} \overset{\text{(def)}}{=} \operatorname{ad}^{j}_{A, A} B|_{\mathcal{K}} \in B_{\mathcal{K}}(\mathcal{H}) \text{ for } j \in \mathbf{N}. \tag{A.5}$$

a) *If $\hat{f}' \in L^1 (\mathbf{R})$, then*

$$\operatorname{ad}^k_{f(A)} B|_{\mathcal{K}} \overset{(\mathrm{def})}{=} \operatorname{ad}^k_{f(A),f(A)} B|_{\mathcal{K}} \in B_{\mathcal{K}}(\mathcal{H}), \tag{A.6}$$

$$\| \operatorname{ad}^k_{f(A)} B|_{\mathcal{K}} \| \le \overline{\omega}_0 (f)^k \| \operatorname{ad}^k_A B|_{\mathcal{K}} \| \tag{A.6$'$}$$

with $\overline{\omega}_n (f)$ *as in* (*A*.4′).

b) *If* $\hat{f}'' \in L^1(\mathbf{R})$ *and* $f' \in L^\infty(\mathbf{R})$, *then (A.6) still holds, but* (*A*.6′) *should be replaced by*

$$\| \operatorname{ad}^k_{f(A)} B|_{\mathcal{K}} \| \le (\overline{\omega}_1 (f) + \|f'\|_{L^\infty(\mathbf{R})})^k \max_{k \le j \le 2k} \| \operatorname{ad}^j_A B|_{\mathcal{K}} \|. \tag{A.6$''$}$$

P r o o f. For $k = 1$ the statement results from Lemma A.1a) with $A = \tilde{A}$. In the following we drop $|_{\mathcal{K}}$ and note that $\operatorname{ad}_{f(A)} \operatorname{ad}^j_A = \operatorname{ad}^j_A \operatorname{ad}_{f(A)}$, which holds for $j = 1$ due to the Jacobi identity and the obvious induction extends the equality to arbitrary $j \in \mathbf{N}$.

Then, by induction with respect to k we may write

$$\| \operatorname{ad}^{k+1}_{f(A)} B \| = \| \operatorname{ad}^k_{f(A)} \operatorname{ad}_{f(A)} B \| \le \overline{\omega}_0 (f)^k \| \operatorname{ad}^k_A \operatorname{ad}_{f(A)} B \| =$$
$$= \overline{\omega}_0 (f)^k \| \operatorname{ad}_{f(A)} \operatorname{ad}^k_A B \| \le \overline{\omega}_0 (f)^k \overline{\omega}_0 (f) \| \operatorname{ad}_A \operatorname{ad}^k_A B \| =$$
$$= \overline{\omega}_0 (f)^{k+1} \| \operatorname{ad}^{k+1}_A B \|.$$

To get b) for $k \doteq 1$, we use Lemma A.1 with $n = 1$ and $A = \tilde{A}$, which gives

$$\operatorname{ad}_{f(A)} B = f'(A) \operatorname{ad}_A B + R_1, \quad \| R_1 \| \le \overline{\omega}_1 (f) \| \operatorname{ad}^2_A B \|,$$

and clearly $\|f'(A) \operatorname{ad}_A B \| \le \|f'\|_{L^\infty} \| \operatorname{ad}_A B \|$. Then as in the proof of a), we get the desired result for every $k \in \mathbf{N}$ by induction with respect to k. ■

Lemma A.3. *Let* f_t *and* f_0 *be such as in the section* 7. *Then there exists a constant* C_0 (*independent on* $t \ge 1$, $\lambda \in \mathbf{R}$) *such that*

$$| f_t (\lambda) - f_0 (\lambda) | \le C_0 t^{-\beta}, \tag{A.7a}$$

$$| \lambda \frac{d}{d\lambda} f_t (\lambda) - \lambda \frac{d}{d\lambda} f_0 (\lambda) | \le C_0 t^{-\beta}, \tag{A.7b}$$

$$| \frac{d}{dt} f_t (\lambda) | \le C_0 t^{-1-\beta}, \tag{A.7c}$$

$$\exists\ c_0 > 0,\ \lambda f_t(\lambda) + \lambda \frac{d}{d\lambda} f_t(\lambda) \geq c_0 F_+(\lambda) - C_0 t^{-\beta}. \tag{A.7d}$$

P r o o f. By the definition of f_t we have

$$(f_t - f_0)(\lambda) = \int (f_0(\lambda + \lambda') - f_0(\lambda))\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda',$$

$$\lambda \frac{d}{d\lambda}(f_t - f_0)(\lambda) = \int (f_0'(\lambda + \lambda')\,\lambda - f_0'(\lambda)\,\lambda)\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda',$$

$$\frac{d}{dt} f_t(\lambda) = \frac{d}{dt}(f_t - f_0)(\lambda) = \int (f_0(\lambda + \lambda') - f_0(\lambda))\,\frac{d}{dt}\gamma_{t^{-\beta}}(\lambda')\,d\lambda'.$$

We get a) using the fact that f_0 is Lipschitz continuous and estimating

$$|(f_t - f_0)(\lambda)| \leq C \int |\lambda'|\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' = CC_1 t^{-\beta}.$$

Also $\lambda \to f_0'(\lambda)\,\lambda$ is Lipschitz continuous, hence

$$\left| \int (f_0'(\lambda + \lambda')(\lambda + \lambda') - f_0'(\lambda)\,\lambda)\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' \right| \leq CC_1 t^{-\beta},$$

and we get b) noting moreover that

$$\left| \int (f_0'(\lambda + \lambda')\,\lambda'\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' \right| \leq C \int |\lambda'|\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' = CC_1 t^{-\beta}.$$

To get c) we use fact that f_0 is Lipschitz continuous and

$$\frac{d}{dt}\gamma_{t^{-\beta}}(\lambda') = \beta t^{\beta-1}\gamma_1(t^{\beta}\lambda') + \beta t^{\beta-1}\lambda'\gamma_1'(t^{\beta}\lambda')\,t^{\beta},$$

hence the first term may be estimated by the same integral as before and the second term gives an estimate by the integral

$$C \int |\lambda'|^2\, t^{\beta-1}\,\gamma_{t^{-\beta}}(\lambda')\,d\lambda' = CC_2 t^{-1-\beta}.$$

Since $\operatorname{supp} f_t(\lambda) \subset [-t^{-\beta}; \infty)$, we have $\lambda f_t(\lambda) \geq Ct^{-\beta}$ and using b) we have

$$\lambda \frac{d}{d\lambda} f_t(\lambda) = \lambda f_0'(\lambda) + O(t^{-\beta}) = F_+(\lambda) + O(t^{-\beta}) \ \text{ if } \lambda \leq \frac{1}{4},$$

hence d) holds for $\lambda \leq \frac{1}{4}$; finally we complete the proof of d) estimating

$$\lambda \geq \frac{1}{4} \Rightarrow \lambda f_t(\lambda) \geq \lambda f_t(\tfrac{1}{4}) \geq \lambda c_1 \quad \text{with} \quad c_1 = \frac{1}{8}\int_0^{1/8} \gamma_1 > 0. \qquad \blacksquare$$

Lemma A.4 (Nirenberg-Trèves). *There exists a constant C (independent on operators A and B) such that*

$$\| \operatorname{ad}_{F_+(A)^{1/2}} B \| \leq C \| B \|^{1/2} \| \operatorname{ad}_A B \|^{1/2}, \tag{A.8a}$$

$$\| \operatorname{ad}^2_{F_+(A)^{1/2}} B \| \leq C^2 \| B \|^{1/4} \| \operatorname{ad}_A B \|^{1/2} \| \operatorname{ad}^2_A B \|^{1/4}. \tag{A.8b}$$

It is possible to give a proof using the Cauchy resolvent formula (cf. [Hö], § 26.8), but here we present a slightly different proof.

P r o o f. Using the same γ_ε as above, we may estimate

$$| (F^s_+ * \gamma_\varepsilon - F^s_+)(\lambda) | \leq$$

$$\leq \int | F_+(\lambda + \lambda')^s - F_+(\lambda)^s | \gamma_\varepsilon(\lambda')\, d\lambda' \leq \int | \lambda' |^s \gamma_\varepsilon(\lambda')\, d\lambda' = C_s \varepsilon^s,$$

if $0 < s \leq 1$, hence

$$\| \operatorname{ad}_{F_+(A)^{1/2}} B \| \leq \| \operatorname{ad}_{(F^{1/2}_+ * \gamma_\varepsilon)(A)} B \| + 2C_{1/2}\, \varepsilon^{1/2} \| B \|.$$

Since $(F^{1/2}_+)'$ is homogeneous of the degree $-\frac{1}{2}$, we have

$$\| \widehat{(F^{1/2}_+)'}\, \hat{\gamma}_\varepsilon \|_{L^1(\mathbf{R})} \leq C' \varepsilon^{-1/2},$$

$$\| \operatorname{ad}_{F_+(A)^{1/2}} B \| \leq C' \varepsilon^{-1/2} \| \operatorname{ad}_A B \| + C_{1/2}\, \varepsilon^{1/2} \| B \|,$$

due to (A.2). We get (A.8a) setting $\varepsilon = \| \operatorname{ad}_A B \| / \| B \|$. Finally

$$\| \operatorname{ad}_{F_+(A)^{1/2}} \operatorname{ad}_{F_+(A)^{1/2}} B \| \leq C \| \operatorname{ad}_{F_+(A)^{1/2}} B \|^{1/2} \| \operatorname{ad}_A \operatorname{ad}_{F_+(A)^{1/2}} B \|^{1/2} \leq$$

$$\leq C^{\frac{3}{2}} \| B \|^{\frac{1}{4}} \| \operatorname{ad}_A B \|^{\frac{1}{4}} \| \operatorname{ad}_{F_+(A)^{1/2}} \operatorname{ad}_A B \|^{1/2} \leq$$

$$\leq C^2 \| B \|^{\frac{1}{4}} \| \operatorname{ad}_A B \|^{\frac{1}{4}} \| \operatorname{ad}_A B \|^{\frac{1}{4}} \| \operatorname{ad}^2_A B \|^{\frac{1}{4}}. \quad \blacksquare$$

Clearly the estimates hold also with $F_-(A)$ or $|A| = (F_+ + F_-)(A)$ instead of $F_+(A)$.

Corollary A.5. *There exists a constant C such that for any operators A, B,*

$$A \geq 0,\ B \geq 0 \Rightarrow AB + hc \geq -C^2 \| B \|^{1/4} \| \mathrm{ad}_A B \|^{1/2} \| \mathrm{ad}_A^2 B \|^{1/4}. \quad \text{(A.9)}$$

P r o o f. The desired lower bound results from (A.8b), because

$$AB + hc = F_+(A)B + hc =$$

$$= F_+(A)^{1/2} B F_+(A)^{1/2} + \frac{1}{2} \mathrm{ad}^2_{F_+(A)^{1/2}} B \geq \frac{1}{2} \mathrm{ad}^2_{F_+(A)^{1/2}} B. \blacksquare$$

P r o o f o f P r o p o s i t i o n 2.4. To get (2.11a) we use (A.2) with $A = \tilde{A} = x_j/t$, $B = \eta(D)$, $f = J$, noting that $[i\eta(D), x_j/t] = \nabla_j \eta(D)/t = O(t^{-1})$. To get (2.11c) we use (A.4) with $A = \tilde{A} = x_j/t$, $B = p(D)$, $f = J$, $\mathcal{K} = H^\infty(\mathbf{R}^d)$, noting that $-[[p(D), x_j/t], x_j/t] = [i\nabla_j p(D), x_j]/t^2 = \nabla_j^2 p(D)/t^2 = O(t^{-2})$.

To get (2.11b) we use (A.9) with $A = \eta(D)$, $B = J(x_j/t)$ and it remains to prove that

$$\left[\eta(D), [\eta(D), J(\frac{x_j}{t})]\right] = O(t^{-2}). \quad \text{(A.10)}$$

Using the expression (A.1) with $A = \tilde{A} = x_j/t$, $B = \eta(D)$, $f = J$, we can see that it suffices to show the existence of a constant C such that for all $t \geq 1$, the estimate

$$\left\| \left[\eta(D), e^{i(1-\sigma)\lambda x_j/t} [\eta(D), \frac{x_j}{t}] e^{i\sigma\lambda x_j/t}\right] \right\| \leq C|\lambda| t^{-2}, \quad \text{(A.11)}$$

holds. But $\eta(D) e^{isx_j} = e^{isx_j} \eta(D + se_j)$, hence (A.11) follows from

$$\| (\eta(D + (1-\sigma)\lambda e_j/t) - \eta(D - \sigma\lambda e_j/t)) \nabla_j^{\circ} \eta(D)/t \| \leq C|\lambda| t^{-2},$$

being a consequence of the Lipschitz continuity of η, i.e. $|\eta(\xi + \nu) - \eta(\xi)| \leq C_0 |\nu|$. $\blacksquare$

R e m a r k c o n c e r n i n g (H_2). Let $\Phi \in L^1([1; \infty); dt)$ be positive, decreasing and assume

$$(1/\Phi(|x|))\, V_k^s E_{[-\bar{r}; \bar{r}]}(H) \text{ extends to a bounded operator,} \quad \text{(A.12a)}$$

$$|\partial^\alpha V_k^\ell(x)| \leq C_\alpha \Phi(|x|) |x|^{(1-|\alpha|)\rho} \text{ with } \rho > 0,\ 1 \leq |\alpha| \leq N_\rho. \quad \text{(A.12b)}$$

Then $(\mathbf{H}_2)$ holds and following [Hö] it is possible to decrease N_ρ modulo additional short range terms. If p is quadratic then $(\mathbf{H}_2\, b)$ is reduced immediately to the condition

$$|\nabla V_k^\ell(x)| \le C\,\Phi(|x|). \qquad (A.12b')$$

Note that $(\mathbf{H}_2 a)$ appeared in [E 1], but many papers consider only the case $\Phi(t) = Ct^{-1-\varepsilon}$, $\varepsilon > 0$. Note also that V_k^ℓ may be a (pseudo) differential operator (especially when $|p(\xi)| \to \infty$ for $|\xi| \to \infty$, cf. [Hö], [If] with suitable decay properties of coefficients.

R e m a r k c o n c e r n i n g $(\mathbf{H}_3)$. If p satisfies the condition $|p(\xi)| + |\nabla p(\xi)| \to \infty$ when $|\xi| \to \infty$ and inequalities $|\partial^\alpha p(\xi)| \le C_\alpha (1 + |p(\xi)| + |\nabla p(\xi)|)$, then it is proved in [DM] that $(\mathbf{H}_3\, a)$ holds for every V_k being H_0-compact. The same hypotheses on p imply (cf. [DM]) the compactness of $f(x)\, h(H_0)$ for any bounded functions f, h such that $f(x) \to 0$ for $|x| \to \infty$ and $h(\lambda) \to 0$ for $|\lambda| \to 0$. Using f, g as before and assuming that f is positive and $A(1/f)(x)$ extends to a bounded operator, one gets $Ah(H_0)$ compact.

If Φ is decreasing and integrable on $\mathbf{R}$ then $\Phi(t) = o(t^{-1})$ for $t \to \infty$ hence $(A.12b')$ [or (A.12b] implies $(1 + |x|)|\nabla V_k^\ell(x)| \to 0$ [or $(1 + |x|)\, \partial^\alpha V_k^\ell(x) \to 0$] for $|x| \to \infty$. Then $(\mathbf{H}_3\, c)$ holds if $V_k^\ell(x) \to 0$ for $|x| \to \infty$ with (A.12b) [or (A.12b') if p is quadratic] satisfied and with p satisfying estimates formulated above.

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Волновые операторы типа Дейфта-Саймона для одного класса пропагаторов Шредингера. I

Лех Зелински

Исследуются вопросы теории рассеяния, связанные с асимптотическим поведением некоторых Шредингеровских пропагаторов. А именно, представлены результаты об их асимптотической полноте, которые получены методом волновых операторов Дейфта–Саймона, развитым в настоящее время в теории *N*-тел. Рассматривается случай 2-х тел для класса общих, зависящих от времени гамильтонианов $H(t) = H_0 + V(t, x)$, где H_0 – дифференциальный оператор второго порядка с постоянными коэффициентами, а $V(t, x)$ убывает при $|x| \to \infty$.

Хвильові оператори типу Дейфта-Саймона для одного класу пропагаторів Шредінгера. I

Лех Зелінські

Досліджуються питання теорії розсіювання, пов'язані з асимптотичною поведінкою деяких Шредингерових пропагаторів. А саме, представлено результати про їх асимптотичну повноту, які одержані методом хвильових операторів Дейфта–Саймона, розвинутим в даний час в теорії N-тіл. Розглядається випадок 2-х тіл для класу загальних гамільтоніанів $H(t) = H_0 + V(t, x)$, що залежать від часу, де H_0 – диференціальний оператор другого порядку з постійними коефіцієнтами, а $V(t, x)$ спадає при $|x| \to \infty$.