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Gårding domains for unitary representations of countable inductive limits of locally compact groups

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Let G be the inductive limit of an increasing sequence of locally compact second countable groups $G_1 \subset G_2 \subset \dots$. Given a strongly continuous unitary representation U of G in a separable Hilbert space $\mathcal{H}$, we construct an U-invariant, separable, nuclear, Montel (DF)-space $\mathcal{F}$ which is densely (topologically) embedded in $\mathcal{H}$ and such that the restriction of U to $\mathcal{F}$ is a weakly continuous representation of G by continuous linear operators in $\mathcal{F}$. Moreover, $\mathcal{F}$ is a domain of essential self-adjointness for the generator of each one-parameter subgroup of G, and all such generators keep $\mathcal{F}$ invariant.

0. Introduction

Modern analysis on functions (and spaces of functions) of infinite variables (see, for example, [BK]) leads naturally to studying the groups of symmetry of infinite dimensional topological vector spaces and their representations. Many of these groups of symmetry are the countable inductive limits of locally compact groups or contain them as dense subgroups. We note that the class of topological groups that can be represented in the form of such inductive limit includes the infinitely dimensional Lie groups, the groups of finite H-currents on a countable set, the group of step H-currents on the segment $[0,1]$ with discontinuities in the binary-rational points, etc., where H is a locally compact group.

The purpose of this paper is to construct a Gårding domain possessing a number of additional "good" properties for a unitary representation of an arbitrary countable inductive limit of locally compact second countable groups. Such spaces endowed with the natural differentiable structure proved to be a powerful tool in studying representations of the groups and their Lie algebras (in a generalized sense).

The author's need for Gårding domains with the special "good" properties became apparent during the course of work on point realization of Boolean actions of non locally compact groups [D], where the linearization approach of A.M. Vershik [V] was used. So, the present work can be also viewed as preparatory to [D].

Let G be a topological group and U a strongly continuous unitary representation of G in a separable Hilbert space $\mathcal{H}$. The family of all one-parameter subgroups of G will be denoted by $\mathfrak{g}$. For each $\partial = \{ \partial(t) | \, t \in R \} \in \mathfrak{g}$, $U(\partial)$ will stand for the self-adjoint generator of the unitary one-parameter subgroup $\{ U(\partial(t)) | \, t \in R \}$. Following the concept of G.C. Hegerfeld [Heg], by Gårding domain for U we shall mean a subspace of $\mathcal{H}$ that is $U(g)$- and $U(\partial)$-invariant, $g \in G$, $\partial \in \mathfrak{g}$, and, besides, which is a domain of essential self-adjointness for all $U(\partial)$. As it was shown in [G], if G is a Lie group, then a Gårding domain consisting of ∞-differentiable vectors for U exists for each continuous unitary representation of G. This result was later extended to arbitrary locally compact group [Ka]. However the situation is different for non locally compact groups (see [Sa, § 2.5] for a counterexample). The main difficulty arising is the lack of Haar measure on G, which was used explicitly in the constructions of [G, Ka]. Nevertheless P. Richter [Ri], Yu.S. Samoilenko [Sa] and A.V. Kosyak [Ko] provided Gårding domains for some ∞-dimensional Lie groups. In this paper we generalize and refine this to arbitrary $G = \operatorname{inj}\lim_{n \to \infty} G_n$ for an increasing sequence $G_1 \subset G_2 \subset \ldots$ of locally compact second countable subgroups.

Main Theorem. *Let $U : G \ni g \mapsto U(g)$ be a strongly continuous unitary representation of G in a separable Hilbert space $\mathcal{H}$. Then there exist a separable nuclear Montel space $\mathcal{F}$ and continuous one-to-one linear map $J : \mathcal{F} \to \mathcal{H}$ such that the following properties are satisfied:*

(i) *$\mathcal{F}'$ endowed with $\beta(\mathcal{F}', \mathcal{F})$ is a separable Fréchet space;*

(ii) *$\operatorname{Im} J$ is dense in $\mathcal{H}$;*

(iii) *$U(g) \operatorname{Im} J = \operatorname{Im} J$ for all $g \in G$;*

(iv) *$U(\partial) \operatorname{Im} J \subset \operatorname{Im} J$ and $\operatorname{Im} J$ is a domain of essential self-adjointness for $U(\partial)$ for all $\partial \in \mathfrak{g}$;*

(v) *$J^{-1} U(g) J \in \mathcal{L}(\mathcal{F}, \mathcal{F})$ for all $g \in G$;*

(vi) *$G \ni g \mapsto J^{-1} U(g) J \in \mathcal{L}(\mathcal{F}, \mathcal{F})$ is a continuous map, where $\mathcal{L}(\mathcal{F}, \mathcal{F})$ is the space of continuous linear operators on $\mathcal{F}$ endowed with the weak operator topology.*

We shall call such $\mathcal{F}$ (respectively $\operatorname{Im} J$) a *strong Gårding space* (respectively *strong Gårding domain*) for U. Note that under the above assumptions on G there is a natural structure of Lie algebra on $\mathfrak{g}$. So, it is relevant to call $\mathfrak{g}$ the *Lie algebra* of G.

To prove the theorem we first define the space $\mathcal{D}(G)$ of "basic functions" on G as the inductive limit of $\mathcal{D}(G_n)$, $n \in N$, with some compact canonical embeddings (depending on U). It should be noted at once that the elements of $\mathcal{D}(G)$ are not functions on G in a proper sense, but they are the sequences of functions on G_n, $n \in N$. Then the strong Gårding space for U appears as the locally convex direct sum of a countable family of some quotient spaces of $\mathcal{D}(G)$. The main technical tools used here are the Gleason–Montgomery–Zippen structure theorem for locally compact groups [MZ] and the theory of (LN*)- and (M*)-spaces developed by J. Sebastião e Silva [SS] and D.A. Raikov [Ra1-Ra3]. Finally, we define a strong Gårding domain for U as the linear span of the set

$$\left\{ \int\limits_{G_n} f_n(g_n)\, U(g_n)\, h_{n,k}\, d\lambda_n(g_n) \;\middle|\; f_n \in \mathcal{D}(G_n), \quad n, k \in N \right\},$$

where λ_n is a left Haar measure on G_n, and $\{ h_{n,k} | \; n, k \in N\} \subset \mathcal{H}$ is a special compatible collection of vectors.

The paper is organized as follows. The first section contains background on topological vector spaces. Here our attention is mainly focused on the properties of (LN*)- and (M*)-spaces. The second section is devoted to the proof of Main Theorem in the most important case: all the subgroups $G_n \subset G$ are Lie groups and $\dim G_{n+1}/G_n > 0$, $n \in N$. The final section begins with some applications of the Gleason–Montgomery–Zippen theory to the inductive limits of locally compact groups. Then we use these applications to adapt the argument of the previous section to the general case.

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1. Preliminaries

In the present paper we consider vector spaces over the field $\mathbb{R}$. Without explanations we shall use standard concepts and facts of the topological vector space theory [P, Sc]. Given a locally convex space $\mathcal{E}$, we denote by $\mathcal{E}'$ the (topologically) dual vector space, namely the space of continuous linear forms on $\mathcal{E}$ with values in $\mathbb{R}$. The notation $\langle \mathcal{E}, \mathcal{F} \rangle$ means a duality system, i.e., two vector spaces $\mathcal{E}$ and $\mathcal{F}$ equipped with an irreducible bilinear form $\langle . , . \rangle$ on the product $\mathcal{E} \times \mathcal{F}$. As usual by $\sigma(\mathcal{E}, \mathcal{F})$ and $\beta(\mathcal{E}, \mathcal{F})$ we denote the standard topologies on $\mathcal{E}$: the weak one and the strong one respectively.

Given two locally convex spaces $\mathcal{E}$ and $\mathcal{F}$, we denote by $\mathcal{L}(\mathcal{E}, \mathcal{F})$ the space of all continuous linear maps $A: \mathcal{E} \to \mathcal{F}$. The *weak operator* topology on $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is determined by the following system of seminorms:

$$\mathcal{L}(\mathcal{E}, \mathcal{F}) \ni A \mapsto |\langle Ae, f \rangle|, \; e \in \mathcal{E}, f \in \mathcal{F}',$$

where $\langle . , . \rangle$ is the natural bilinear form on $\mathcal{F} \times \mathcal{F}'$. Everywhere below $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is endowed with the weak operator topology, unless another is stated explicitly. Given an operator $A \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, we denote by $A' : \mathcal{F}' \to \mathcal{E}'$ the adjoint operator. Then $A' \in \mathcal{L}(\mathcal{F}', \mathcal{E}')$ if $\mathcal{F}'$ and $\mathcal{E}'$ are endowed with $\beta(\mathcal{F}', \mathcal{F})$ and $\beta(\mathcal{E}', \mathcal{E})$ respectively.

The following two definitions were introduced by J. Sebastião e Silva [SS] and D.A. Raikov [Ra1].

A locally convex space $\mathcal{E}$ is said to be an (LN*)-*space* if $\mathcal{E} = \operatorname{inj} \lim_{n \to \infty} \mathcal{E}_n$ for an increasing sequence $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots$ of Hausdorff locally convex spaces with the compact canonical embeddings $\mathcal{E}_n \subset \mathcal{E}_{n+1}$, $n \in \mathbb{N}$.

A locally convex space $\mathcal{F}$ is said to be an (M*)-*space* if $\mathcal{F} = \operatorname{proj}\lim_{n\to\infty} \mathcal{F}_n$ for a sequence $\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \ldots$ of Hausdorff locally convex spaces with the compact canonical mappings $\mathcal{F}_n \leftarrow \mathcal{F}_{n+1}$, $n \in N$.

We remind some basic facts concerning these two classes of locally convex spaces. For more information and proofs see [SS, Ra1–Ra3].

Every (LN*)-space is a complete, Hausdorff, Montel space. Moreover, $\mathcal{E}$ is a free union of $\mathcal{E}_n$, $n \in N$, i. e., an arbitrary (not necessarily absolutely convex) subset $A \subset \mathcal{E}$ is closed if and only if $A \cap \mathcal{E}_n$ is closed in $\mathcal{E}_n$ for all $n \in N$. Every (M*)-space is a Fréchet–Montel space. It follows that an (M*)-space is separable [Sc, IV, Exercise 19(d)]. A closed subspace as well as a Hausdorff quotient space of an (LN*)-space ((M*)-space) is an (LN*)-space ((M*)-space). The strongly dual space to an (LN*)-space ((M*)-space) is an (M*)-space ((LN*)-space).

2. Case of infinitely dimensional Lie groups

2.1. Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{A}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. For every functional space $\mathcal{E}$ on G we denote by L_g and R_g, $g \in G$, the operators of left and right shift respectively:

$$(L_g f)(h) = f(g^{-1}h), \ (R_g f)(h) = f(hg), \ h \in G, \ f \in \mathcal{E}.$$

Given a compact subset $K \subset G$, we set

$$\mathcal{D}(G, K) = \{ f\colon G \to R \mid f \in C^\infty(G) \ \text{and} \ \operatorname{Supp} f \subset K \}.$$

Then $\mathcal{D}(G) = \cup_{K \in \mathcal{K}} \mathcal{D}(G, K)$ is the space of compactly supported ∞-differentiable functions, where $\mathcal{K}$ stands for the family of all compact subsets of G. For each function $f \in \mathcal{D}(G)$ and an element $\partial \in \mathfrak{g}$ there exists a uniform limit

$$\tilde{\partial} f = \lim_{t\to 0} t^{-1}(R_{\partial(t)} f - f) \in \mathcal{D}(G), \tag{2.1}$$

where $\partial(t) = \exp(t\partial)$, $t \in R$. We see that (2.1) determines the structure of $\mathfrak{A}(\mathfrak{g})$-module on $\mathcal{D}(G)$. Note that each $\tilde{\partial}$ remains invariant any subspace $\mathcal{D}(G, K)$, $K \in \mathcal{K}$, and $\tilde{\partial} L_g = L_g \tilde{\partial}$ for every $g \in G$.

It is convenient for us to introduce the standard nuclear topology on $\mathcal{D}(G)$ in the following way. We choose a left Haar measure λ on G and a basis $\{ \partial_1, \dots, \partial_n \}$ of $\mathfrak{g}$. Now consider the sequence of norms on $\mathcal{D}(G, K)$ defined by

$$\| f \|_l^2 = \sum_{k \le l} \| \tilde{\partial}_{\alpha_1} \dots \tilde{\partial}_{\alpha_k} f \|_0^2 \quad (\alpha_1, \dots, \alpha_k \in \{ 1, \dots, n \}),$$

$$f \in \mathcal{D}(G, K), \; l \in N,$$

where $\| \cdot \|_0$ is the norm determined by the inner product in $L^2(G, \lambda)$. The completion of $\mathcal{D}(G, K)$ by $\| \cdot \|_l$ will be denoted by $\mathring{W}_2^l(G, K)$. Since the map $G \ni g \mapsto L_g$ is a weakly continuous unitary representation of G in $L^2(G, \lambda)$, it follows that

(i_1) $L_g \in \mathcal{L}(\mathring{W}_2^l(G, K), \mathring{W}_2^l(G, gK))$ for all $g \in G$ and $K \in \mathcal{K}$;

(ii_1) the map $K \ni g \mapsto L_g \in \mathcal{L}(\mathring{W}_2^l(G, K_1), \mathring{W}_2^l(G, gK_2))$ is continuous for all $K, K_1, K_2 \in \mathcal{K}$ with $KK_1 \subset K_2$.

Since $\| \cdot \|_1 \le \| \cdot \|_2 \le \dots$, we have a sequence of canonical continuous embeddings of separable Hilbert spaces:

$$\mathring{W}_2^0(G, K) \supset \mathring{W}_2^1(G, K) \supset \mathring{W}_2^2(G, K) \supset \dots, \tag{2.2}$$

where $\mathring{W}_2^0(G, K) = \{ f \in L^2(G, \lambda) \mid f(g) = 0 \text{ for a.e. } g \notin K \}$. It follows from the embedding theorems for Sobolev spaces that $\bigcap_l \mathring{W}_2^l(G, K) = \mathcal{D}(G, K)$. Now we endow $\mathcal{D}(G, K)$ with the projective limit topology of the sequence (2.2). In other words, $\mathcal{D}(G, K)$ is a countable-Hilbert space and its topology is compatible with the family of Hilbert norms $\| \cdot \|_l$, $l \in N$. It follows from the Sobolev spaces theory that there exists a sequence $l_1 < l_2 < \dots$ such that the canonical embeddings $\mathring{W}_2^{l_m}(G, K) \supset \mathring{W}_2^{l_{m+1}}(G, K)$ are nuclear operators, $m \in N$. Hence $\mathcal{D}(G, K) =$

$= \operatorname{proj}\lim_{m\to\infty} \mathring{W}_2^{l_m}(G, K)$ is a nuclear space. Furthermore, since a nuclear operator is compact, $\mathcal{D}(G, K)$ is an (M^*)-space.

Denote by $\mathring{W}_2^{-l}(G, K)$ the dual vector space for $\mathring{W}_2^{l}(G, K)$, $l \in N$, $K \in \mathcal{K}$. Then we have an increasing sequence of separable Hilbert spaces

$$\mathring{W}_2^{0}(G, K) \subset \mathring{W}_2^{-1}(G, K) \subset \mathring{W}_2^{-2}(G, K) \subset \ldots$$

with the continuous canonical embeddings. This sequence is dual to (2.2). Then $\mathcal{D}(G, K)'$ endowed with $\beta(\mathcal{D}(G, K)', \mathcal{D}(G, K))$ is a (LN^*)-space and $\mathcal{D}(G, K)' = \operatorname{inj}\lim_{l\to\infty} \mathring{W}_2^{-l}(G, K)$. By [P, Theorem 4.3.3] $\mathcal{D}(G, K)'$ is nuclear. Note that $\mathcal{D}(G, K)' = \cup_{l\in N} \mathring{W}_2^{-l}(G, K)$ as a set. It is straightforward that $\mathcal{D}(G, K)'$ is separable. Now we deduce from (i_1) and (ii_1) that

(i_2) $L_g \in \mathcal{L}(\mathcal{D}(G, K), \mathcal{D}(G, gK))$ for all $g \in G$ and $K \in \mathcal{K}$;

(ii_2) the map $K \ni g \mapsto L_g \in \mathcal{L}(\mathcal{D}(G, K_1), \mathcal{D}(G, K_2))$ is continuous for all $K, K_1, K_2 \in \mathcal{K}$ with $KK_1 \subset K_2$.

Note that for each $\partial \in \mathfrak{g}$ and $f \in \mathcal{D}(G)$ there exists a limit

$$\hat{\partial} f = \lim_{t\to 0} t^{-1}(L_{\partial(t)} f - f) \in \mathcal{D}(G).$$

Moreover,

(iii_2) $\hat{\partial} \in \mathcal{L}(\mathcal{D}(G, K), \mathcal{D}(G, K))$ for all $\partial \in \mathfrak{g}$ and $K \in \mathcal{K}$.

Now we introduce a topology on $\mathcal{D}(G)$. Consider a sequence $\{K_p\}_{p=1}^{\infty}$ of compact subsets of G with $\cup_{p\in N} K_p = G$ and $K_p \subset \operatorname{Int}(K_{p+1})$, where Int (.) means the topological interior. Then we have an increasing sequence

$$\mathcal{D}(G, K_1) \subset \mathcal{D}(G, K_2) \subset \mathcal{D}(G, K_3) \subset \ldots \tag{2.3}$$

such that all the canonical embeddings are homomorphisms [Sc, IV. 1]. It follows that there is a Hausdorff locally convex topology on $\mathcal{D}(G) = \cup_{p\in N} \mathcal{D}(G, K_p)$ such that $\mathcal{D}(G) = \operatorname{inj}\lim_{p\to\infty} \mathcal{D}(G, K_p)$. This topology is unaffected if we replace

$\{ K_p \}_{p=1}^{\infty}$ by another sequence with the similar properties. By the definition, $\mathcal{D}(G)$ is an (LF)-space. It is nuclear, since every $\mathcal{D}(G, K_p)$ is nuclear, $p \in N$ [P, Theorem 5.2.4]. By [Sc, IV, Exercise 19(a)], $\mathcal{D}(G)$ is a Montel space. Consider the dual sequence for {2.3):

$$\mathcal{D}(G, K_1)' \leftarrow \mathcal{D}(G, K_2)' \leftarrow \mathcal{D}(G, K_3)' \leftarrow \dots .$$

Note that all the canonical projections in this sequence are onto. Furnish $\mathcal{D}(G)'$ with $\beta(\mathcal{D}(G)', \mathcal{D}(G))$. Since $\mathcal{D}(G, K_p)$ is a reflexive Fréchet space for all $p \in N$, it follows that $\mathcal{D}(G)' = \operatorname{proj}\lim_{p \to \infty} \mathcal{D}(G, K_p)'$. Hence $\mathcal{D}(G)'$ is a separable nuclear Montel space. We deduce from (i_2) – (iii_2) that

(i_3) $L_g \in \mathcal{L}(\mathcal{D}(G), \mathcal{D}(G))$ for all $g \in G$;

(ii_3) the map $G \ni g \mapsto L_g \in \mathcal{L}(\mathcal{D}(G), \mathcal{D}(G))$ is continuous;

(iii_3) $\hat{\partial} \in \mathcal{L}(\mathcal{D}(G), \mathcal{D}(G))$ for all $\partial \in \mathfrak{g}$.

2.2. Let H be a closed subgroup of G with $\dim(G/H) > 0$, λ_H a left Haar measure on H, and $\mathfrak{h}$ the Lie algebra of H. Denote by e the identity of G.

Lemma 2.1. *Let $W \subset G$ be a compact neighborhood of e. Then there exists a nonnegative function $c \in \mathcal{D}(G, W)$ such that for each neighborhood $V \subset G$ of e there is $g \in G$ with*

$$\varnothing \neq \{ h \in H \mid c(hg) > 0 \} \subset V.$$

The linear operator $C : C_0(H) \to C_0(G)$ given by

$$Cf(g) = \int_H f(h) c(h^{-1} g)\, d\lambda_H(h), \quad f \in C_0(H),$$

is one-to-one and $L_h C = C L_h$ for all $h \in H$.

P r o o f. We denote the quotient space $H \setminus G$ by X and the quotient map $G \to X$ by p. It is well known that there exists a Borel cross-section $s : X \to G$ of p. Thus the map

$$m : H \times X \ni (h, x) \mapsto hs(x) \in G$$

is one-to-one and onto. Moreover, s may be chosen in such a way that for some open subset $O_X \subset X$ and open neighborhoods O_H and O_G of e in H and G respectively we have: m restricted to $O_H \times O_X$ is a diffeomorphism onto O_G (remark that X endowed with the quotient topology is a ∞-differentiable manifold). Without loss in generality, we may assume that $O_G \subset W$. Let $\{ W_n \}_n$ be a fundamental system of neighborhoods of e in G and $\{ X_n \}_n$ be a sequence of nonempty open subsets of X with $\mathrm{Cl}(\cup_n X_n) \subset O_X$ and $\mathrm{Cl}(X_n) \cap \mathrm{Cl}(X_m) = \varnothing$ for all $n, m \in \mathcal{N}$, where Cl means the topological closure. One can choose nontrivial functions $f_n : X \to [0,1]$ and $l_n : H \to [0,1]$ such that $f_n \in \mathcal{D}(X, X_n)$, $l_n \in \mathcal{D}(H, W_n \cap H)$, and the function $c : G \to [0,1]$ defined by

$$c(g) = \begin{cases} \sum_n f_n(x)\, l_n(h) & \text{if } g = hs(x) \in O_G \\ 0, & \text{otherwise} \end{cases}$$

is ∞-differentiable. It is clear that $c \in \mathcal{D}(G, W)$. Given a neighborhood $V \subset G$ of e, we find $n \in \mathcal{N}$ with $W_n \subset V$ and take $x_n \in X_n$ such that $f_n(x_n) \neq 0$. Then $c(hs(x_n)) = f_n(x_n)\, l_n(h)$ for every $h \in H$ and the first conclusion of the lemma follows. To prove the second one we suppose that there are a function $f \in C_0(H)$, a neighborhood $V \subset G$ of e, and $h_0 \in H$ such that $f(g) > 0$ for all $g \in h_0 V$. Then there is $g_0 \in G$ with

$$\varnothing \neq O \overset{\text{def}}{=} \{ h \in H \mid c(hg_0) > 0 \} \subset V^{-1}.$$

Since $\lambda_H(h_0 O^{-1}) > 0$, we have

$$Cf(h_0 g_0) = \int_{h_0 O^{-1}} f(h) c(h^{-1} h_0 g_0)\, d\lambda_H(h) > 0,$$

as desired. A routine verification implies $L_h C = C L_h$ for all $h \in H$. □

2.3. In this subsection we consider a sequence of Lie groups $G_0 \subset G_1 \subset G_2 \subset \ldots$ such that G_n is closed in G_{n+1}, $n \in \mathbb{N}$, and $G_0 = \{e\}$. We furnish the group $G = \cup_{n=1}^{\infty} G_n$ with the inductive limit topology. Then G is a separable, complete, σ-compact, Hausdorff topological group. Note that G is not metrizable, unless there is $N \in \mathbb{N}$ such that G_n is open in G_{n+1} for all $n > N$. Since G is isomorphic to $\operatorname{inj\,lim}_{m \to \infty} G_{n_m}$ for each increasing sequence $n_1 < n_2 < \ldots$, we can assume, without loss in generality, that one of the following is satisfied:

(I) $\dim (G_{n+1}/G_n) > 0$ for all $n \in \mathbb{Z}_+$;

(II) G is a Lie group.

Here we consider only the case (I). The main purpose of this subsection is to produce the space of "basic functions" on G, namely an analogue of the space of compactly supported ∞-differentiable functions on a Lie group. We assume that a strongly continuous unitary representation

$$U : G \ni g \mapsto U(g)$$

of G in a separable Hilbert space $\mathcal{H}$ is given. Let λ_n be a left Haar measure on G_n and let $\mathfrak{g}_n$ be the Lie algebra of G_n, $n \in \mathbb{N}$. Notice that the Lie algebra $\mathfrak{g}$ of G can be identified naturally with the inductive limit of the sequence $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots$. Denote by $\mathfrak{B}_1(\mathcal{H})$ the unit ball of the algebra of all bounded operators in $\mathcal{H}$. It is known that the strong operator topology on $\mathfrak{B}_1(\mathcal{H})$ is Polish and compatible with the metric r:

$$r(A, B) = \sum_{k=1}^{\infty} 2^{-k} \|(A - B)h_k\|_{\mathcal{H}}, \quad A, B \in \mathfrak{B}_1(\mathcal{H}),$$

where $\{ h_k \mid k \in N \}$ is a dense subset of the unit sphere of $\mathcal{H}$. Moreover, the following two maps are continuous:

$$[0, 1] \times \mathfrak{B}_1(\mathcal{H}) \times \mathfrak{B}_1(\mathcal{H}) \ni (\alpha, A, B) \mapsto \alpha A + (1 - \alpha)B \in \mathfrak{B}_1(\mathcal{H}),$$

$$\mathfrak{B}_1(\mathcal{H}) \times \mathfrak{B}_1(\mathcal{H}) \ni (A, B) \mapsto AB \in \mathfrak{B}_1(\mathcal{H}).$$

Since $U : G \to U(g) \in \mathcal{U}(\mathcal{H})$ is a strongly continuous map, there is a compact neighborhood $W_n \subset G_n$ of e with

$$r(U(g), I) < 0.1 \cdot 2^{-n} \text{ for all } g \in W_n, \ n \in N. \tag{2.4}$$

We apply Lemma 2.1 to the triple $G_{n-1} \subset G_n$, W_n. Denote by $c_n \in \mathcal{D}(G_n, W_n)$ the function with the properties asserted in this lemma. Without loss in generality, we may assume that $\| c_n \|_0 \overset{\text{def}}{=} \int_{G_n} | c_n(g) | \, d\lambda_n(g) = 1$, $n \in N$.. We set

$$Q_n = \int_{G_n} c_n(g_n) U(g_n) \, d\lambda_n(g_n).$$

Then $Q_n \in \mathfrak{B}_1(\mathcal{H})$ for all $n \in N$. It is easy to verify that

$$r(\alpha A + (1 - \alpha)B, C) < \alpha r(A, C) + (1 - \alpha) r(B, C)$$

for all $\alpha \in [0, 1]$ and $A, B, C \in \mathfrak{B}_1(\mathcal{H})$. So we deduce from (2.4) that $r(Q_n, I) \leq$ $\leq 0.1 \cdot 2^{-n}$ for all $n \in N$.

Lemma 2.2. *The infinite product* $\overrightarrow{\prod_{n \in N}} Q_n = Q_1 Q_2 \dots$ *is strongly convergent.*

P r o o f. We have to show that a sequence of partial products is fundamental:

$$r(Q_1 \dots Q_n, Q_1 \dots Q_n \dots Q_p) \leq \| Q_1 \dots Q_n \| r(I, Q_{n+1} \dots Q_p) \leq$$
$$\leq r(I, Q_{n+1}) + r(Q_{n+1}, Q_{n+1} Q_{n+2}) + \dots + r(Q_{n+1} \dots Q_{p-1}, Q_{n+1} \dots Q_p) \leq$$
$$\leq 0.1(2^{-n-1} + \dots + 2^{-p}). \ \square$$

It follows from Lemma 2.2 that the operator $P_n = \overrightarrow{\prod_{k>n}} Q_k$ is well defined for every $n \in \mathbb{Z}_+$. Note that $r(P_0, I) \leq 0.1$. Since $r(I, 0) = 1$, we have $P_0 \neq 0$. It is also easy to verify that $0 < \| P_n \| \leq 1$ for all $n \in \mathbb{Z}_+$.

Now we select a sequence $\{ E_n \}_{n=1}^{\infty}$ of subsets of G such that the following properties are satisfied for all $n \in \mathbb{N}$:

(a) $E_n \subset G_n$ is a compact neighborhood of e;

(b) $E_1 \subset E_2 \subset \dots$, and $\cup_{n=1}^{\infty} E_n = G$;

(c) $E_n E_n \subset E_{n+1}$;

(d) $E_n W_m \subset E_{n+1}$ if $m \leq n+1$.

We set $\mathcal{D}_m(G_n) = \mathcal{D}(G_n, E_m \cap G_n)$ for $m \geq n$. Then $\mathcal{D}(G_n) = \operatorname{inj}\lim_{m \to \infty} \mathcal{D}_m(G_n)$ for every $n \in \mathbb{N}$. We define linear operators $C_n : \mathcal{D}(G_n) \to \mathcal{D}(G_{n+1})$ by setting

$$(C_n f)(g_{n+1}) = \int_{G_n} f(g_n) c_{n+1}(g_n^{-1} g_{n+1}) \, d\lambda_n(g_n) =$$

$$= \int_{G_n} f(g_n)(L_{g_n} c_{n+1})(g_{n+1}) \, d\lambda_n(g_n),$$

$f \in \mathcal{D}(G_n)$, $n \in \mathbb{N}$. Let us verify that C_n is well defined and continuous. For this, it suffices to prove

$$C_n(\mathcal{D}_m(G_n)) \subset \mathcal{D}_{m+1}(G_{n+1}), \tag{2.5}$$

$$C_n \in \mathcal{L}(\mathcal{D}_m(G_n), \mathcal{D}_{m+1}(G_{n+1})) \tag{2.6}$$

for all $m \geq n$. Actually, for a function $f \in \mathcal{D}_m(G_n)$, we have

$$\operatorname{Supp}(C_n f) \subset (E_m \cap G_n) \operatorname{Supp} c_{n+1} \subset (E_m W_{n+1}) \cap G_{n+1} \subset E_{m+1} \cap G_{n+1}.$$

Since $c_{n+1} \in \mathcal{D}(G_{n+1})$, it follows from ($i_3$) (see Subsection 2.1) that $L_{g_n} c_{n+1} \in \mathcal{D}(G_{n+1})$ for all $g_n \in G_n$, and hence $C_n f \in \mathcal{D}(G_{n+1})$. So (2.5) is done.

Now given an element ∂ of the universal enveloping algebra $\mathfrak{A}(\mathfrak{g}_n)$, then we have for every $f \in \mathcal{D}_m(G_n)$ and $m \geq n$:

$$\| \tilde{\partial}(C_n f) \|_0 \leq \int_{G_n} | f(g_n) | \, \| \tilde{\partial} L_{g_n} c_{n+1} \|_0 \, d\lambda_n(g_n) =$$

$$= \int_{G_n} | f(g_n) | \, \| L_{g_n} \tilde{\partial} c_{n+1} \|_0 \, d\lambda_n(g_n) =$$

$$= \int_{G_n} | f(g_n) | \, d\lambda_n(g_n) \cdot \| \tilde{\partial} c_{n+1} \|_0 \leq \| \tilde{\partial} c_{n+1} \|_0 \sqrt{\lambda_n(E_m \cap G_n)} \, \| f \|_0 \, .$$

One can easily deduce (2.6) from this. Moreover, it follows that C_n takes the neighborhood $\{ f \in \mathcal{D}_m(G_n) | \, \| f \|_0 < 1 \}$ of zero onto a bounded subset of $\mathcal{D}_{m+1}(G_{n+1})$. Since the closure of each bounded subset of $\mathcal{D}_{m+1}(G_{n+1})$ is compact, C_n is a compact operator. By Lemma 2.1, Ker C_n is trivial. It is straightforward that the following infinite diagram is commutative:

$$\begin{array}{ccccccc}
\mathcal{D}_1(G_1) & \longrightarrow & \mathcal{D}_2(G_1) & \longrightarrow & \dots & \longrightarrow & \mathcal{D}(G_1) \\
C_1 \downarrow & & C_1 \downarrow & & & & C_1 \downarrow \\
\mathcal{D}_2(G_2) & \longrightarrow & \mathcal{D}_3(G_2) & \longrightarrow & \dots & \longrightarrow & \mathcal{D}(G_2) \\
C_2 \downarrow & & C_2 \downarrow & & & & C_2 \downarrow \\
\dots & & \dots & & \dots & & \dots \, .
\end{array} \tag{2.7}$$

Consider the diagonal subsequence:

$$\mathcal{D}_1(G_1) \to \mathcal{D}_3(G_2) \to \mathcal{D}_5(G_3) \to \dots \, .$$

This is an increasing sequence of Fréchet spaces with the compact embeddings, since the product of a continuous linear operator and a compact one is a compact operator. So there is an (LN^*)-space $\mathcal{D}(G)$ with $\mathcal{D}(G) = \operatorname{inj}\lim_{n\to\infty} \mathcal{D}_{2n+1}(G_n)$. Since $\mathcal{D}_{2n+1}(G_n)$ is nuclear and separable for all $n \in N$, it follows that $\mathcal{D}(G)$ is also nuclear [P, Theorem 5.2.4] and separable. In view of (2.7), $\mathcal{D}(G) = \operatorname{inj}\lim_{n\to\infty} \mathcal{D}(G_n)$ (with the canonical embeddings C_n, $n \in N$). Moreover, $\mathcal{D}(G)$ is a free union of $\mathcal{D}(G_n)$, $n \in N$, i.e., an arbitrary subset $F \in \mathcal{D}(G)$ is closed if and only if $F \cap \mathcal{D}(G_n)$ is closed for each $n \in N$. By the duality theorem (see Section 1), $\mathcal{D}(G)'$ endowed with $\beta(\mathcal{D}(G)', \mathcal{D}(G))$ is an (M^*)-space. Furthermore, by [P, Theorem 4.3.3], $\mathcal{D}(G)$ is nuclear. Since C_n commutes with L_{g_n} by Lemma 2.1 and $L_{g_n} \in \mathcal{L}(\mathcal{D}(G_n), \mathcal{D}(G_n))$ (see (i_3)) for all $g_n \in G_n$ and $n \in N$, it follows from (c) that

(i_4) $L_g \in \mathcal{L}(\mathcal{D}(G), \mathcal{D}(G))$ for all $g \in G$.

Furthermore, since the map $G_n \in g_n \mapsto L_{g_n} \in \mathcal{L}(\mathcal{D}(G_n), \mathcal{D}(G_n))$ is continuous for each $n \in N$ (see (ii_3)) and $\mathcal{D}(G)' = \operatorname{proj}\lim_{n\to\infty} \mathcal{D}_{2n+1}(G_n)'$, we have that

(ii_4) the map $G \in g \mapsto L_g \in \mathcal{L}(\mathcal{D}(G), \mathcal{D}(G))$ is also continuous.

Thus the space $\mathcal{D}(G)$ of "basic functions" on G possesses a number of "good" properties of the classical space of compactly supported ∞-differentiable functions on a Lie group. But unlike the classical case, now $\mathcal{D}(G)$ depends on U.

2. 4. This subsection is devoted solely to the proof of Main Theorem for countable inductive limits of Lie groups. Suppose the hypotheses of this theorem hold. We use the notation of the previous subsection and consider only the case (I).

Assume first that there is a unit vector $h \in \mathcal{H}$ such that $P_0 h$ is cyclic for U, i.e., the linear span of the set $\{ U(g)P_0 h \mid g \in G \}$ is dense in $\mathcal{H}$. We define an operator $R_n : \mathcal{D}(G_n) \to \mathcal{H}$ by setting

$$R_n f = \int_{G_n} f(g_n) U(g_n)\, d\lambda_n(g_n) P_n h, f \in \mathcal{D}(G_n), \quad n \in N.$$

Since

$$\| R_n f \|_{\mathcal{H}} \le \left\| \int_{G_n} f(g_n) U(g_n)\, d\lambda_n(g_n) \right\| \; \| P_n h \|_{\mathcal{H}} \le$$

$$\le \int_{G_n} | f(g_n) |\, d\lambda_n(g_n) \le \| f \|_0 \sqrt{\lambda_n(\mathrm{Supp} f)}$$

for all $f \in \mathcal{D}(G_n)$, it follows that $R_n \in \mathcal{L}(\mathcal{D}(G_n), \mathcal{H})$, $n \in N$. Now we prove that $R_{n+1} C_n = R_n$ for each $n \in N$:

$$R_{n+1} C_n f = \int_{G_{n+1}} (C_n f)(g_{r+1}) U(g_{n+1})\, d\lambda_{n+1}(g_{n+1}) P_{n+1} h =$$

$$= \int_{G_{n+1}} \int_{G_n} f(g_n) c_{n+1}(g_n^{-1} g_{n+1}) U(g_{n+1})\, d\lambda_n(g_n)\, d\lambda_{n+1}(g_{n+1}) P_{n+1} h =$$

$$= \int_{G_n} f(g_n) \int_{G_{n+1}} c_{n+1}(g_{n+1}) U(g_n) U(g_{n+1})\, d\lambda_{n+1}(g_{n+1})\, d\lambda_n(g_n) P_{n+1} h =$$

$$= \int_{G_n} f(g_n)\, U(g_n)\, d\lambda_n(g_n) Q_{n+1} P_{n+1} h = C_{n+1} f$$

for every function $f \in \mathcal{D}(G_n)$. So a linear operator $R : \mathcal{D}(G) \to \mathcal{H}$ given by

$$\mathcal{D}(G) \supset \mathcal{D}(G_n) \ni f \mapsto Rf \overset{\text{def}}{=} R_n f$$

is well-defined and continuous. Moreover, since

$$U(g)R_n f = \int_{G_n} f(g_n)\, U(gg_n)\, d\lambda_n(g_n) P_n h = \int_{G_n} f(g^{-1} g_n)\, U(g_n)\, d\lambda_n(g_n) P_n h =$$

$$= \int_{G_n} (L_g f)(g_n)\, U(g_n)\, d\lambda_n(g_n) P_n h = R_n L_g f$$

for all $f \in \mathcal{D}(G_n)$ and $g \in G_n$, it follows that

$$U(g)R = RL_g \quad \text{for each } g \in G. \tag{2.8}$$

Next, for each $f \in \mathcal{D}(G_n)$ and $\partial \in \mathfrak{g}_n$, we have

$$\lim_{t \to 0} t^{-1} (U(\partial(t)) - I) R_n f = \lim_{t \to 0} t^{-1} \int_{G_n} (L_{\partial(t)} - I) f(g_n)\, d\lambda_n(g_n) P_n h = R_n \tilde{\partial} f,$$

where $\partial(t) = \exp(t\partial)$. It follows that

$$U(\partial)\, \mathrm{Im}\, R_n \subset \mathrm{Im}\, R_n \tag{2.9}$$

for all $\partial \in \mathfrak{g}_n$, $n \in \mathbb{N}$. Denote by $\mathcal{F}$ the quotient space $\mathcal{D}(G)/\mathrm{Ker}\, R$ and by $S : \mathcal{D}(G) \to \mathcal{F}$ the canonical projection map. Then R can be decomposed as $R = JS$, where the linear operator $J \in \mathcal{L}(\mathcal{F}, \mathcal{H})$ is one-to-one. Since $\mathrm{Ker}\, R$ is a closed subspace of a separable nuclear (LN*)-space, it follows that $\mathcal{F}$ is also a separable nuclear [P, Theorem 5.1.3] (LN*)-space, as desired. Then the strongly dual vector space $\mathcal{F}'$ is an (M*)-space and the property (i) of Main Theorem follows (see Section 1). Furthermore, it follows directly from (2.8) that $L_g(\mathrm{Ker}\, R) \subset \mathrm{Ker}\, R$ for all $g \in G$. Since $L_g L_g^{-1} = I$, we obtain $L_g(\mathrm{Ker}\, R) = \mathrm{Ker}\, R$ for all $g \in G$. So the linear operator $\tilde{L}_g : \mathcal{F} \to \mathcal{F}$ given by

$$\tilde{L}_g S f = S L_g f, \quad f \in \mathcal{D}(G)$$

is well-defined. We deduce from (i_4) that $\tilde{L}_g \in \mathcal{L}(\mathcal{F}, \mathcal{F})$ for all $g \in G$. It follows from (2.8) that $U(g) J = J \tilde{L}_g$ for all $g \in G$, which yields (iii) and (v). It is easy to derive (iv) from (2.9), (iii_3), and the fact that $\mathfrak{g} = \operatorname{inj} \lim_{n \to \infty} \mathfrak{g}_n$. Since by the assumption $J(Sc_1) = Rc_1 = P_0 h$ is a cyclic vector for U, the property (ii) follows. Finally, to prove (vi) we observe that $\mathcal{F}'$ can be identified with the polar

$$(\operatorname{Ker} R)^\circ \overset{\text{def}}{=} \{ \zeta \in \mathcal{D}(G)' \mid \langle f, \zeta \rangle = 0 \text{ for all } f \in \operatorname{Ker} R \}$$

endowed with the relative topology $\beta(\mathcal{D}(G)', \mathcal{D}(G)) \mid (\operatorname{Ker} R)^\circ$ [P, Theorem 5.1.9]. Then for all $f \in \mathcal{D}(G)$ and $\zeta \in \mathcal{F}'$, we have

$$\langle \tilde{L}_g S f, \zeta \rangle = \langle L_g f, \zeta \rangle, \quad g \in G.$$

So the map

$$G \ni g \mapsto \langle \tilde{L}_g \tilde{f}, \zeta \rangle \in R$$

is continuous for all $\tilde{f} \in \mathcal{F}$ and $\zeta \in \mathcal{F}'$. This is equivalent to (vi). Thus $\mathcal{F}$ is a strong Gårding space for U.

In the general case $\mathcal{H}$ can be decomposed into an orthogonal Hilbert sum $\mathcal{H} = \oplus_{n=1}^{N} \mathcal{H}_n$, $N \in \mathbb{N} \cup \{\infty\}$, in such a way that the following properties are satisfied for all $n = 1, \dots, N$:

(1) $\mathcal{H}_n$ is an $U(G)$-invariant closed subspace of $\mathcal{H}$;

(2) there exists $h_n \in \mathcal{H}$ such that $P_0 h \in \mathcal{H}_n$ is a cyclic vector for $U(G) \upharpoonright \mathcal{H}_n$. Let $\mathcal{F}_n$ be a strong Gårding (LN^*)-space for $U(G) \upharpoonright \mathcal{H}_n$, $n = 1, \dots, N$. Then $\mathcal{F} \overset{\text{def}}{=} \odot_{n=1}^{N} \mathcal{F}_n$ is a separable nuclear [P, Theorem 5.2.2] Montel space (but it is not, in general, an (LN^*)-space), where $\odot$ means the algebraic direct sum. Furthermore, $\mathcal{F}' = \prod_{n=1}^{N} \mathcal{F}'_n$ is a separable Fréchet space. Now one can easily deduce (ii)-(vi). Thus Main Theorem is demonstrated for the infinite dimensional Lie groups.

R e m a r k 2.3. It follows from the proof that $\mathcal{F}$ is an (LN^*)-space if U is irreducible.

3. Inductive limits of locally compact groups

3.1. Let G be a locally compact second countable group. By the Gleason–Montgomery–Zippen theorem [MZ] there exist an open subgroup $O \subset G$ and a sequence $N_1 \supset N_2 \supset \ldots$ of compact normal subgroups of O such that $\bigcap_{n=1}^{\infty} N_n = \{ e \}$ and O/N_n is a Lie group for all $n \in \mathbb{N}$. The family $\{ O; N_n \mid n \in \mathbb{N}\}$ will be called an *L-system* for G. We will use the following properties of L-systems:

Lemma 3.1. [Ka, Lemma 3.1]. *Let $\{ O; N_n \mid n \in \mathbb{N}\}$ be an L-system for G. If there are an open subgroup $O_1 \subset G$ and a normal subgroup $N \subset O_1$ such that O_1/N is a Lie group, then $N \supset N_n$ for some $n \in \mathbb{N}$.*

Lemma 3.2. *Let $\{ O ; N_n \mid n \in \mathbb{N}\}$ be an L-system for G and let H be a closed subgroup of G. Then $\{ O \cap H; N_n \cap H \mid n \in \mathbb{N}\}$ is an L-system for H.*

P r o o f. We first observe that $N_n \cap H$ is a compact normal subgroup of $O \cap H$, $n \in \mathbb{N}$. By [HR, Theorem 5.33], the groups $(O \cap H)/(N_n \cap H)$ and $(O \cap H)N_n/N_n$ are topologically isomorphic. The last one is a closed subgroup of the Lie group O/N_n, since the natural projection $O \to O/N_n$ is a closed map [HR, Theorem 5.18]. It follows that $(O \cap H)/(N_n \cap H)$ is a Lie group. The rest of the properties of L-system are evident. □

Now we remind the definition of the Lie algebra of G [L; Hey, 4.4.1–4. 4.5]. Let $\mathfrak{g}_n$ be the Lie algebra of O/N_n and let $\exp_n : \mathfrak{g}_n \to O/N_n$ be the exponential map, $n \in \mathbb{N}$. Denote by p_n and $p_{m,n}$ the canonical homomorphisms $O \to O/N_n$ and $O/N_n \to O/N_m$, $n > m$, respectively. It is evident that $p_m = p_{m,n}p_n$ for all $n > m$. Then we have a projective system of Lie algebras

$$\mathfrak{g}_1 \xleftarrow{d p_{1,2}} \mathfrak{g}_2 \xleftarrow{d p_{2,3}} \mathfrak{g}_3 \cdots . \tag{3.1}$$

Notice that all the homomorphisms $d\,p_{m,\,n} : \mathfrak{g}_n \to \mathfrak{g}_m$ are onto. By the Lie algebra of G, namely $\mathfrak{g}$, we mean the (algebraic) projective limit of (3.1). Note that there is a one-to-one correspondence between $\mathfrak{g}$ and the family of all one-parameter subgroups of G. Moreover, there are a map $\exp : \mathfrak{g} \to G$ and a collection of (onto) homomorphisms $d\,p_n : \mathfrak{g} \to \mathfrak{g}_n$ such that

$$p_n \circ \exp = \exp_n \circ \, d\,p_n\,, \text{ and } d\,p_{m,\,n}\, d\,p_n = d\,p_m$$

for all $n > m$. Note that $\mathfrak{g}$ and exp are independent of the choice of L-system [L].

We define a dimension of G by setting $G = \dim \mathfrak{g}$. Let us generalize the notion "dimension" to homogeneous spaces of locally compact groups. We need an auxiliary

Lemma 3.3. *Let G be a locally compact group, H a closed subgroup of G, and N a compact normal subgroup of G. Then there is a G-equivariant (topological) isomorphism of the homogeneous space $(G/N)/(NH/N)$ onto $G/(NH)$.*

We omit the proof of the above statement, since it is a close analogue of [HR, Theorem 5.35].

Now let H be a closed subgroup of G and let $\{\, O;\, N_n \mid n \in \mathbb{N} \,\}$ be an L-system for G. It follows from the proof of Lemma 3.2 that $(O \cap H)/(N_n \cap H)$ can be identified with a closed subgroup of O/N_n. Denote by X_n the homogeneous space $(O/N_n)/((O \cap H)/(N_n \cap H))$. By Lemma 3.3, X_n is isomorphic (as a topological O-space) to $O/((O \cap H)N_n)$, $n \in \mathbb{N}$. Moreover, a routine verification implies the commutativity of the infinite diagram

$$\begin{array}{ccccccccc}
\{e\} & \longrightarrow & (O \cap H)/(N_1 \cap H) & \longrightarrow & O/N_1 & \longrightarrow & X_1 & \longrightarrow & \{*\} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\{e\} & \longrightarrow & (O \cap H)/(N_2 \cap H) & \longrightarrow & O/N_2 & \longrightarrow & X_2 & \longrightarrow & \{*\}\,, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

where each line is a short exact sequence and each vertical arrow is an onto map. Now we set

$$\dim G/H \overset{\text{def}}{=} \lim_{n\to\infty} \dim X_n = \lim_{n\to\infty} [\dim O/N_n - \dim (O \cap H)/(N_n \cap H)]. \quad (3.2)$$

Note that the sequence $\{ \dim X_n \}_{n=1}^{\infty}$ is nondecreasing.

Proposition 3.4. dim G/H *does not depend on the choice of an L-system for G, and* $\dim G = \dim H + \dim G/H$. (*We mean* $\infty + n = \infty$ *for all* $n \in N \cup \{ \infty \}$.)

P r o o f. Let two L-systems for G be given: (I) $= \{ O_1 ; N_n \mid n \in N\}$ and (II) $= \{ O_2 ; M_n \mid n \in N\}$. Since $\bigcap_{n=1}^{\infty} N_n = \bigcap_{m=1}^{\infty} M_n = \{ e \}$ and N_n, M_n are compact, we may assume without loss of generality that $N_n \subset O_2$ and $M_n \subset O_1$ for all $n \in N$. Then, by Lemma 3.2, one has that the families (III) $= \{ O_1 \cap O_2 ; N_n \}$ and (IV) $= \{ O_1 \cap O_2 ; M_n \}$ are L-systems for the group $O_1 \cap O_2$ as well as for G (since $O_1 \cap O_2$ is open in G). Denote by $d_I - d_{IV}$ the dimensions of G/H with respect to (I)–(IV). It follows from (3. 2) that $d_I = d_{III}$ and $d_{II} = d_{IV}$. By Lemma 3.1, there is an increasing sequence $n_1 < n_2 < \ldots$ such that $N_{n_1} \subset M_{n_2} \subset N_{n_3} \subset M_{n_4} \ldots$. Then $\{ O_1 \cap O_2 ; N_{n_1}, M_{n_2}, N_{n_3}, \ldots \}$ is also an L-system for G, and hence $d_{III} = d_{IV}$. That proves the first part of the proposition. The second one follows directly from Lemma 3. 2 and (3.2). □

Corollary 3.5. *Let H be a closed subgroup of G and let* $\{ O; N_n \mid n \in N \}$ *be an L-system for G. If* $\dim G/H > 0$, *then there exists* $m \in N$ *such that* $\dim (O \cap H)/(N_n \cap H) < \dim O/N$ *for all* $n > m$.

3.2. Let O be an open subgroup of G and N a compact normal subgroup of O such that O/N is a Lie group. Denote by $\mathfrak{o}$ the Lie algebra of O/N and by $\mathfrak{U}(\mathfrak{o})$ the universal enveloping algebra of $\mathfrak{o}$. There is a unique structure of ∞-differentiable manifold on the homogeneous space G/N with the following two properties [Ka; Hey, 4.4.1–4.4.5]:

(1) its restriction to O/N coincides with the ∞-differentiable structure of the Lie group,

(2) for each $g \in G$ the transformation $G/N \ni x \mapsto gx \in G/N$ is a diffeomorphism.

Thus one can define the spaces $\mathcal{D}(G/N)$ and $\mathcal{D}(G/N, K)$ in a usual way for any compact subset K of G/N. Notice that G/N is not only left G-space but also right O/N-space. These actions (left and right) are mutually commuting and the second one is ∞-differentiable. It follows that $\mathcal{D}(G/N)$ is an $\mathfrak{U}(\mathfrak{o})$-module and $\tilde{\partial} L_g = L_g \tilde{\partial}$ for all $g \in G$ and $\partial \in \mathfrak{U}(\mathfrak{o})$. Furthermore, $\mathcal{D}(G/N, K)$ is $\mathfrak{U}(\mathfrak{o})$-invariant for every compact subset $K \in G/N$. We need to introduce a nuclear topology on $\mathcal{D}(G/N)$ like it has been done for the case of Lie groups. To this end let us choose a left Haar measure λ on G and a basis $\{\partial_1, \dots, \partial_n\}$ in $\mathfrak{o}$. Denote by $\tilde{\lambda}$ the projection of λ onto G/N. Since N is compact, $\tilde{\lambda}$ is a G-invariant measure. Consider the increasing sequence of norms on $\mathcal{D}(G/N)$ defined by

$$\| f \|_l = \sum_{k \le l} \| \tilde{\partial}_{\alpha_1} \dots \tilde{\partial}_{\alpha_k} f \| \quad (\alpha_1, \dots, \alpha_k \in \{1, \dots, n\}), \; f \in \mathcal{D}(G/N), \; l \in N,$$

where $\| \cdot \|$ is the norm determined by the inner product in $L^2(G/N, \tilde{\lambda})$. Then $\mathcal{D}(G/N, K)$ endowed with $\{ \| \cdot \|_l \}_{l=1}^{\infty}$ is a nuclear (M*)-space and the natural analogues of (i_2)-(iii_2) from Subsection 2.1 are satisfied. Next, having a countable exhaustion of G/N by compact subsets, we furnish $\mathcal{D}(G/N)$ with the inductive limit topology. It is easy to make sure that the analogues of (i_3)-(iii_3) are also satisfied.

Given a closed subgroup $H \subset G$, we set $O_H = O \cap H$ and $N_H = N \cap H$. Notice that there is a one-to-one H-equivariant continuous map from the homogeneous space H/N_H into G/N such that the diagram

$$\begin{array}{ccc} O_H/N_H & \longrightarrow & O/N \\ \downarrow & & \downarrow \\ H/N_H & \longrightarrow & G/N \end{array} \tag{3.3}$$

is commutative. Moreover, this map is closed and hence H/N_H is a closed subset of G/N.

We need to obtain some generalization of Lemma 2.1. Denote by p and p_H the canonical projections $G \to G/N$ and $H \to H/N_H$ respectively. Let P^* and P^*_H be the linear operators associated with p and p_H, namely

$$P^* : C(G/N) \ni f \mapsto P^*(f) = f \circ p \in C(G),$$

$$P^*_H : C(H/N_H) \ni f \mapsto P^*_H(f) = f \circ p_H \in C(H).$$

Let us also denote by λ_H a left Haar measure on H.

Lemma 3.6. *Let $W \subset G$ be a compact neighborhood of the identity and $N \subset W$. If $O_H/N_H <$ dim O/H, there is a non-negative function $c \in P^*(\mathcal{D}(G/N))$ with* Supp $c \subset W$ *and such that the linear operator $C : P^*_H(C_0(H/N_H)) \to P^*(C_0(G/N))$ given by*

$$(Cf)(g) = \int_H f(h)c(h^{-1}g)\, d\lambda_H(h), \quad f \in P^*_H(C_0(H/N_H)),$$

is one-to-one and $L_hC = CL_h$ for all $h \in H$.

P r o o f. Since $N \subset W$, it follows from [HR, Theorem 4.10] that there is a compact neighborhood $\tilde{W} \subset O/N$ of the identity with $p^{-1}(\tilde{W}) \subset W$. Let us now make use of Lemma 2.1: there exists a non-negative function $\tilde{c} \in \mathcal{D}(O/N, \tilde{W})$ such that for each neighborhood $\tilde{U} \subset O/N$ of the identity there is $\tilde{o} \in O/N$ with

$$\varnothing \neq \{\tilde{h} \in O_H/N_H \mid \tilde{c}(\tilde{h}\tilde{o}) > 0\} \subset \tilde{U}. \tag{3.4}$$

We define the function $c \in P^*(\mathcal{D}(G/N))$ by setting

$$c(g) = \begin{cases} \tilde{c}\,(p(g)) & \text{for } g \in O \\ 0 & \text{otherwise.} \end{cases}$$

Then Supp $c \subset W$. Let us prove that C is one-to-one. Suppose that there are a function $f \in C_0(H/N_H)$, an element $h_0 \in H$, and a neighborhood $\tilde{V} \subset O_H/N_H$ of the identity with

$$f(h_0\,\tilde{\upsilon}) > 0 \ \text{ for all } \tilde{\upsilon} \in \tilde{V}. \tag{3. 5}$$

We deduce from (3. 4) that there is $a \in O$ with

$$\varnothing \neq M \overset{\text{def}}{=} \{\tilde{h} \in O_H/N_H \mid \tilde{c}\,(\tilde{h}\,\tilde{a}\,) > 0\} \subset \tilde{V}^{-1}\,, \tag{3. 6}$$

where $\tilde{a} = p(a)$. It follows from (3. 3), (3. 5), and (3. 6) that

$$CP_H^* f(h_0 a) = \int_H f(p_H(h))c(h^{-1}h_0 a)\,d\lambda_H(h) =$$

$$= \int_{h_0 O_H} f(p_H(h))\,\tilde{c}\,(p(h^{-1}h_0)\,\tilde{a}\,)\,d\lambda_H(h) = \int_A f(p_H(h))\,\tilde{c}\,(h^{-1}h_0\tilde{a}\,)\,d\lambda_H(h) > 0,$$

since the integrand is strictly positive on the set

$$A \overset{\text{def}}{=} \{h \in H \mid p_H(h^{-1}h_0) \in M\} = \{h \in H \mid p_H(h^{-1}h_0)^{-1} \in M^{-1}\} =$$

$$= \{h \in H \mid p_H(h_0^{-1}h) \in M^{-1}\} = \{h \in H \mid h_0^{-1}p_H(h) \in M^{-1}\} =$$

$$= \{h \in H \mid p_H(h) \in h_0 M^{-1}\} \subset \{h \in H \mid p_H(h) \in h_0\tilde{V}\},$$

(we use that $h^{-1}h_0 \in O_H$). Moreover, since

$$CP_H^* f(gn) = \int_{H \cap gO} f(p_H(h))c(p(h^{-1}gn))\,d\lambda_H(h) = CP_H^* f(g)$$

for all $g \in G$ and $n \in N$, we deduce that $CP^*_H f \in P^*(C_0(G/N))$ for every $f \in C_0(H/N_H)$, as desired. The last assertion of the lemma is obvious. □

3.3. Now let a sequence $G_0 \subset G_1 \subset G_2 \subset \ldots$ of locally compact second countable groups be given so that G_{n-1} is closed in G_n, $n \in N$, and $G_0 = \{ e \}$. We furnish the group $G = \bigcup_{n \in N} G_n$ with the inductive limit topology. Without loss in generality, we may assume that one of the following is fulfilled:

$$\dim (G_n / G_{n-1}) > 0 \text{ for all } n \in N;$$

$$\dim (G_n / G_{n-1}) = 0 \text{ for all } n \in N.$$

In this subsection we prove Main Theorem for G which satisfies (I). Let $U: G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of G in a Hilbert space $\mathcal{H}$. We first choose a compact neighborhood $W_n \subset G_n$ of the identity such that (2.4) is held. Then using Lemmas 3.1 and 3.2 and Corollary 3.5, we proceed by induction to construct a sequence $\{ O_n , N_n \}_{n=1}^{\infty}$ such that

(1) O_n is an open subgroup of G_n and N_n is a normal subgroup of O_n;

(2) O_n / N_n is a Lie group;

(3) $\mathrm{Int}\,(W_n) \supset N_n \supset N_{n+1} \cap G_n$;

(4) $\dim (O_{n+1} \cap G_n)/(N_{n+1} \cap G_n) < \dim O_{n+1}/N_{n+1}$;

(5) $\bigcap_{m=n}^{\infty} (N_n \cap G_n) = \{ e \}$.

Denote by $\mathfrak{g}_{n,m}$, $m \geq n$, the Lie algebra of the Lie group $(O_n \cap O_m)/(N_m \cap G_n)$ and by $\mathfrak{g}_n$ the Lie algebra of G_n. Then we have the natural projective system

$$\mathfrak{g}_{n,n} \leftarrow \mathfrak{g}_{n,n+1} \leftarrow \mathfrak{g}_{n,n+2} \cdots$$

with $\operatorname{proj}\lim_{m \to \infty} \mathfrak{g}_{n,m} = \mathfrak{g}_n$ (compare with (3.1)). Moreover, the following infinite triangular diagram is commutative:

$$\begin{array}{ccccccc} \mathfrak{g}_{1,1} & & & & & & \\ \uparrow & & & & & & \\ \mathfrak{g}_{1,2} & \longrightarrow & \mathfrak{g}_{2,2} & & & & \\ \uparrow & & \uparrow & & & & \\ \mathfrak{g}_{1,3} & \longrightarrow & \mathfrak{g}_{2,3} & \longrightarrow & \mathfrak{g}_{3,3} & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \cdots & & \cdots & & \cdots & & \cdots . \end{array} \tag{3.7}$$

Consider the increasing sequence of Lie algebras $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots$. By the *Lie algebra* of G will be meant $\mathfrak{g} \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathfrak{g}_n$. It is straightforward that $\mathfrak{g}$ can be identified with the family of one-parameter subgroups of G; moreover, the natural exponential map $\exp : \mathfrak{g} \to G$ is well defined and $\exp \restriction \mathfrak{g}_n = \exp_n : \mathfrak{g}_n \to G_n$. We need some notation. Let

$$p_n : G_n \to G_n / N_n ,$$

$$q_n : G_n \to G_n /(N_{n+1} \cap G_n),$$

$$t_n : G_n /(N_{n+1} \cap G_n) \to G_n / N_n$$

be the canonical projection maps, and let

$$P_n^* : C(G_n / N_n) \to C(G_n),$$

$$Q_n^* : C(G_n /(N_{n+1} \cap G_n)) \to C(G_n),$$

$$T_n^* : C(G_n / N_n) \to C(G_n /(N_{n+1} \cap G_n))$$

be the associated one-to-one linear operators respectively. Then the homomorphism $\mathfrak{g}_{n, n+1} \to \mathfrak{g}_{n, n}$ is exactly dt_n (see (3.7). It is obvious that $P_n^* = Q_n^* T_n^*$. We endow the space $P_n^*(\mathcal{D} (G_n / N_n))$ [respectively $Q_n^*(\mathcal{D} (G_n /(N_{n+1} \cap G_n)))$] with the topology such that the restriction of P_n^* to $\mathcal{D} (G_n / N_n)$ [respectively Q_n^* to $\mathcal{D} (G_n /(N_{n+1} \cap G_n))$] is a homeomorphism. Then $P_n^*(\mathcal{D} (G_n / N_n))$ is a closed subspace of $Q_n^*(\mathcal{D}(G_n/(N_{n+1}\cap G_n)))$. It is straightforward that $T_n^*(\mathcal{D}(G_n/N_n))\subset$ $\subset\mathcal{D}(G_n/(N_{n+1}\cap G_n))$ and $T_n^*(\widetilde{dt_n(\partial)})=\tilde{\partial}\,T_n^*$ for all $\partial\in\mathfrak{g}_{n,\,n+1}$. From this we deduce that

$$T_n^*\in\mathcal{L}(\mathcal{D}(G_n/N_n),\mathcal{D}(G_n/(N_{n+1}\cap G_n)))$$

Denote by λ_n a left Haar measure on G_n, $n\in N$. Let us make use of Lemma 3.6: there exists a non-negative function $c_n\in P_n^*(\mathcal{D}(G_n/N_n))$ with Supp $c_n\subset W_n$ and such that the linear operator

$$\tilde{C}_n:Q_n^*(C_0(G_n/(N_{n+1}\cap G_n)))\ni f\mapsto\tilde{C}_nf\in P_{n+1}^*(C_0(G_{n+1}/N_{n+1})),$$

determined by

$$(\tilde{C}_nf)(g_{n+1})=\int_{G_n}f(g_n)c_{n+1}(g_n^{-1}g_{n+1})\,d\lambda_n(g_n),\quad g_{n+1}\in G_{n+1},$$

is one-to-one. One can prove by a slight modification of the argument in Subsection 2.3 that $\tilde{C}_n(Q_n^*(\mathcal{D}(G_n/(N_{n+1}\cap G_n))))\subset P_{n+1}^*(\mathcal{D}(G_{n+1}/N_{n+1}))$ and $\tilde{C}_n\in\mathcal{L}(Q_n^*(\mathcal{D}(G_n/(N_{n+1}\cap G_n))),P_{n+1}^*(\mathcal{D}(G_{n+1}/N_{n+1})))$. Now we define the operator $C_n:P_n^*(\mathcal{D}(G_n/N_n))\to P_{n+1}^*(\mathcal{D}(G_{n+1}/N_{n+1}))$ as follows:

$$C_n(P_n^*f)=\tilde{C}_nQ_n^*(T_n^*f),\quad f\in\mathcal{D}(G_n/N_n).$$

Then $C_n\in\mathcal{L}(P_n^*(\mathcal{D}(G_n/N_n)),P_n^*(\mathcal{D}(G_{n+1}/N_{n+1})))$ and Ker $C_n=\{0\}$.

Now we select a sequence $\{E_n\}_{n+1}^{\infty}$ of subsets of G such that the properties (a)-(d) from Subsection 2.3 are satisfied and $p_n^{-1}(p_n(E_n))=E_n$ for all $n\in N$. It follows that $p_n^{-1}(p_n(E_m))=E_m$ for all $m\geq n$. We set

$$\mathcal{D}_m^*(G_n)=P_n^*(\mathcal{D}(G_n/N_n,p_n(E_m\cap G_n)))\subset P_n^*(\mathcal{D}(G_n/N_n)).$$

Then we have:

So,

(i$_3$) $\mathcal{D}^\bullet(G) \overset{\text{def}}{=} \operatorname{inj}\lim_{n\to\infty} \mathcal{D}^\bullet_{2n+1}(G_n)$ is a separable nuclear (LN*)-space;

(ii$_3$) $L_g, \tilde{\partial} \in \mathcal{L}(\mathcal{D}^\bullet(G), \mathcal{D}^\bullet(G))$ for all $g \in G$ and $\partial \in \mathfrak{g}$;

(iii$_3$) $G \ni g \mapsto \mathcal{L}(\mathcal{D}^\bullet(G), \mathcal{D}^\bullet(G))$ is a continuous homomorphism.

It is worthwhile to observe that (3.7) is used to prove (ii$_3$). Now we set

$$Q_n = \int_{G_n} c_n(g_n) U(g_n)\, d\lambda_n(g_n) \quad \text{and} \quad P_n = \overrightarrow{\prod_{k>n}} Q_k$$

for $n \in N$. Arguing as in Subsection 2. 4, we complete the proof of Main Theorem.

3.4. In this subsection the final step of the proof of Main Theorem will be done. It remains to consider the case of the group $G = \bigcup_{n=1}^{\infty} G_n$ satisfying (II) (see Subsection 3.3). We set $\tilde{G}_n = G_n \times R^n$, $n \in N$, and $\tilde{G} = G \times R_0^\infty$, where $R_0^\infty = \operatorname{inj}\lim_{n\to\infty} R^n$ with the natural embeddings $R^n \to R^m$, $m > n$. Next, we extend U to the unitary representation $\tilde{U}$ of $\tilde{G}$ by setting $\tilde{U}(g, t) = U(g)$ for all $(g, t) \in \tilde{G}$. Since $\dim \tilde{G}_{n+1}/\tilde{G}_n = 1 > 0$ for all $n \in N$, we may apply the argument of the previous subsection. It is obvious that the strong Gårding domain for $\tilde{U}$ is a strong Gårding domain for U. Thus the Main Theorem is proved completely.

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Области Гординга для унитарных представлений счетных индуктивных пределов локально компактных групп

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Пусть G — индуктивный предел возрастающей последовательности локально компактных групп $G_1 \subset G_2 \subset \dots$. Для произвольного сильно непрерывного унитарного представления U группы G в сепарабельном гильбертовом пространстве $\mathcal{H}$ построено U-инвариантное, сепарабельное, ядерное, Монтелевское (DF)-пространство $\mathcal{F}$, которое (топологически) плотно вложено в $\mathcal{H}$, и такое, что сужение U на $\mathcal{F}$ есть слабо непрерывное представление G непрерывными линейными операторами пространства $\mathcal{F}$. Кроме того, $\mathcal{F}$ является областью существенной самосопряженности для генераторов всех однопараметрических подгрупп в G, и все эти генераторы оставляют $\mathcal{F}$ инвариантной.

Області Гордінга для унітарних зображень зчисленних індуктивних границь локально компактних груп

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Нехай G є індуктивною границею зростаючої послідовності локально компактних груп $G_1 \subset G_2 \subset \dots$. Для довільного сильно неперервного унітарного зображення U групи G в сепарабельному просторі Гільберта $\mathcal{H}$ побудовано U-інваріантний, сепарабельний, ядерний, монтелевський (DF)-простір $\mathcal{F}$, який (топологічно) щільно вкладений в $\mathcal{H}$, та такий, що звуження U на $\mathcal{F}$ є слабко неперервним зображенням G неперервними лінійними операторами простору $\mathcal{F}$. Крім того, $\mathcal{F}$ — область істотної самоспряженності для генераторів усіх однопараметричних підгруп в G, і всі ці генератори зберігають $\mathcal{F}$ інваріантною.