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Projection method and unconditional bases

A. M. Minkin

Department of Mechanics and Mathematics, Saratov State University, 83, Astrakhanskay Str., 410071, Saratov, Russia
E-mail:minkin@scnit.saratov.su

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Necessary and sufficient conditions are obtained for unconditional basicity with brackets of a family of exponentials in the space $L^2(-a, a)$ or in its span. The subspaces corresponding to the brackets are spanned by exponentials with "neighboring" exponents.

1. Let us consider a family $\{e_\lambda^a\}_{\lambda\in\Lambda}$ of exponentials $e_\lambda^a := \exp(-i\bar{\lambda}x)$, $-a\le x\le a$ on a finite interval. The question of its unconditional basisness (notation: (UB)) has been completely solved in [1–3] provided that

$$\inf \operatorname{Im}\Lambda := \inf\{\operatorname{Im}\lambda \mid \lambda\in\Lambda\} > -\infty; \tag{1}$$

in the case

$$\inf |\operatorname{Im}\Lambda| := \inf\{|\operatorname{Im}\lambda| \mid \lambda\in\Lambda\} > 0; \tag{2}$$

in [4] and for arbitrary spectrum in [5]. All these works are based on the *projection method* due to B. S. Pavlov. In the case $\operatorname{Im}\lambda \ge h > 0$, it consists of reducing the UB property to the question of whether there is an isomorphism between *span* $(e_\lambda \mid (-a,\infty))$ and $L^2(-a, a)$ under the orthogonal projection P_a on $L^2(-a, a)$ in $L^2(-a, \infty)$. Here and below $e_\lambda \equiv \exp(i\lambda x)$. Let us try to apply the result of [5] to the spectrality problem of the operator generated by boundary value problem:

$$-idy/dx = f,\quad -a\le x\le a,\ f\in L^2(-a, a), \tag{3}$$

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$$U(y) \equiv \int_{-a}^{a} y(t)d\sigma(t) = \sum_j c_j y(a_j), \; c_j \neq 0. \tag{4}$$

Here $d\sigma$ is a discrete measure of finite variation, $\sum |c_j| < \infty$, $\{a_j\} \subset [-a, a]$. Let $\mathcal{L}(\lambda) \equiv \int_{-a}^{a} \exp(i\lambda x)d\sigma(x)$ be the generating function of this problem, $\Lambda = \{\lambda_n\}$ be the set of its zeros with multiplicities $k(\lambda_n)$. Suppose that

$$\pm a \in \{a_j\}. \tag{5}$$

Then it is known [6] that the following assertions are valid:

i) Λ lies in a strip of finite width:

$$|\mathrm{Im}\,\lambda_n| \leq h = \mathrm{const}; \tag{6}$$

ii) in every rectangle $R(t, h) := \{ |\mathrm{Re}\, z - t| \leq h, \; |\mathrm{Im}\, z| \leq h \}$ the number of zeros counting multiplicities is uniformly bounded with respect to $t \in (-\infty, \infty)$; moreover, $\forall \delta > 0$ the following double-sided estimate holds true:

$$|\mathcal{L}(z)| \asymp \exp(a|y|), \; y = \mathrm{Im}\, z; \; dist\,(z, \Lambda) \geq \delta; \tag{7}$$

iii) the spectrum Λ can be partitioned into a set of disjoint clusters Λ_n, $Card\,\Lambda_n := \sum_{\lambda \in \Lambda} k(\lambda) = O(1)$ and the corresponding family $N = \{N_n\}$ constitutes an unconditional basis of subspaces in $L^2(-a, a)$ (notation: $N \in (UB)$). Here N_n stands for the *span* $(x^k e_\lambda^a \mid k = 0, \dots, k(\lambda) - 1; \; \lambda \in \Lambda_n)$.

Whenever relation (5) is broken it is only known [6] that

$$+a \notin \{a_j\} \to \inf \mathrm{Im}\,\lambda_n = -\infty, \tag{8}$$

$$-a \notin \{a_j\} \to \sup \mathrm{Im}\,\lambda_n = +\infty, \tag{9}$$

and no basisness results were obtained.

2. In view of (iii) it is useful to introduce the following notation. Let $k(\lambda)$ be a divisor in the complex plane C with a discrete support Λ which has a single limiting point at ∞ and let

$$\Lambda = \bigcup \Lambda_n \tag{10}$$

be its partition into disjoint subsets called clusters. Further we set

$$k(\lambda) = \sum k_n(\lambda),\ k_n(\lambda) = k(\lambda),\ \lambda \in \Lambda_n \textit{ and} = 0 \textit{ otherwise} \tag{11}$$

and assume that

$$1.\ \Lambda_n \in K_n := \{\, |z - \xi_n| \le r_n \,\};\ \ r_n \le C \cdot (1 + |\operatorname{Im} \xi_n|); \tag{12}$$

$$2.\ \textit{Card}\ \Lambda_n = O(1). \tag{13}$$

D e f i n i t i o n 1. Let N_n be as in ii), $N := N(\Lambda) = \{ N_n \}$. Suppose (10)–(13) are valid and $N \in (UB)$. Then we shall say that N is a block-basis in $L^2(-a, a)$ (notation: $N \in (BB)$).

Theorem 1. *Let* (10) *be a certain partition of the problem's* (3) *spectrum and* $N = N_n$ *be the corresponding family of spectral subspaces. Then* $N \in (BB)$ *iff* (5) *is valid*.

The statement of this theorem rests on a *block-basis* generalization of the criterion in [5].

Let Λ_- be a union of all Λ_n such that $\operatorname{Im} K_n < 0$, Λ_+ be its complement in Λ and denote their divisors $k_\pm(\lambda)$.

Theorem 2. *Let* $N = \{ N_n \}$ *be defined by some partition* (11) *of the divisor* $k(\lambda)$ *in Section 2. Then* $N \in (BB)$ *iff the following relations hold*:

A) $k(\lambda)$ *is a zero divisor of some entire function* $\mathcal{L}(z)$ *of exponential type* (*e.f.e.t.*); (14)

B) *dist* $(\Lambda_n, \Lambda_n) \ge \varepsilon > 0$, $n \ne m$; (*notation*: $\Lambda \in (GS)$); (15)

C) $\Lambda_h^{\pm} := \cup \{ \Lambda_n \mid \inf (\pm \operatorname{Im} K_n \geq h) \} \in (CV)$; (16)

D) $| M(\cdot - iy) |^2 \in (A_2)$, $y > 0$ *and sufficiently large.* (17)

Here *dist* $(X, Y) := \inf \{ |x - y| \mid x \in X, y \in Y \}$, (A_2) is the Muckenhoupt's condition, (CV) – the Carleson–Vasjunin condition [8] and $M(z)$ is an entire function of the first order with zero divisor

$$k_M(\lambda) := k(\lambda) + k(-\lambda),\ \operatorname{Im} \lambda \geq 0;\ = 0 \ \textit{otherwise}. \tag{18}$$

R e m a r k 1. Let $\Theta \equiv \exp(iaz)$; $B(z, k)$ be the Blaschke product with divisor k. Then in Theorem 2 one can replace D) by

$$dist_{L^\infty} (\Theta^2 \overline{B} (\cdot , k_M (\cdot - iy)), H_{\pm}^{\infty}) < 1. \tag{19}$$

Let us also mention that (BB) yields completeness and the latter implies A).

R e m a r k 2. Theorem 2 is also valid for the block-basisness in the span. Herewith it is only needed to remove the second condition from (19) (it corresponds to the minus sign) together with A).

3. P r o o f o f T h e o r e m 2. It repeats essentially that one given in [5]. Therefore we shall mention below only the corrections in the proof preserving notation from that article.

3.1. Corrections to §1. Let $\Lambda_-' = \Lambda_h^-$ for some $h > 0$, $\Lambda_+' := \Lambda \setminus \Lambda_-'$. Hence $\operatorname{Im} K_n \geq -h_1 \ \forall \ \Lambda_n \subset \Lambda_+'$ with some $h_1 > 0$. We take $\beta_- = 0$, $\beta_+ > -h_1$. Then all the statements in [5, §1] except Lemma 1.4 are readily checked. One must only replace Carleson condition (C) by (CV). For instance, Lemma 1.2 is reduced to the question of norm equivalence for "packets" of exponentials on $(0, 2a)$ and $(0, \infty)$. But the latter has already been proved in [5, Theorem 9.1].

3.2. Necessity of (15)–(16). Without loss of generality we may consider some subsystem $N(\Lambda') \subset N$ with spectrum Λ' such that $\ell = \inf \operatorname{Im} \Lambda' > -\infty$. Moreover, multiplying it by $\exp(-yt)$, $y > |\ell|$ we come to a new subsystem $N(\Lambda' + iy)$. Then it is a block-basis in its span. This yields that the continued subsystem constitutes an unconditional basis of subspaces in its span in $L^2(-a, \infty)$. The latter implies that $\Lambda' + iy \in (CV)$.

3.3. Corrections to Section 4.2. Now we set $\delta > \beta_+$ and let $\Lambda_1 = \Lambda'_- \setminus \Lambda_2$, Λ_2 be the union of all $\Lambda_n \in \Lambda'$ such that $\sup \operatorname{Im} K_n \le -\delta$.

3.4. Corrections to §6. We replace everywhere Carleson condition (C) with Carleson–Vasjunin one (CV). Proposition 6.1 has respectively an analogous generalization to the case of the union of two (CV) sets. We notice that

$$\Gamma = \Gamma_+ \cup \Gamma_- \in (CV), \ \Gamma_- = \Lambda_1 , \ \Gamma_+ = \overline{\Lambda}_2 \cup \Lambda'_+$$

because $\Gamma_- \in (CV)$ as a part of Λ'_- and the second set Γ_+ also belongs to (CV) (maybe with some other partition into clusters). Then *dist* $(\Gamma_+, \Gamma_-) > 0$ due to our choice of these sets.

3.5. Corrections to §7. Now we fix y such that

$$\inf \operatorname{Im} (\Gamma + iy) > 0. \tag{20}$$

In 7.2, we set

$$\gamma_+ = \bigcup_{\inf \operatorname{Im} \Gamma_n > 0} \Gamma_n , \qquad \gamma_- = \Gamma \setminus \gamma_+ . \tag{21}$$

We obtain set M_0 from Λ reflecting all Λ_n's with K_n intersecting C_- to upper halfplane, namely such Λ_n's are replaced by $\overline{\Lambda}_n$. All other Λ_n's stay in M_0 unchanged. Therefore $M_0 = \gamma_+ \cup \overline{\gamma}_-$. Setting $\gamma = \gamma_- \cup \overline{\gamma}_-$, we see that γ lies in the strip $C(-\delta_1, \delta_1)$ with sufficiently big $\delta_1 > 0$. We choose y such that $y - \delta_1 > \varepsilon_1$ (cf. [5, Section 7.4]) and repeat all the considerations in §7 with δ replaced by δ_1 . Instead of (7.11) we set $S_2 = \bigcup (K_n \cup \overline{K}_n)$, the union is taken over all $\Lambda_n \subset \gamma_-$. At last we come to a new relation

$$\Theta^{-1} B(\,\cdot\,, M_0 + iy) H_+^2 \dotplus \Theta H_+^2 = L^2 \tag{22}$$

instead of (7.16). Let us consider the function

$$h(z) = B(z, M + iy)/B(z, M_0 + iy).$$

It is easy to calculate that

$$h(z) = \prod_{\lambda \in \nu} b_{\lambda + iy}(z) / \prod_{\lambda \in \nu} b_{\bar{\lambda} + iy}(z),$$

where $\nu = \{ \lambda \in \gamma_- \mid \operatorname{Im} \lambda \geq 0 \mid \}$ and b_λ stands for elementary Blaschke factor. Since $\gamma_- \in (CV)$ we get that $\nu + iy$, $\bar{\nu} + iy \in (CV)$. Reasoning as in [5, Lemma 7.2], we deduce that $| h | \asymp 1$, $\operatorname{Im} z \leq 0$. Therefore $hH_-^2 = H_-^2$ and $hL^2 = L^2$. At last multiplying (22) by h, we establish (7.16).

3.6. Corrections to §8. It suffices only to repeat word by word all the considerations in this paragraph. Thus we proved Theorem 2 with sufficiently big $y > 0$. But in the half-bounded case, namely for the spectrum $M + iy$ it is well-known that the exponents can be shifted downwards as much as one wants to subject to inequality $\operatorname{Im}(M + iy) \geq \varepsilon > 0$. The proof is finished.

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Метод проєкций и безусловные базисы

А.М. Минкин

Получены необходимые и достаточные условия безусловной базисности со скобками семейства экспонент в пространстве $L^2(-a, a)$ или в своей замкнутой линейной оболочке. При этом отвечающие скобкам подпространства натянуты на экспоненты с "близкими" показателями.

Метод проекцій та безумовні базиси

А.М. Мінкін

Отримано необхідні та достатні умови безумовної базисності з дужками сім'ї експонент у просторі $L^2(-a, a)$ або в своїй замкненій лінійній оболонці. При цьому відповідні дужкам підпростори натягнені на експоненти з "близькими" показниками.