Matematicheskaya fizika, analiz, geometriya
1996, v. 3, No 3/4, p. 406-411

Eigenfunction completeness for a third-order ordinary differential bundle of operators

V.S. Rykhlov

Department of Mechanics and Mathematics, Saratov State University,
83, Astrakhanskaya Str., 410026, Saratov, Russia
E-mail:rykhlov@scnit.saratov.su

Received May 22, 1996

In the paper one-fold eigenfunction completeness and two-fold eigenfunction incopleteness with infinite defect are proved for a non-normal third-order ordinary differential bundle of operators with non-separated two-boundary conditions.

Many questions of modern mathematics, mechanics and physics lead to the problem of the expansion a given function in biorthogonal Fourier series in terms of the eigenfunctions (e.f.) of nonselfadjoint ordinary differential bundles of operators. In particular, one of the important problem here is the problem of the e.f. completeness. Up to the present this problem has been far from its complete solving. In this paper we study the problem of e.f. completeness for one bundle of differential operators not considered before.

Let us consider the ordinary differential bundle of third-order operators generated over finite interval $[0,1]$ by the differential expression

$$\ell(y,\lambda) = y^{(3)} - \lambda\, y^{(2)} + \lambda^2 y^{(1)} - \lambda^3 y \tag{1}$$

and two-point non-separated boundary conditions

$$\begin{aligned} U_1(y) &:= y(0) + y(1) = 0, \\ U_2(y) &:= y^{(1)}(0) + i y^{(1)}(1) = 0, \\ U_3(y) &:= y^{(2)}(0) - y^{(2)}(1) = 0. \end{aligned} \tag{2}$$

The roots of the characteristic equation here are $\omega_1 = -i$, $\omega_2 = 1$, $\omega_3 = i$. Let us denote

$$y_j(x,\lambda) := \exp(\lambda\, \omega_j x), \quad j = 1, 2, 3.$$

These functions are linearly independent solutions of the equation $\ell(y,\lambda) = 0$.

Let us denote

$$u_{kj} := (v_{kj} + e^{\lambda\omega_j} w_{kj}) := U_k(y_j), \quad k, j = 1, 2, 3,$$

where $v_{kj} = \omega_j^{k-1}$, $w_{kj} = \beta_k \omega_j^{k-1}$, $\beta_1 = 1$, $\beta_2 = i$, $\beta_3 = -1$. Also we introduce the column vectors

$$V_j = (v_{1j}, v_{2j}, v_{3j})^T, \; W_j = (w_{1j}, w_{2j}, w_{3j})^T,$$

i.e.,

$$V_1 = \begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, V_3 = \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix},$$

$$W_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, W_2 = \begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}, W_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

The characteristic determinant has the following form:

$$\Delta(\lambda) := \det (u_{kj})_{k,j=1,2,3} =$$

$$= \lambda^3 \det (V_1 + e^{\lambda\omega_1} W_1, V_2 + e^{\lambda\omega_2} W_2, V_3 + e^{\lambda\omega_3} W_3) =$$

$$= \lambda^3 \Big(A_0 + A_1 e^{\lambda\omega_1} + A_2 e^{\lambda\omega_2} + A_3 e^{\lambda\omega_3} +$$

$$+ A_{12} e^{\lambda(\omega_1+\omega_2)} + A_{13} e^{\lambda(\omega_1+\omega_3)} + A_{23} e^{\lambda(\omega_2+\omega_3)} + A_{123} e^{\lambda(\omega_1+\omega_2+\omega_3)} \Big) =$$

$$= \lambda^3 \Big((A_0 + A_{13}) + A_1 e^{-\lambda i} + (A_2 + A_{123}) e^{\lambda} + A_3 e^{\lambda i} + A_{12} e^{\lambda(1-i)} + A_{23} e^{\lambda(1+i)} \Big),$$

where

$$A_0 := \det (V_1, V_2, V_3) = -4i, \quad A_{12} := \det (W_1, W_2, W_3) = 0,$$

$$A_1 := \det (W_1, V_2, V_3) = 0, \quad A_{13} := \det (W_1, V_2, W_3) = 0,$$

$$A_2 := \det (V_1, W_2, V_3) = 0, \quad A_{23} := \det (V_1, W_2, W_3) = 4i,$$

$$A_3 := \det (V_1, V_2, W_3) = 4, \quad A_{123} := \det (W_1, W_2, W_3) = -4.$$

Hence

$$\Delta(\lambda) = 4\lambda^3(-i - e^{\lambda} + e^{\lambda i} + ie^{\lambda(1+i)}) = 4\lambda^3 \hat{\Delta}(\lambda).$$

Because of $A_1 = A_{12} = 0$, the bundle (1), (2) is not normal (see the definition in [1]).

Let us denote the roots of the equation $\Delta(\lambda) = 0$ as λ_k and let $\Lambda = \{\lambda_k\} \setminus \{0\}$. It is known that $\lambda \in \Lambda$ are eigenvalues of the bundles (1), (2).

We consider the functions $f_j(x,\lambda)$, $j = 1, 2, 3$, obtained from $\Delta(\lambda)$ by replacing the j-th row with $(y_1(x,\lambda), y_2(x,\lambda), y_3(x,\lambda))$. It is known that $f_j(x,\lambda)$ are e.f. of our bundle if $\lambda \in \Lambda$. Hence the functions $f_j(x,\lambda)$, $\lambda \in C$, are linearly independent analytic continuations of the e.f. Traditional method of proving eigenfunction multiple completeness in $L_2[0,1]$ (this method uses only the functions $f_j(x,\lambda)$ as analytic continuations of the e.f.) does not give here any results. We use another approach. Let us attempt to find an analytic continuation for the e.f. in the following form:

$$g(x,\lambda) = \sum_{j=1}^{3} \lambda^{j-1} \gamma_j(\lambda) f_j(x,\lambda),$$

where $\gamma_j(\lambda)$ are entire functions, which will be defined later. If we denote $\Gamma = (\gamma_1, \gamma_2, \gamma_3)^T$, we obtain

$$g(x,\lambda) = \lambda^3 \Big(y_1(x,\lambda) \det(\Gamma, V_2 + e^{\lambda\omega_2} W_2, V_3 + e^{\lambda\omega_3} W_3) +$$
$$+ y_2(x,\lambda) \det(V_1 + e^{\lambda\omega_1} W_1, \Gamma, V_3 + e^{\lambda\omega_3} W_3) +$$
$$+ y_3(x,\lambda) \det(V_1 + e^{\lambda\omega_1} W_1, V_2 + e^{\lambda\omega_2} W_2, \Gamma) \Big).$$

We need to select such a Γ (if it is possible) that in the expression for $g(x,\lambda)$ the "bad members" will be absent, i.e.,

$$\det(\Gamma, V_2, V_3) = \det(\Gamma, W_2, V_3) = \det(W_1, \Gamma, V_3) =$$
$$= \det(W_1, V_2, \Gamma) = \det(W_1, W_2, \Gamma) = 0.$$

Evidently, this property will be valid if $\Gamma = W_1$ and $\Gamma = V_3$. Hence we find the two-parametric representation for the sought function Γ

$$\Gamma = c_1 W_1 + c_2 V_3,$$

where c_1, c_2 are arbitrary constants which are not equal zero simultaneously. Thus we obtain for the function $g(x,\lambda)$ the two-parametric representation

$$g(x,\lambda) = 4\lambda^3 \Big((-c_1 e^{\lambda(1+i)} + c_2 e^{\lambda i}) y_1(x,\lambda) +$$
$$+ \big((c_1 + ic_2)e^{\lambda i} - (ic_1 + c_2)\big) y_2(x,\lambda) + (ic_1 e^{\lambda} - ic_2) y_3(x,\lambda) \Big), \qquad (3)$$
$$\text{for all } c_1, c_2 \in C.$$

Theorem 1. *The eigenfunction system of the bundle* (1), (2) *is one-fold complete in* $L_2[0,1]$.

P r o o f. Let $h_1 \in L_2[0,1]$ and $h_1 \perp g(\cdot, \lambda)$ for all $\lambda \in \Lambda$. Then we denote $H(\lambda) = \int_0^1 g(x,\lambda)h_1(x)dx$ and consider the function $F(\lambda) := H(\lambda)/\Delta(\lambda)$. Evidently, this function is an entire function bounded in the whole complex plane. Taking into consideration the theorem of Liouville, we obtain $F(\lambda) \equiv const.$ But because $F(\lambda) \to 0$ when λ tends to infinity along some set of points belonging to imagine axis, we obtain finally $F(\lambda) \equiv 0$, i.e.,

$$\left(-c_1 e^{\lambda(1+i)} + c_2 e^{\lambda i}\right) Y_1(\lambda) + \left((c_1 + ic_2)e^{\lambda i} - (ic_1 + c_2)\right) Y_2(\lambda) + \\ + (ic_1 e^{\lambda} - ic_2) Y_3(\lambda) \equiv 0, \quad \text{for all } c_1, c_2 \in \mathrm{C}, \tag{4}$$

where

$$Y_j(\lambda) = \int_0^1 y_j(x,\lambda)h_1(x)dx, \quad j = 1, 2.$$

Taking in (4) firstly $c_1 = 0$, $c_2 = 1$ and then $c_1 = -1$, $c_2 = 0$, we obtain the system

$$e^{\lambda i} Y_1(\lambda) + (ie^{\lambda i} - 1)Y_2(\lambda) - iY_3(\lambda) \equiv 0, \tag{5}$$

$$e^{\lambda(1+i)} Y_1(\lambda) + (-e^{\lambda i} + i)Y_2(\lambda) - ie^{\lambda} Y_3(\lambda) \equiv 0. \tag{6}$$

Multiplying (5) by e^{λ} and subtracting (6), we find

$$\hat{\Delta}(\lambda) Y_2(\lambda) \equiv 0,$$

from here

$$Y_2(\lambda) = 0, \quad \text{for all } \lambda \in \mathrm{C} \setminus \Lambda. \tag{7}$$

In view of (5) we find also

$$ie^{\lambda i} Y_1(\lambda) + Y_3(\lambda) = 0, \quad \text{for all } \lambda \in \mathrm{C} \setminus \Lambda. \tag{8}$$

It can be concluded evidently from (7) that $h_1(x) \equiv 0$. Thus the system of the e.f. of the bundle (1), (2) is one-fold complete in $L_2[0,1]$ (we did not use here the property (8)).□

Theorem 2. *The eingenfunction system of the bundle (1), (2) is not two-fold complete in* $L_2^2[0,1] := L_2[0,1] \oplus L_2[0,1]$ *and has an infinite defect there.*

P r o o f. Let $\hat{h} = (h_1, h_2)^T \in L_2^2[0,1]$ and be orthogonal to two-derived chains of e.f. for our bundle (1), (2) (see the definition of the derived chains of e.f. in [1]). Denoting $H(\lambda) = \int\limits_0^1 g(x,\lambda)h(x,\lambda)dx$, where $h(x,\lambda) = h_1(x) + \lambda h_2(x)$, we consider the function $F(\lambda) = H(\lambda)/\Delta(\lambda)$. Evidently, this function is an entire function and $|F(\lambda)| \le C|\lambda|$, for all $\lambda \in \mathbb{C}$. From the theorem of Liouville we obtain

$$F(\lambda) \equiv A + B\lambda = \langle c_1\Phi_1 + c_2\Phi_2, \hat{h}\rangle + \lambda\langle c_1\Phi_3 + c_2\Phi_4, \hat{h}\rangle,$$

where $\Phi_k = (\phi_{1k}, \phi_{2k})^T$, $k = \overline{1,4}$. The functions ϕ_{jk} can be easily found. If we suppose in addition $\hat{h} \perp \Phi_k$, $k = \overline{1,4}$, we obtain $F(\lambda) \equiv 0$. Hence we can derive the formulae (7), (8), where $Y_j(\lambda) = \int\limits_0^1 y_j(x,\lambda)h(x,\lambda)dx$, as we did it before.

From (7), when $\lambda = 0$, we find $\int\limits_0^1 h_1(x)dx = 0$. Taking it into consideration, we can obtain

$$Y_2(\lambda) = \int\limits_0^1 e^{\lambda x}\, d\left(\int\limits_0^x h_1(\tau)\,d\tau\right) + \lambda\int\limits_0^1 e^{\lambda x}h_2(x)\,dx =$$

$$= -\lambda\int\limits_0^1 e^{\lambda x}\left(\int\limits_0^x h_1(\tau)\,d\tau - h_2(x)\right)dx = 0, \quad \text{for all } \lambda \in \mathbb{C}\setminus\Lambda.$$

Thus

$$h_2(x) \equiv \int\limits_0^x h_1(\tau)\,d\tau, \quad \text{for a.e. } x \in [0,1]. \tag{9}$$

Using (9) in (8), we find

$$0 = \int_0^1 e^{i\lambda x}((h_1(x) + ih_1(1 - x)) + \lambda\,(h_2(x) + ih_2(1 - x)))dx =$$

$$= \int_0^1 e^{i\lambda x}((h_1(x) + ih_1(1 - x)) + i(h_1(x) - ih_1(1 - x)))dx =$$

$$= (1 + i)\int_0^1 e^{i\lambda x}(h_1(x) + h_1(1 - x))dx, \quad \text{for all} \quad \lambda \in \mathbf{C} \setminus \Lambda.$$

From this we obtain

$$h_1(x) \equiv -h_1(1 - x)\,, \quad \text{for a.e.} \quad x \in [0, 1]. \tag{10}$$

Hence the conditions (9), (10) give the linear manifold of all functions which orthogonal to the two-derived chains of e.f. and some fixed finite set of vector functions. Evidently, this linear manifold has infinite dimension. □

References

1. *A. A. Shkalikov*, Boundary value problems for ordinary differential equations with a parameter in the boundary conditions.— Tr. Semin. Imeni Petrovskogo (1983), v. 9, p. 190–229 [J. Sov. Math. (1986), v. 33, p. 1311–1342].

Полнота собственных функций пучка дифференциальных операторов 3-го порядка

В.С. Рыхлов

В статье рассматривается ненормальный обыкновенный дифференциальный пучок операторов третьего порядка в пространстве суммируемых с квадратом на отрезке [0, 1] функций с нераспадающимися двухточечными краевыми условиями. Доказывается, что система собственных функций этого пучка однократно полна и имеет бесконечный дефект относительно двукратной полноты.

Повнота власних функцій в'язка диференціальних операторів 3-го порядку

В.С. Рихлов

У статті розглядається ненормальна звичайна диференціальна в'язка операторів третього порядку у просторі сумовних з квадратом на відрізку [0, 1] функцій з нерозпадними двоточковими крайовими умовами. Доводиться, що система власних функцій цієї в'язки однократно повна і має нескінченний дефект відносно двократної повноти.