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A remark to the definition of Nevanlinna matrices

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We prove that the unimodular entire matrix-function

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

with real entries is a Nevanlinna matrix provided that the three quotients B/A, A/C, and D/C have positive imaginary part in the upper half-plane.

Introduction. A unimodular entire matrix-function

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is called a Nevanlinna matrix if for each fixed z from the upper (lower) half-plane the Möbius transformation $w \mapsto (A(z)w + B(z))(C(z)w + D(z))^{-1}$ maps the upper (lower) half-plane into a disc lying with its closure in the same half-plane. It is readily seen that equivalently one can say that the entries of the unimodular matrix $M(z)$ are real entire functions and for all real values t

$$\frac{A(z)t + B(z)}{C(z)t + D(z)} \in (\mathcal{R}). \tag{1}$$

Here, $(\mathcal{R})$ is a class of resolvent-type functions, i.e., functions $\theta(z)$ holomorphic in the both upper and lower half-planes and satisfying $\theta(\bar{z}) = \overline{\theta(z)}$, and $\operatorname{Im} \theta(z) \operatorname{Im} z > 0$ for $\operatorname{Im} z \neq 0$.

Nevanlinna matrices play a fundamental role in the description of solutions of problems of analysis connected with self-adjoint extensions of special classes of symmetric operators with deficiency index (1,1), see [AG], [Akh], [dBr, Chapter 2], and [Kr1]. We mention here the Hamburger moment problem [Akh] (probably, the term "Nevanlinna matrix" was introduced for the first time in that book) and [Berg], the spectral theory of Sturm–Liouville differential operators [Kr 2], Krein's problem of continuation of positive definite functions from an interval [Ber, Chapter VIII, § 3], and de Branges' theory of Hilbert spaces of entire functions [dBr]. Analytical properties of Nevanlinna matrices were investigated in the paper [Kr 2] as well as in [dBr, Chapter 2].

Condition (1) immediately yields that

$$A/C \in (\mathcal{R}), \text{ and } B/D \in (\mathcal{R}).$$

If the Möbius transformation $w \mapsto (Aw+B)(Cw+D)^{-1}$, $AD-BC=1$ maps the upper half-plane into a disc lying in the upper half-plane, then the Möbius transformation obtaining by replacement of B and C by $-C$ and $-B$ or by changing positions of A and D have the same property. Therefore the matrices

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}, \begin{pmatrix} A(z) & -C(z) \\ -B(z) & D(z) \end{pmatrix}, \text{ and } \begin{pmatrix} D(z) & -C(z) \\ -B(z) & A(z) \end{pmatrix}$$

simultaneously belong or do not belong to the class of Nevanlinna matrices. Thus (1) also yields that $B/A \in (\mathcal{R})$, and $D/C \in (\mathcal{R})$.

It is worth to remind, that the Möbius transformation given by a unimodular matrix M maps the upper half-plane into itself if and only if

$$\frac{MJM^* - J}{i} \geq 0, \tag{2}$$

where M^* is the conjugate matrix, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; or, equivalently, if and only if

$$\operatorname{Im}(B/A) \geq 0, \ \operatorname{Im}(D/C) \geq 0, \text{ and } \operatorname{Re}[A\overline{D} - B\overline{C}] \geq 1. \tag{3}$$

See, e.g., [dBr, Problems 75 and 81]. We will use another equivalent form of the same condition (see [Kr 2, Lemma 3] or [Akh, Chapter 3, Addenda and Problems]):

$$\operatorname{Im}(D/C) \geq 0, \text{ and } \operatorname{Im}(D/C) \cdot \operatorname{Im}(A/C) > (\operatorname{Im}(1/C))^2. \tag{3a}$$

Theorem. *Let*

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

be a matrix-function whose entries are real entire functions satisfying the conditions:

$$\det M(z) = A(z)D(z) - B(z)C(z) \equiv 1 \tag{4}$$

and

$$B/A \in (\mathcal{R}),\ A/C \in (\mathcal{R}),\ D/C \in (\mathcal{R}). \tag{5}$$

Then $M(z)$ is a Nevanlinna matrix.

We do not insist on the novelty of this result, though we have not met it before. It can be easily seen that there are complex values A, B C, and D such that

$$AD - BC = 1,\ \mathrm{Im}\ (B/A) > 0,\ \mathrm{Im}\ (A/C) > 0, \text{ and } \mathrm{Im}\ (D/C) > 0,$$

but nevertheless neither (3) nor (3a) are fulfilled.

Krein's theorems. In the proof of the Theorem we follow arguments from [Kr2] (see also [Akh, Chapter 3, Addenda and Problems]). For convenience of the reader, we formulate here the analytical results which are used below. First, we use Krein's multiplicative representation of functions from the class $(\mathcal{R})$ which are meromorphic in the whole complex plane (see [Lev, Chapter VII]).

Theorem K1. *In order that a meromorphic function in the complex plane $\Psi(z)$ belong to the class $(\mathcal{R})$ it is neccessary and sufficient that*

$$\Psi(z) = c\,\frac{z-\lambda_0}{z-\mu_0}\prod_{n\neq 0}\left(1-\frac{z}{\lambda_n}\right)\left(1-\frac{z}{\mu_n}\right)^{-1},$$

where $\mu_n < \lambda_n < \mu_{n+1}$, $\lambda_{-1} < 0 < \mu_1$, $c > 0$. The product in the right-hand side converges absolutely and uniformly on each compact subset of the complex plane.

Another analytical device we need is also due to Krein ([Kr 2], [Lev, Chapter V]). A real entire function $F(z)$ is said to be a function of Krein's class $(\mathcal{K})$ if $F^{-1}(z)$ can be represented as a series of simple fractions

$$\frac{1}{F(z)} = c_0 + \frac{c_1}{z} + \sum_n a_n \left\{ \frac{1}{z - \lambda_n} + \frac{1}{\lambda_n} \right\}$$

with real poles λ_n, and the series converges absolutely and uniformly on each compact set of the complex plane, i.e., $\sum_n | a_n | \lambda_n^{-2} < \infty$.

An entire function $F(z)$ is said to be a function of Cartwright class if F has at most finite exponential type and the logarithmic integral converges:

$$\int_{-\infty}^{\infty} \frac{\log^+ | F(x) |}{1 + x^2} dx < \infty.$$

Theorem K 2. *An entire function F belongs to the Cartwright class if and only if* $\log | F(z) |$ *has positive harmonic majorants in the upper and lower half-planes.*

Theorem K 3. *Each entire function of the class* ($\mathcal{K}$) *belongs to the Cartwright class.*

Remark (Krein, de Branges). Using the Theorems K1 and K3, Krein proved in [Kr 2] that *a real entire function* $C(z)$ *can serve as an entry in a Nevanlinna matrix if and only if it belongs to the class* ($\mathcal{K}$). He also described the pairs of real entire functions $C(z)$ and $D(z)$ which can serve as the second row in a Nevanlinna matrix: *the product* $C \cdot D$ *should belong to the Krein class and*

$$D/C \in (\mathcal{R}).$$

In the framework of the theory of Hilbert spaces of entire functions another form of the same condition was given by de Branges [dBr, Chapter 2]: *the pair of real entire functions* $C(z)$ *and* $D(z)$ *is the second row in a Nevanlinna matrix if and only if the entire function* $E(z) = C(z) - iD(z)$ *has at most exponential type,*

$$| E(x + iy) | < | E(x - iy)| \quad \textit{for} \ y > 0,$$

and

$$\int_{-\infty}^{\infty} \frac{1}{|E(x)|^2} \frac{dx}{1+x^2} < \infty.$$

If these conditions are fulfilled, then, certainly, E must belong to the Cartwright class.

If the second row of the Nevanlinna matrix is given then one can easily restore the whole matrix: at zeros of $C(z)$ the function $A(z)$ equals $1/D(z)$ and therefore, assuming that $C(z)$ does not vanish at the origin, the Nevanlinna–Chebotarev representation (see e. g., [Lev, Chapter VII]) of the function $A/C \in (\mathcal{R})$ has a form

$$\frac{A(z)}{C(z)} = az + b + \sum_{\{\lambda : C(\lambda) = 0\}} \frac{1}{C'(\lambda)D(\lambda)} \left\{\frac{1}{\lambda - z} - \frac{1}{\lambda}\right\}$$

with a positive a and real b as parameters. In fact, that series is nothing but the Lagrange interpolation series for the function $A(z)$ with nodes at the zeros of $C(z)$. Then the function $B(z)$ may be found either directly from equation (4) or using a similar series.

P r o o f o f t h e T h e o r e m. We split the proof into several elementary steps.

1. *The zeros of all four entire functions $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are real.* Indeed, if say $A(z)$ has a non-real zero, then $C(z)$ and hence $\det M(z)$ would have the same non-real zero. Without lost of generality, we assume that all these four functions do not vanish in the origin.

Conditions (5) yield that zeros of the pairs of entire functions (B, A), (A, C), and (D, C) are interlacing.

2. *The zeros of functions $B(z)$ and $D(z)$ are interlacing as well.* It follows immediately from identity (4) and conditions (5). Indeed, let λ_n and λ_{n+1} be two consequitive zeros of $B(z)$. Then $A(\lambda_n)D(\lambda_n) = A(\lambda_{n+1})D(\lambda_{n+1}) = 1$ and since the zeros of A and B are interlacing, we conclude, that D has an odd number of zeros between each pair of consequitive zeros of B. In the same way, B has an odd number of zeros between each pair of consequitive zeros of D.

3. *The entire functions $A \cdot C$ and $B \cdot D$ belong to the Cartwright class.* Indeed, identity (4) may be rewritten in the form

$$\frac{1}{AC} = \frac{D}{C} - \frac{B}{A}.$$

Thus $A \cdot C$ belongs to the Krein class ($\mathcal{K}$) and by Theorem K 3 it belongs to the Cartwright class.

4. Let $\Lambda_F = \{ \lambda_n \}$ be a zero set of a real entire function $F(z)$ from Cartwright's class with real zeros. Then the limit

$$\Pi_F(z) = \lim_{R \to \infty} \prod_{|\lambda_n| \le R} \left(1 - \frac{z}{\lambda_n}\right)$$

exists uniformly on each compact set of the complex plane (see [Lev, Chapter V]), and

$$F(z) = \text{const}\ \Pi_F(z).$$

Thus we can write

$$A = A(0)e^{H}\, \Pi_A\,,\ B = B(0)e^{G}\, \Pi_B\,,\ C = C(0)e^{-H}\, \Pi_C\,,\ D = D(0)e^{-G}\, \Pi_D\,,$$

where $H(z)$ and $G(z)$ are real entire functions vanishing at the origin. All four canonical products also belong to the Cartwright class. Indeed, consider, for example, the products Π_A and Π_C. The logarithmic integrals

$$\int_{-\infty}^{\infty} \frac{\log^+ | \Pi(x \pm i) |}{1 + x^2}\, dx < \infty$$

converge for $\Pi = \Pi_{AC}$. Since the zeros of A and C are interlacing,

$$C_1(1 + |x|)^{-2} \le \frac{| \Pi_A(x \pm i) |}{| \Pi_C(x \pm i) |} \le C_2(1 + |x|)^{-2},$$

and hence

$$\log|\Pi_A(x\pm i)| = \frac{1}{2}\log|\Pi_{AC}(x\pm i)| + O(\log|x|),$$

$$\log|\Pi_C(x\pm i)| = \frac{1}{2}\log|\Pi_{AC}(x\pm i)| + O(\log|x|),$$

as x tends to infinity. That is, the logarithmic integrals converge also for $\Pi = \Pi_A$ and $\Pi = \Pi_C$. Thus, by Theorem K 2, the logarithm of the modulus of the both canonical products has harmonic majorants in the half-planes $\operatorname{Im} z < 1$ and $\operatorname{Im} z > -1$ and hence in the lower and upper half-planes. Therefore the products Π_A and Π_C also belong to the Cartwright class.

Since $B/A \in (\mathcal{R})$, Theorem K1 implies that $G(z) - H(z) \equiv \mathrm{const}$. Taking into account that both functions vanish at the origin, we find $G(z) \equiv H(z)$. The same argument applied to the functions $A(z)$ and $C(z)$ says that $H \equiv 0$. That is, *all four entries of the matrix-functions $M(z)$ are entire functions of Cartwright class.*

5. *The entries of $M(z)$ belong to the Krein class.* We prove it for $C(z)$. First, we will show that the series

$$\sum_{\lambda\in\Lambda_C} \frac{1}{C'(\lambda)}\left\{\frac{1}{\lambda} + \frac{1}{z-\lambda}\right\} \tag{6}$$

converges absolutely on compact subsets of the complex plane, and then, that it converges to $C^{-1}(z) - C^{-1}(0)$. As before, we assume for simplicity, that $C(z)$ does not vanish at the origin.

The convergence of the series (6) is equivalent to

$$\sum_{\lambda\in\Lambda_C} \frac{1}{|C'(\lambda)|\lambda^2} < \infty. \tag{7}$$

To prove (7) we use the Cauchy–Bunyakovskii inequality

$$\left\{\sum_{\lambda\in\Lambda_C} \frac{1}{|C'(\lambda)|\lambda^2}\right\}^2 = \left\{\sum_{\lambda\in\Lambda_C} \sqrt{-\frac{A(\lambda)}{C'(\lambda)\lambda^2}}\,\sqrt{-\frac{1}{A(\lambda)C'(\lambda)\lambda^2}}\right\}^2 \le$$

$$\leq \sum_{\lambda \in \Lambda_C} \left(-\frac{A(\lambda)}{C'(\lambda)\lambda^2} \right) \cdot \sum_{\lambda \in \Lambda_C} \left(-\frac{1}{A(\lambda)C'(\lambda)\lambda^2} \right) < \infty.$$

The first series converges since $A/C \in (\mathcal{R})$, and the second series converges since $A \cdot C \in (\mathcal{K})$.

The entire function of exponential type

$$F(z) = \frac{1}{z} \left\{ \frac{1}{C(z)} - \frac{1}{C(0)} - z \sum_{\lambda \in \Lambda_C} \frac{1}{C'(\lambda)} \frac{1}{\lambda(z-\lambda)} \right\}$$

tends to zero on the boundary of angles

$$\frac{(2j-1)\pi}{4} \leq \arg z \leq \frac{(2j+1)\pi}{4}.$$

as z tends to ∞, and the Phragmén–Lindelöf's theorem applying in each of these four angles immediately yields that $F(z) \equiv 0$, proving that $C \in (\mathcal{K})$.

6. Now, having the representation

$$\frac{1}{C(z)} = \frac{1}{C(0)} + \sum_{\lambda \in \Lambda_C} \frac{1}{C'(\lambda)} \left\{ \frac{1}{z-\lambda} + \frac{1}{\lambda} \right\}, \tag{8}$$

we prove (3a).

To this end, we separate the imaginary part in (8):

$$\operatorname{Im} \frac{1}{C(z)} = \sum_{\lambda \in \Lambda_C} \frac{1}{C'(\lambda)} \operatorname{Im} \frac{1}{z-\lambda} = -\operatorname{Im} z \sum_{\lambda \in \Lambda_C} \frac{1}{C'(\lambda)|z-\lambda|^2}$$

and estimate its square using again the Cauchy–Bunyakovskii inequality

$$\left\{ \operatorname{Im} \frac{1}{C(z)} \right\}^2 = (\operatorname{Im} z)^2 \left\{ \sum_{\lambda \in \Lambda_C} \frac{1}{\sqrt{-A(\lambda)C'(\lambda)}} \sqrt{-\frac{A(\lambda)}{C'(\lambda)}} \frac{1}{|z-\lambda|^2} \right\}^2 \leq$$

$$\leq (\operatorname{Im} z)^2 \left\{ \sum_{\lambda \in \Lambda_C} \frac{-1}{A(\lambda)C'(\lambda)} \frac{1}{|z-\lambda|^2} \right\} \cdot \left\{ \sum_{\lambda \in \Lambda_C} \left(-\frac{A(\lambda)}{C'(\lambda)} \right) \frac{1}{|z-\lambda|^2} \right\}. \tag{9}$$

Since $A/C \in (\mathcal{R})$, we have

$$\operatorname{Im} \frac{A}{C}(z) \geq \sum_{\lambda \in \Lambda_C} \left(-\frac{A(\lambda)}{C'(\lambda)} \right) \frac{1}{|z-\lambda|^2}, \tag{10}$$

and since $D/C \in (\mathcal{R})$, we have

$$\operatorname{Im} \frac{D}{C}(z) \geq \sum_{\lambda \in \Lambda_C} \left(-\frac{D(\lambda)}{C'(\lambda)} \right) \frac{1}{|z-\lambda|^2} = \sum_{\lambda \in \Lambda_C} \frac{-1}{A(\lambda)C'(\lambda)} \frac{1}{|z-\lambda|^2}. \tag{11}$$

Substituting (10) and (11) into (9), we obtain (3a) completing the proof of the theorem.

R e m a r k. I.V. Ostrovskii called author's attention to the fact that the arguments used above prove the following. *Let $F(z)$ be a function from Krein's class. Then it can be factorized into the product*

$$F(z) = F_1(z)\, F_2(z) \ldots F_n(z),$$

where each entire functions $F_j(z)$ also belong to Krein's class $(\mathcal{K})$ and the zeros of each pair of functions F_j and F_k are interlacing. This factorization may be useful in various situations since the asymptotic behaviour of the modulus of the factors $|F_j|$ is very close to $|F|^{1/n}$.

The converse to that statement requires certain additional assumptions since the product of two functions of Krein's class with interlacing zeros does not always belong to Krein's class (since zeros of the factors may come too close to each other, see, e.g., an example in [Koo]). The same combination of the Cauchy–Bunyakovskii inequality and the Phragmén–Lindelöf theorem as used above gives the following statement. *Let the entire functions F and G have interlacing zeros and let $G \in (\mathcal{K})$. Then $FG \in (\mathcal{K})$ if and only if*

$$\sum_{\lambda\, :\, G(\lambda)=0} \left(-\frac{1}{F(\lambda)G'(\lambda)} \right) \frac{1}{1+\lambda^2} < \infty. \tag{12}$$

Condition (12) is also sufficient for $F \in (\mathcal{K})$.

This fact (as, in a sense, everything presented here) is contained implicitly in [Kr 2]. We also mention that condition (12) arises naturally in Hamburger's moment problem ([Akh, Chapter 4, Addenda and Problems; Berg]), as well as in the de Branges theory [dBr, Chapter 2].

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Замечание к определению Неванлинновских матриц

Михаил Содин

Доказано, что унимодулярная целая матрица-функция

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

является Неванлинновской матрицей, если ее элементы — вещественные целые функции и три отношения B/A, A/C и D/C имеют положительную мнимую часть в верхней полуплоскости.

Зауваження до визначення Неванлінновських матриць

Михайло Содін

Доведено, що унімодулярна ціла матриця-функція

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

є Неванлінновською матрицею, якщо її елементи — дійсні цілі функції і три відношення B/A, A/C та D/C мають позитивну уявну частину у верхній півплощині.