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The Cauchy problem for nonlinear Schrödinger equation with bounded initial data

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An analog of scattering data for the operators which are strong limits of reflectionless Dirac operators is introduced and the corresponding inverse problem is solved. On this basis a method of solving the Cauchy problems for nonlinear Schrödinger equation with initial data from a wide set of non-vanishing at infinity functions is developed.

Introduction

Let $\vec{L}_2(a,b)$ denote the Hilbert space of vector-functions (column matrices)

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = (f_1(x), f_2(x))^T$$

with the standard scalar product

$$(f,g) = \int_a^b (f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)})dx.$$

Let D be the differential operator

$$D = i\sigma_3 \frac{d}{dx} + V(x) \qquad (-\infty < x < \infty),$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0 & \overline{i\psi(x)} \\ i\psi(x) & 0 \end{pmatrix},$$

where $\psi(x)$ is a continuous *complex-valued potential.* D defines a symmetric operator in $\vec{L}_2(-\infty,\infty)$ with domain of definition the set of C^1 functions with compact support. The closure of this operator is the *Dirac operator*, which is a self-adjoint operator and will still be denoted by D.

A complete analysis of the direct and inverse scattering problems for the Dirac operators with potential $\psi(x)$ which satisfy the *finite density conditions*

$$\lim_{x\to\mp\infty} x^k \frac{d^l}{dx^l}\left(\psi(x)-\frac{1}{2}\rho e^{i\varphi_\mp}\right)=0 \tag{1}$$

($\rho>0$, $\varphi_\mp\in[-\pi,\,\pi)$ are fixed, $k,l=0,1,2,\ldots$) is given in the monography [1]. Among the potentials satisfying the finite density condition an important class is the class of reflectionless potentials, which generate the Dirac operators with vanishing reflection coefficients.

Let us introduce the following notations:

$B(\rho,\varphi_-,\varphi_+)$ is the set of all reflectionless potentials satisfying condition (1);

$B(\omega)$ is the union of all sets $B(\rho,\varphi_-,\varphi_+)$ for all $\rho\in(0,\omega]$, $\varphi_-,\varphi_+\in[-\pi,\pi)$;

$\overline{B(\omega)}$ is the closure of $B(\omega)$ for the topology of compact convergence on the real axis.

The characteristic properties of potentials belonging to the set $\overline{B(\omega)}$ and of the corresponding Dirac operators are described in [2].The set $\overline{B(\omega)}$ contains a broad class of bounded functions. For example, all finite-gap potentials and all potentials which satisfy the condition of finite density and define Dirac operators with compactly supported reflection coefficients belong to it.

In this paper we prove, under some additional conditions, the following results:

Results. 1. *We obtain a linear integral equation which, for Dirac operators with potentials in $\overline{B(\omega)}$, plays the same role as the main equation of the inverse scattering problem for Dirac operators with potentials satisfying the finite density condition.*

2. *The solution of the Cauchy problem for the nonlinear Schrödinger equation*

$$\begin{cases} i\frac{\partial}{\partial t}\psi(x,t)=-\frac{\partial^2}{\partial x^2}\psi(x,t=)+2\left(|\psi(x,t)|^2-\frac{\omega^2}{4}\right)\psi(x,t)\\ \psi(x,0)=\psi(x)\end{cases}$$

with the initial data $\psi(x)$ from the set $\overline{B(\omega)}$ is reduced to the resolution of a system of linear integral equations, and we prove this is uniquely solvable.

1. Auxiliary results

According to the classical Weyl theorem, the equation

$$i\sigma_3 y'+V(x)y=\frac{\lambda}{2}y \tag{1.1}$$

has two solutions $\Phi_+(\lambda, x) \in \vec{L}_2(0,\infty)$, $\Phi_-(\lambda, x) \in \vec{L}_2(-\infty, 0)$ for all non real values of λ, where

$$\Phi_\pm(\lambda, x) = F(\lambda, x) + m_\pm(\lambda) G(\lambda, x),$$

and $F(\lambda, x)$, $G(\lambda, x)$ are the solutions of equation (1.1) with initial data

$$F(\lambda, 0) = (1, i)^T, \quad G(\lambda, 0) = (i, 1)^T,$$

and $m_\pm(\lambda)$ are holomorphic functions outside of the real axis. The solutions $\Phi_\pm(\lambda, x)$ are called the Weyl solutions and the functions $m_\pm(\lambda)$ are called the Weyl functions of equation (1.1), or of the corresponding Dirac operator.

Choosing an arbitrary positive number ω, we may replace the Weyl functions $m_\pm(\lambda)$ by one function

$$n(z) = \begin{cases} m_-(\omega\frac{z+z^{-1}}{2}) & \text{if } \operatorname{Im} z > 0,\ |z| \neq 1, \\ m_+(\omega\frac{z+z^{-1}}{2}) & \text{if } \operatorname{Im} z < 0,\ |z| \neq 1, \end{cases} \tag{1.2}$$

and the Weyl solutions $\Phi_\pm(\lambda, x)$ by one function

$$\Phi(z, x) = \begin{cases} \Phi_-(\omega\frac{z+z^{-1}}{2}, x) & \operatorname{Im} z > 0,\ |z| \neq 1, \\ \Phi_+(\omega\frac{z+z^{-1}}{2}, x) & \operatorname{Im} z < 0,\ |z| \neq 1, \end{cases} \tag{1.3}$$

which satisfies the equation

$$i\sigma_3\Phi' + V(x)\Phi = \frac{\omega}{4}(z + z^{-1})\Phi, \tag{1.4}$$

and belongs to the space $\vec{L}_2(-\infty, 0)$ if $\operatorname{Im} z > 0, |z| \neq 1$ and to the space $\vec{L}_2(0,\infty)$ if $\operatorname{Im} z < 0,\ |z| \neq 1$.

The function $n(z)$ and the solution $\Phi(z, x)$ defined by (1.2), (1.3) are also called the Weyl function and Weyl solution of equation (1.4), or of the corresponding Dirac operator.

Definition. *The number $\omega > 0$ and the function $n(z)$ defined by the equalities* (1.2) are called *the spectral data* of the Dirac operator.

It is obvious that the Weyl functions $m_\pm(\lambda)$, and therefore also the potential $\psi(x)$, can be uniquely reconstructed from the spectral data. Characteristic properties of the spectral data of operators with potentials in $\overline{B(\omega)}$ are obtained in [2]:

The number $\omega > 0$ and the function $n(z)$ are spectral data of some Dirac operator with potential in $\overline{B(\omega)}$ if and only if the function $n(z)$ has an integral representation of the form

$$n(z) = i\int_T \frac{\xi + z}{\xi - z} d\mu(\xi), \tag{1.5}$$

where μ is a Borel measure on the unit circle $T = \{\xi \mid |\xi| = 1\}$ satisfying the condition

$$\int_T d\mu(\xi) = 1. \tag{1.5'}$$

(The measures which satisfy condition (1.5′) are probability measures.)

In particular, it follows from this that *the potential $\psi = \psi(x)$ belongs to $\overline{B(\omega)}$ if and only if the Weyl functions $m_+(\lambda), m_-(\lambda)$ have continuous limits on both semi-axes $(-\infty, -\omega), (\omega, \infty)$ satisfying the equality*

$$m_+(\lambda - i0) = m_-(\lambda + i0) \quad \text{if} \;\; \lambda \in (-\infty, -\omega) \cup (\omega, \infty).$$

Thus $\overline{B(\omega_2)} \subset \overline{B(\omega_1)}$ if $\omega_2 > \omega_1$.

It is often more convenient to use, instead of (1.5), the equivalent representation

$$n(z) = i \int_{-\pi}^{\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha), \tag{1.6}$$

where $\mu(\alpha)$ is a non-decreasing function which satisfies the condition

$$\mu(\pi) - \mu(-\pi) = 1. \tag{1.6'}$$

According to the F. Riesz–Herglotz theorem (see, for example, [3, p. 116]), the function $n(z)$ can be represented in the form (1.6) if and only if it is holomorphic in the circle $|z| < 1$, its imaginary part is positive there, $n(0) = i$, and $n(\overline{z}) = \overline{n(z^{-1})}$. In this case the function $\mu(\alpha)$ is determined by the formula

$$\mu(\alpha) = \lim_{r \to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\alpha} \operatorname{Im} n(re^{i\varphi}) d\varphi. \tag{1.7}$$

If the function $n(z)$ satisfies the conditions of the F. Riesz–Herglotz theorem then (for some choice of the branch of the logarithm) the function

$$\frac{2}{\pi} \ln n(z) = \frac{2}{\pi} \ln |n(z)| + i \frac{2}{\pi} \arg n(z)$$

also satisfies these conditions.

Therefore we have

$$\frac{2}{\pi} \ln n(z) = i \int_{-\pi}^{\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\nu(\alpha)\,,$$

where

$$\nu(\alpha) = \lim_{r \to 1-0} \frac{1}{\pi^2} \int_{-\pi}^{\alpha} \arg n(re^{i\varphi}) d\varphi\,.$$

Since $0 \le \arg n(re^{i\varphi}) \le \pi$ for $r < 1$, and since for almost all $\varphi \in [-\pi, \pi]$ the limit

$$\eta(\varphi) = \lim_{r \to 1-0} \arg n(re^{i\varphi}) \tag{1.8}$$

exists, the function $\nu(\alpha)$ is absolutely continuous and $d\nu(\alpha) = \pi^{-2}\eta(\alpha)d\alpha$, where

$$0 \le \eta(\alpha) \le \pi, \qquad \frac{1}{\pi^2}\int_{-\pi}^{\pi} \eta(\alpha)d\alpha = 1 \ . \tag{1.9}$$

Hence

$$\ln n(z) = \frac{i\pi}{2}\int_{-\pi}^{\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z} d\nu(\alpha) = -\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\ \eta(\alpha)d\alpha$$

and

$$n(z) = \exp\Big\{-\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\ \eta(\alpha)d\alpha\Big\} \ . \tag{1.10}$$

Elementary checking shows that for any measurable function $\eta(\alpha)$ satisfying (1.9), the function (1.10) satisfies the conditions of the F. Riesz–Herglotz theorem and hence can be represented in the form (1.6).

Therefore *the spectral data of the Dirac operators with the potentials from the set* $\overline{B(\omega)}$ *are defined by the positive number* ω *and either by an arbitrary non-decreasing function* $\mu(\alpha)$ *which satisfies* (1.6′) *or an arbitrary measurable function* $\eta(\alpha)$ *which satisfies* (1.9).

Here the functions $\mu(\alpha)$, $\eta(\alpha)$ and $n(z)$ are connected by the relations (1.6)–(1.8), (1.10).

We emphasize that the spectral data depends on the chosen value of the parameter ω. It follows from (1.3) that the pairs $\{\omega, n(z)\}$ and $\{\omega_1, n(w(z))\}$ are the spectral data of the same Dirac operator, if

$$w(z) = \frac{\omega_1}{\omega}\frac{z+z^{-1}}{2} + \sqrt{\Big(\frac{\omega_1}{\omega}\frac{z+z^{-1}}{2}\Big)^2 - 1} \ .$$

On the other hand, if the vector-function $\Phi(z,x)$ is the Weyl solution of equation (1.4) then the vector-function $\Phi(z, \frac{\omega_1}{\omega}x)$ will be the Weyl solution of the equation

$$i\sigma_3 y' + \frac{\omega_1}{\omega}V\Big(\frac{\omega_1}{\omega}x\Big)y = \omega_1\frac{z+z^{-1}}{4}y$$

from which it follows that the Weyl function $n(z)$ of this equation is the same as the Weyl function of equation (1.4). Therefore the potentials $\psi_\omega(x)$ and $\psi_{\omega_1}(x)$

of the Dirac operators with the spectral data $\{\omega, n(z)\}$ and $\{\omega_1, n(z)\}$ are related by the formula

$$\psi_{\omega_1}(x) = \frac{\omega_1}{\omega}\psi_\omega(\frac{\omega_1}{\omega}x).$$

Hence the transformation $\psi(x) \mapsto \omega^{-1}\psi(\omega^{-1}x)$ is a one–to–one mapping of the set $\overline{B(\omega)}$ onto the set $\overline{B(1)}$ which preserves the Weyl functions of the corresponding operators. This allows, without loss of generality, to deal only with Dirac operators with potentials in $\overline{B(1)}$ and their spectral data $\{1, n(z)\}$.

From the results of [2] (see formulae (5.2) and (5.6)–(5.8) on pages 22–24) it follows that for any x, the Weyl solutions $\Phi(z, x)$ of Dirac operators of the class we consider are analytic with respect to z outside of the circle $|z| = 1$ and the points $z = 0, \infty$, and we have

$$\begin{aligned} \lim_{|z|\to\infty} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} e^{\frac{ik(z)x}{2}}\Phi(z, x) &= (2, 4i\psi(x))^T, \\ \lim_{z\to 0} \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} e^{\frac{ik(z)x}{2}}\Phi(z, x) &= (4\overline{\psi(x)}, 2i)^T, \end{aligned} \tag{1.11}$$

with

$$k(z) = \frac{1}{2}(z - z^{-1}). \tag{1.12}$$

Let $R(z)$ denote a scalar function, analytic outside of the circle $|z| = 1$, tending to a finite limit $R(\infty)$ for $|z| \to \infty$, and having simple pole with residue $R_{-1}(0)$ at the origin. It follows from (1.11) that the vector-function

$$g(z, x) = e^{i\frac{k(z)x}{2}}R(z)\Phi(z, x) \tag{1.13}$$

is also analytic outside of the circle $|z| = 1$, tends to $2(R(\infty), 0)^T$ when $|z| \to \infty$ and has a simple pole at 0 with residue $2(0, iR^{-1}(0))^T$, and the vector–function

$$g(z, x) - 2(R(\infty), iR_{-1}(0)z^{-1})^T$$

tends to zero when $|z| \to \infty$ and to a finite limit when $z \to 0$. According to Cauchy's theorem,

$$g(z, x) - 2(R(\infty), iR_{-1}(0)z^{-1})^T = \frac{1}{2\pi i}\int_\Gamma \frac{g(\xi, x) - 2(R(\infty), iR_{-1}(0)\xi^{-1})^T}{\xi - z}d\xi,$$

where Γ is the oriented boundary of the annulus $r_2 < |\xi| < r_1$, and z lies outside of this annulus. Because the function $\xi \mapsto (\xi - z)^{-1}(R(\infty), iR_{-1}(0)\xi^{-1})^T$ is holomorphic in this annulus, its integral along Γ vanishes and

$$g(z, x) = 2(R(\infty), iR_{-1}(0)z^{-1})^T + \frac{1}{2\pi i}\int_\Gamma \frac{g(\xi, x)}{\xi - z}d\xi\,. \tag{1.14}$$

It follows further from equalities (1.11) that

$$\begin{aligned} \lim_{|z|\to\infty} \; zg_2(z,x) &= 4iR(\infty)\psi(x)\,, \\ \lim_{z\to 0} \; g_1(z,x) &= 4R_{-1}(0)\overline{\psi(x)}\,. \end{aligned} \tag{1.15}$$

We note that any vector-function analytic outside of the circle $|z| = 1$, having a finite limit $2(R(\infty),0)^T$ when $|z| \to \infty$ and a simple pole with residue $2(0, iR_{-1}(0))^T$ at the origin, satisfies equality (1.14). By a suitable choice of the function $R(z)$ we obtain a uniquely solvable integral equation from equality (1.14).

2. Factorization and choice of the function $R(z)$

The choice of the factor $R(z)$ in formula (1.13) is connected with a suitable factorization of the function

$$N(z) = \frac{1}{2}(n(z) - n(z^{-1})), \tag{2.1}$$

the study of which is the purpose of this section.

It follows from (1.6) that

$$N(z) = i\int_{-\pi}^{\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z} d\Big(\frac{\mu(\alpha)-\mu(-\alpha)}{2}\Big), \tag{2.2}$$

and because the function $N(z)$, together with $n(z)$, satisfies the conditions of the F. Riesz–Herglotz theorem

$$N(z) = \exp\Big\{-\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\nu(\alpha)d\alpha\Big\} \tag{2.3}$$

with

$$\nu(\alpha) = \lim_{r\to 1-0} \arg N(re^{i\alpha}). \tag{2.3$'$}$$

Let us introduce the following notations:

$$\begin{aligned} S(z,\rho) &= i\int_{-\pi}^{\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z} d\rho(\alpha)\,, \\ P(z,\delta) &= \exp\left\{-\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\delta(\alpha)d\alpha\right\}, \end{aligned}$$

where $\rho(\alpha)$ is a nondecreasing function and $\delta(\alpha) = \overline{\delta(\alpha)}$ a bounded and measurable function on the interval $[-\pi, \pi]$.

Let V be the involutive map of sets $A \subset [-\pi, \pi]$ or of functions $p(\alpha)$ $(-\pi \leq \alpha \leq \pi)$ such that

$$\begin{aligned} V(A) &= \{\alpha \mid -\alpha \in A\}\,, \\ V(p)(\alpha) &= p(-\alpha). \end{aligned}$$

It is easy to verify that the functions $S(z,\rho), P(z,\delta)$ satisfy the equalities

$$\begin{aligned} S(z, \rho_1 + \rho_2) &= S(z,\rho_1) + S(z,\rho_2)\,, \\ S(z^{-1}, \rho) &= \overline{S(\overline{z}, \rho)} = S(z, -V(\rho))\,, \qquad (2.4) \\ P(z, \delta_1 + \delta_2) &= P(z,\delta_1)P(z,\delta_2)\,, \end{aligned}$$

$$P(z^{-1}, \delta) = \overline{P(\overline{z}, \delta)} = P(z, -V(\delta))\,. \qquad (2.4')$$

We denote the non-tangential limits of the function $f(z)$ when $z \to e^{i\alpha} \in T$ by

$$f^{\pm}(\alpha) = \lim_{r \to 1 \mp 0} f(re^{i\alpha})$$

or $f(\alpha)$ if $f^{+}(\alpha) = f^{-}(\alpha)$.

It follows from classical complex analysis that there exist, for almost all α (with respect to the Lebesgue mesure), finite limits

$$S^{\pm}(\alpha, \rho) = \mathrm{Re} S(\alpha, \rho) \pm 2\pi i \rho'(\alpha)\,, \qquad (2.5)$$

$$P^{\pm}(\alpha, \delta) = \exp\left\{\pm i\delta(\alpha) - \frac{1}{2\pi}\text{v.p.}\int_{-\pi}^{\pi} \cot\frac{\alpha - \tau}{2}\delta(\tau)d\tau\right\}. \qquad (2.6)$$

If the function $\delta(\alpha)$ satisfies a Hölder condition on the interval $\alpha_1 < \alpha < \alpha_2$ then the convergence to the limit values is uniform on any interior segment of this interval, the functions $P^{\pm}(\alpha, \delta)$ satisfy the same Hölder condition, and their absolute values

$$|P(\alpha, \delta)| = \exp\left\{-\frac{1}{2\pi}\text{v.p.}\int_{-\pi}^{\pi} \cot\frac{\alpha - \tau}{2}\delta(\tau)d\tau\right\}$$

are separated from zero and bounded:

$$0 < M^{-1} \leq |P(\alpha, \delta)| \leq M < \infty\,. \qquad (2.7)$$

(The number M depends on the chosen segment.)

Using the notations above, we obtain the following formulas for the functions $n(z)$ and $N(z)$:

$$n(z) = S(z,\mu) = P(z,\eta), \tag{2.8}$$

$$N(z) = S(z, \frac{\mu - V(\mu)}{2}) = P(z,\nu), \tag{2.9}$$

where

$$\eta(\alpha) = \arg n^+(\alpha), \qquad \nu(\alpha) = \arg N^+(\alpha),$$

and the function $\mu(\alpha)$ is defined by (1.7).

Let us denote by Ω the support of the measure $d\mu(\alpha)$ defined on the segment $[-\pi,\pi]$ which derive from the non-decreasing function $\mu(\alpha)$. According to the definition the point $\alpha \in (-\pi,\pi)$ belongs to the support if and only if any interval which contains this point has positive measure and the points $-\pi$ and π are identified and belong to the support if and only if for any $\varepsilon > 0$ the measure of the set $[-\pi, -\pi+\varepsilon) \cup (\pi-\varepsilon, \pi]$ is positive.

The support of the measure $\frac{1}{2}d(\mu(\alpha) - \mu(-\alpha))$ is evidently the compact set $\Omega \cup V(\Omega)$.

Because the open set $(-\pi,\pi)\backslash\Omega$ does not contain points of the support of the measure $d\mu(\alpha)$ we can pass to the limit under the integral sign in formula (1.6) when $z \to e^{i\varphi}, \varphi \in (-\pi,\pi)\backslash\Omega$, and it follows that for any $\varphi \in (-\pi,\pi)\backslash\Omega$

$$\begin{cases} n^+(\varphi) = n^-(\varphi) = n(\varphi) = \int \cot \frac{\alpha-\varphi}{2} d\mu(\alpha), \\ \frac{dn(\varphi)}{d\varphi} = \int\limits_{-\pi}^{\pi} \frac{d\mu(\alpha)}{\sin^2 \frac{\alpha-\varphi}{2}} > 0. \end{cases} \tag{2.10}$$

Similar statements are true for the function $N(\varphi)$ on the set $(-\pi,\pi)\backslash\,(\Omega \cup V(\Omega))$. Therefore *on the set* $\{\xi \mid \xi = e^{i\varphi}, \varphi \in (-\pi,\pi)\backslash(\Omega \cup V(\Omega))\}$ *the functions* $n(\xi), N(\xi)$ *take on real values which increase monotonously with* φ.

Let us divide the support Ω into its symmetric and asymmetric parts:

$$\Omega_2 = \Omega \cap V(\Omega) = \{\alpha \mid \alpha \in \Omega, -\alpha \in \Omega\},$$

$$\Omega_1 = \Omega \backslash V(\Omega) = \{\alpha \mid \alpha \in \Omega, -\alpha \notin \Omega\}.$$

These are disjoint, and the symmetric part Ω_2 is always compact whereas the asymmetric part is compact if and only if the distance $\operatorname{dist}(\Omega_1, \Omega_2)$ is positive.

According to the Lebesgue theorem a non-decreasing function $\mu(\alpha)$ is the sum of an absolutely continuous function $\mu^a(\alpha)$ and a singular function $\mu^s(\alpha)$:

$$\mu(\alpha) = \mu^a(\alpha) + \mu^s(\alpha); \qquad d\mu^a(\alpha) = \mu'(\alpha)d\alpha.$$

The support of the measure which is generated by the singular function $\mu^s(\alpha)$ is denoted by Ω^s. Supposing

$$\Omega_2^s = \Omega_2 \cap \Omega^s \ ; \qquad \Omega_2^a = \Omega_2 \backslash \Omega_2^s,$$

we obtain the following expansion of Ω in nonintersecting subsets:

$$\Omega = \Omega_1 \cup \Omega_2^s \cup \Omega_2^a. \tag{2.11}$$

In what follows we shall consider only the measures $d\mu(\alpha)$ whose support satisfy the condition

A) *In* (2.11) *all three sets have positive mutual distances and the set* Ω_2^s *is finite or empty.*

In particular, from this condition it follows that these sets are compact and are at positive distance from the set $V(\Omega_1)$. Note that the points $-\pi, \pi$ either do not belong to the support or belong to one of the subsets Ω_2^s, Ω_2^a. This is also true of course for the point 0. So there are 9 possible combinations for their location with respect to the support Ω. All of them can be examined likewise, so we shall restrict ourselves to the simplest case when the points $-\pi, \pi$ and 0 do not belong to the support. Moreower, we shall suppose that the argument $\eta(\alpha)$ of the function $n^+(\alpha)$ satisfies the condition

B) *The set* Ω_2^a *can be covered by finite number of mutually disjoint intervals* δ_l *on each of which the following inequalities are true:*

$$\operatorname*{ess\,sup}_{\alpha\in\delta_l} \eta(\alpha) - \operatorname*{ess\,inf}_{\alpha\in\delta_l} \eta(\alpha) < \pi\,, \tag{2.12}$$

$$0 < \eta(\alpha) + \eta(-\alpha) < 2\pi\,, \tag{2.13}$$

and the function $\eta(\alpha) + \eta(-\alpha)$ *satisfies a Hölder condition. The inequalities*

$$0 < \varepsilon < \eta(\alpha) < \pi - \varepsilon \tag{2.14}$$

hold almost ewerywhere on the set Ω_2^a *(with respect to the Lebesgue measure), for some* $\varepsilon > 0$.

Let

$$\Delta = (0,\pi)\backslash(\Omega_1 \cup V(\Omega_1) \cup \Omega_2^s) = \cup_k \Delta_k\,,$$

where $\Delta_k = (\alpha_k, \beta_k)$ are mutually disjoint intervals.

It is evident that

$$\begin{aligned} V(\Delta) &= \cup_k V(\Delta_k) = (-\pi,0)\backslash(\Omega_1 \cup V(\Omega_1) \cup \Omega_2^s)\,, \\ \Delta \cup V(\Delta) &= ((-\pi,0)\cup(0,\pi))\backslash(\Omega_1 \cup V(\Omega_1) \cup \Omega_2^s)\,, \end{aligned}$$

$$\Delta \cup V(\Delta) \supset \Omega_2^a\,.$$

Let us clarify the behavior of the argument $\nu(\alpha)$ of the function $N^+(\alpha)$ on the set $\Delta \cup V(\Delta)\backslash\Omega_2^a$.

Lemma 2.1. *The arguments* $\eta(\alpha)$, $\nu(\alpha)$ *of the functions* $n^+(\alpha)$, $N^+(\alpha)$ *take on the same constant value, equal either to* 0 *or* π, *on every set* $(\delta_l \cap \Delta)\backslash\Omega_2^a$.

P r o o f. Because the set $(\delta_l \cap \Delta)\backslash\Omega_2^a$ lies in the complement of $\Omega \cup V(\Omega)$ it follows from the preceeding arguments that the functions $n^+(\alpha) = n^-(\alpha) = n(\alpha)$, $N^+(\alpha) = N^-(\alpha) = N(\alpha)$ are real on this set and their arguments can only take the values 0 or π. Condition B) implies that the oscillation of the argument $\eta(\alpha)$ of $n^+(\alpha)$ on the interval δ_l is less than π, and it follows that the range of the function $\eta(\alpha)$ on the set $(\delta_l \cap \Delta)\backslash\Omega_2^a$ cannot contain both 0 and π. Therefore the function $\eta(\alpha)$ is constant, equal to 0 or π, on the whole set $(\delta_l \cap \Delta)\backslash\Omega_2^a$. Further, since the values of $\eta(\alpha) + \eta(-\alpha)$ on this set are multiples of π and because of condition B), the inequalities (2.13) hold there, we have

$$\eta(\alpha) + \eta(-\alpha) = \pi, \qquad \alpha \in (\delta_l \cap \Delta)\backslash\Omega_2^a. \tag{2.15}$$

According to (2.8), (2.4′)

$$\begin{aligned} N(z) &= \frac{1}{2}(n(z) - n(z^{-1})) = \frac{1}{2}(P(z,\eta) - P(z^{-1},\eta)) \\ &= \frac{1}{2}P(z,\eta)(1 - P(z,-\eta)P(z^{-1},\eta)) \\ &= n(z)\frac{1}{2}\Big(1 - P(z, -\eta - V(\eta))\Big). \end{aligned}$$

Therefore

$$\arg N^+(\alpha) = \arg n^+(\alpha) + \arg(1 - P^+(\alpha, -\eta - V(\eta)))\,,$$

that is,

$$\nu(\alpha) = \eta(\alpha) + \arg(1 - P^+(\alpha, -\eta - V(\eta))), \tag{2.16}$$

and since we have $P^+(\alpha, -\eta - V(\eta)) < 0$ on $(\delta_l \cap \Delta)\backslash\Omega_2^a$ (according to (2.6), (2.15)), it follows that we also have $\arg\{1 - P^+(\alpha, -\eta - V(\eta))\} = 0$ and $\nu(\alpha) = \eta(\alpha)$ on this set. ■

Lemma 2.2. *Each interval* $(\alpha_k, \beta_k) = \Delta_k$ *splits into two pieces*

$$\Delta_k^- = (\alpha_k, \varphi_k), \qquad \Delta_k^+ = (\varphi_k, \beta_k) \tag{2.17}$$

so that

$$\begin{aligned} N^+(\alpha) &< 0 \quad \textit{when } \alpha \in \Delta_k^-\backslash\Omega_2^a\,, \\ N^-(\alpha) &< 0 \quad \textit{when } \alpha \in \Delta_k^+\backslash\Omega_2^a\,. \end{aligned} \tag{2.18}$$

(One of the intervals Δ_k^-, Δ_k^+ may be empty.)

P r o o f. If the interval Δ_k contains no point of Ω_2^a, then it lies in the complement of $\Omega \cup V(\Omega)$ and the function $N^+(\alpha) = N^-(\alpha) = N(\alpha)$ is real, and increasing. Hence in this interval there is at most one point, where $N^+(\alpha)$ changes sign from $-$ to $+$. Let us denote this point φ_k (if $N^+(\alpha)$ is positive, resp. negative, on the whole interval, we set $\varphi_k = \alpha_k$ (resp. β_k)) we obtain the intervals (2.17) on which the inequalities (2.18) hold.

If the interval Δ_k contains points of Ω_2^a then because of condition B) the set $\Delta_k \cap \Omega_2^a$ is covered by a finite number of mutually disjoint intervals $\delta_l \cap \Delta_k = (\alpha_k^{(l)}, \beta_k^{(l)})$, on which the inequalities (2.12), (2.13) are fulfilled. Obviously, these intervals can be labelled so that

$$\alpha_k = \beta_k^{(0)} \le \alpha_k^{(1)} < \beta_k^{(1)} \le \alpha_k^{(2)} < \beta_k^{(2)} \le \dots \\ \dots \le \alpha_k^{(n)} < \beta_k^{(n)} \le \alpha_k^{(n+1)} = \beta_k$$

(for the sake of convenience the endpoints α_k, β_k of Δ_k are denoted by $\beta_k^{(0)}, \alpha_k^{(n+1)}$). According to Lemma 2.1 the argument $\nu(\alpha)$ of the function $N^+(\alpha)$ is constant, equal to 0 or to π, on each set

$$\Phi_p = (\alpha_k^{(p)}, \beta_k^{(p)}) \backslash \Omega_2^a \qquad (1 \le p \le n).$$

Therefore the function $N^+(\alpha)$ is real, of constant sign, on each set Φ_p. Let us denote by Φ_{p^+} (resp. Φ_{p^-}) the set with the smallest (resp. greatest) number p^+ (resp. p^-) at which the function $N^+(\alpha)$ is positive (resp. negative) and $\Phi_0 = (\beta_k^{(0)}, \alpha_k^{(1)}) = (\alpha_k, \alpha_k^{(1)})$(resp. $\Phi_{n+1} = (\beta_k^{(n)}, \alpha_k^{(n+1)}) = (\beta_k^{(n)}, \beta_k)$) if this function is positive (resp. negative) on all sets $\Phi_p \quad (1 \le p \le n)$.

Since the function $N^+(\alpha)$ grows monotonically on the segment $[\beta_k^{(p^+)}, \alpha_k^{(p^+ +1)}]$ ($[\beta_k^{(p^- -1)}, \alpha_k^{(p^-)}]$) and in its neighborhood, it remains positive (negative) on this segment and on the next (previous) set $\Phi_{p^+ +1}$ (resp. $\Phi_{p^- -1}$). Hence $p^+ = p^- + 1$, and the function $N^+(\alpha)$ is positive on the set $(\alpha_k^{(p^+)}, \beta_k) \backslash \Omega_2^a$ and negative on the set $(\alpha_k, \beta_k^{(p^+ -1)}) \backslash \Omega_2^a$. Because of the monotonicity on the segment $[\beta_k^{(p^+ -1)}, \alpha_k^{(p^+)}]$ there is one point φ_k at which the function $N^+(\alpha)$ changes sign from $-$ to $+$. Setting $\varphi_k = \alpha_k$ when $p^+ = 0$ ($\varphi_k = \beta_k$ when $p^- = n+1$), we obtain the intervals (2.17) on which the inequalities (2.18) are fulfilled. ■

Corollary. *The argument $\nu(\alpha)$ of $N^+(\alpha)$ satisfies the equalities*

$$\nu(\alpha) = \begin{cases} \pi & \text{if } \alpha \in (V(\Delta^+) \cup \Delta^-) \backslash \Omega_2^a, \\ 0 & \text{if } \alpha \in (V(\Delta^-) \cup \Delta^+) \backslash \Omega_2^a, \end{cases} \tag{2.19}$$

with

$$\Delta^- = \cup_k \Delta_k^-, \quad \Delta^+ = \cup_k \Delta_k^+.$$

P r o o f. In fact, if $\alpha \in \Delta^-\backslash\Omega_2^a$ or $\alpha \in \Delta^+\backslash\Omega_2^a$ these equalities are equivalent to the inequalities (2.18) proved in Lemma 2.2. If $\alpha \in V(\Delta^+)\backslash\Omega_2^a$ then $-\alpha \in \Delta^+\backslash\Omega_2^a$, $\nu(-\alpha) = 0$ and

$$\nu(\alpha) = \nu(\alpha) + \nu(-\alpha), \qquad \alpha \in V(\Delta^+)\backslash\Omega_2^a,$$

and if $\alpha \in V(\Delta^-)\backslash\Omega_2^a$ then $-\alpha \in \Delta^-\backslash\Omega_2^a$, $\nu(-\alpha) = \pi$, and

$$\nu(\alpha) = \nu(\alpha) + \nu(-\alpha) - \pi, \qquad \alpha \in V(\Delta^-)\backslash\Omega_2^a.$$

Hence to prove the equalities (2.19) it is sufficient to prove the equality

$$\nu(\alpha) + \nu(-\alpha) = \pi, \qquad -\pi \le \alpha \le \pi. \tag{2.20}$$

According to (2.1) we have $N(z) = -N(z^{-1})$, hence $N^+(\alpha) = -N^-(-\alpha) = -\overline{N^+(-\alpha)}$, so $\nu(\alpha) = \pi - \nu(-\alpha)$. This proves (2.20). ∎

Let us denote by $\chi_1(\alpha), \chi_2^a(\alpha)$, and $\chi(\alpha)$ the indicators of the sets Ω_1, Ω_2^a, and $\Delta^- \cup V(\Delta^+)$. It is evident that $\chi_1(-\alpha)$, $\chi_2^a(-\alpha) = \chi_2^a(\alpha)$ and $\chi(-\alpha)$ are the indicators of the sets $V(\Omega_1), V(\Omega_2^a) = \Omega_2^a$ and $V(\Delta^- \cup V(\Delta^+)) = V(\Delta^-) \cup \Delta^+$, and $\chi(\alpha) + \chi(-\alpha)$ is the indicator of the set $\Delta^- \cup \Delta^+ \cup V(\Delta^-) \cup V(\Delta^+)$ which contains Ω_2^a. Therefore

$$\chi_2^a(\alpha) = \chi_2^a(\alpha)(\chi(\alpha) + \chi(-\alpha))\,. \tag{2.21}$$

Lemma 2.3. *We have, almost everywhere on the segment* $[-\pi, \pi]$ *(with respect to Lebesgue measure), the equality*

$$\nu(\alpha) = \Big(1 - \frac{\alpha}{|\alpha|}\Big)\frac{\pi}{2} + (\varphi(\alpha) + \delta(\alpha)) - (\varphi(-\alpha) + \delta(-\alpha)), \tag{2.22}$$

where

$$\varphi(\alpha) = \Big(\chi_1(\alpha) + \frac{1}{2}\chi_2^a(\alpha)\Big)\frac{\alpha}{|\alpha|}(\nu(|\alpha|) - \chi(|\alpha|)\pi)\,, \tag{2.22$'$}$$

$$\delta(\alpha) = \Big(1 + \frac{\alpha}{|\alpha|}\Big)\chi(|\alpha|)\frac{\pi}{2}\,. \tag{2.22$''$}$$

P r o o f. By definition of the sets $\Delta^-, \Delta^+, \Omega_1$, we have the equality

$$\chi_1(\alpha) + \chi_1(-\alpha) + \chi(\alpha) + \chi(-\alpha) = 1 \tag{2.23}$$

for all $\alpha \in (-\pi, \pi)$, which do not belong to the union of the finite set Ω_2^s or the set of numbers $\pm\varphi_k$, which is at most countable. Hence almost everywhere

$$\begin{aligned}\nu(\alpha) &= (\chi_1(\alpha) + \chi_1(-\alpha))\nu(\alpha) + (\chi(\alpha) + \chi(-\alpha))\nu(\alpha)\\ &= (\chi_1(\alpha) + \chi_1(-\alpha))\nu(\alpha) + \chi(\alpha)(1 - \chi_2^a(\alpha))\nu(\alpha)\\ &\quad + \chi(-\alpha)(1 - \chi_2^a(\alpha))\nu(\alpha) + \chi_2^a(\alpha)(\chi(\alpha) + \chi(-\alpha))\nu(\alpha).\end{aligned}$$

Since $\chi(\alpha)(1-\chi_2^a(\alpha))$ (resp. $\chi(-\alpha)(1-\chi_2^a(\alpha))$) is the indicator of the set $(\Delta^- \bigcup V(\Delta^+))\backslash\Omega_2^a$ (resp. $(V(\Delta^-)\bigcup \Delta^+)\backslash\Omega_2^a$) on which $\nu(\alpha)=\pi$ (resp. $\nu(\alpha)=0$), according to (2.19), then

$$\begin{aligned}\chi(\alpha)(1-\chi_2^a(\alpha))\nu(\alpha) &= \chi(\alpha)(1-\chi_2^a(\alpha))\pi\,,\\ \chi(-\alpha)(1-\chi_2^a(\alpha))\nu(\alpha) &= 0.\end{aligned}$$

From these equalities, equality (2.21), and the preceding arguments, it follows that we have, almost everywhere

$$\nu(\alpha)=(\chi_1(\alpha)+\chi_1(-\alpha))\nu(\alpha)+\chi(\alpha)(1-\chi_2^a(\alpha))\pi+\chi_2^a(\alpha)\nu(\alpha). \tag{2.24}$$

Further, from the identity

$$f(\alpha)=\frac{\alpha}{|\alpha|}f(|\alpha|)+\Big(1-\frac{\alpha}{|\alpha|}\Big)\frac{f(\alpha)+f(-\alpha)}{2}$$

which is true for all functions on the interval $(-\pi,\pi)$, and from equality (2.20), it follows that

$$\begin{aligned}\chi(\alpha) &= \frac{\alpha}{|\alpha|}\chi(|\alpha|)+\Big(1-\frac{\alpha}{|\alpha|}\Big)\frac{\chi(\alpha)+\chi(-\alpha)}{2}\,,\\ \nu(\alpha) &= \frac{\alpha}{|\alpha|}\nu(|\alpha|)+\Big(1-\frac{\alpha}{|\alpha|}\Big)\frac{\pi}{2}.\end{aligned}$$

Substituting these expressions in (2.24) and taking into account (2.21), (2.23), we obtain the equality

$$\begin{aligned}\nu(\alpha) &= (\chi_1(\alpha)+\chi_1(-\alpha))\frac{\alpha}{|\alpha|}\nu(|\alpha|)+\chi_2^a(\alpha)\frac{\alpha}{|\alpha|}(\nu(|\alpha|)\\ &\quad -\chi(|\alpha|)\pi)+\chi(|\alpha|)\frac{\alpha}{|\alpha|}\pi+\Big(1-\frac{\alpha}{|\alpha|}\Big)\frac{\pi}{2}\end{aligned}$$

which is equivalent to (2.22) because

$$\begin{aligned}\varphi(\alpha)-\varphi(-\alpha) &= (\chi_1(\alpha)+\chi_1(-\alpha)+\chi_2(\alpha))\frac{\alpha}{|\alpha|}(\nu(|\alpha|)-\chi(|\alpha|)\pi),\\ \delta(\alpha)-\delta(-\alpha) &= \chi(|\alpha|)\frac{\alpha}{|\alpha|}\pi\,,\end{aligned}$$

and

$$(\chi_1(\alpha)+\chi_1(-\alpha))\chi(|\alpha|)=0$$

(the sets $\Omega_1\cup V(\Omega_1)$ and $\Delta\cup V(\Delta)$ are disjoint). ■

Let us denote by β (resp. α_0) the smallest (resp. greatest) positive number of the compact set $\Omega_1\cup V(\Omega_1)\cup\Omega_2^s$. Since the support Ω does not contain the points $-\pi, \pi, 0$, then $0 < \beta \le \alpha_0 < \pi$ and the intervals $(0, \beta)$ (resp. (α_0, π)) are contained in Δ. Therefore

$$\Delta = (0, \beta) \cup (\alpha_0, \pi) \cup (\cup_k (\alpha_k, \beta_k)),$$

where $0 < \beta \le \alpha_k < \beta_k \le \alpha_0 < \pi$. Let us note now that in a neighborhoods of the points $0, \pi$ the function $N^+(\alpha)$ is continuous, monotonously increasing, and $N^+(0) = N^+(\pi) = 0$. It follows that in right neighborhoods of 0 the function $N^+(\alpha)$ is positive and in the left neighborhood of π it is negative so according to Lemma 2.2 the interval $(0, \beta)$ is contained in Δ^+ and the interval (α_0, π) in Δ^-:

$$\begin{aligned} \Delta^- &= (\alpha_0, \pi) \cup_k (\alpha_k, \varphi_k); \qquad 0 < \beta \le \alpha_k < \varphi_k \le \beta_k \le \alpha_0 < \pi\,; \\ \Delta^+ &= (0, \beta) \cup_k (\varphi_k, \beta_k); \qquad 0 < \beta \le \alpha_k \le \varphi_k < \beta_k \le \alpha_0 < \pi\,. \end{aligned}$$

It follows from the definition of Δ that the endpoints of the intervals (α_k, β_k) belong to one of the three disjoint sets $\Omega_1, V(\Omega_1), \Omega_2^s$. And moreover, the number of the intervals whose endpoint belong to different sets is finite. In fact, if the endpoints of the interval (α_k, β_k) belong to different sets then

$$\beta_k - \alpha_k \ge \min\{\mathrm{dist}(\Omega_1, V(\Omega_1)), \mathrm{dist}(\Omega_1, \Omega_2^s)\}$$

and if there were infinitely many such intervals then some of them would have arbitrary small length so that one of the distances $\mathrm{dist}(\Omega_1, V(\Omega_1))$, $\mathrm{dist}(\Omega_1, \Omega_2^s)$ would vanish, in contradiction with condition A). In the same way, the set Ω_2^a is contained in a finite union of intervals (α_k, β_k) because the distances $\mathrm{dist}(\Omega_1, \Omega_2^a)$, $\mathrm{dist}(\Omega_2^s, \Omega_2^a)$ are positive.

To make a factorization of $N(z)$ with the properties we will need later on, we must choose one number α_k^* (resp. φ_k^*) in each pair of $(\alpha_k, -\alpha_k)$ (resp. $(\varphi_k, -\varphi_k)$), according to the following rule: let

$$\alpha_k^* = \begin{cases} \alpha_k & \text{if } \alpha_k \notin V(\Omega_1)\,; \\ -\alpha_k & \text{if } \alpha_k \in V(\Omega_1)\,. \end{cases} \tag{2.25}$$

If both endpoints of an interval (α_k, β_k) belong to the set Ω_1 or $V(\Omega_1)$, we set

$$\varphi_k^* = \begin{cases} \varphi_k & \text{if } \alpha_k \text{ and } \beta_k \text{ belong to } \Omega_1; \\ -\varphi_k & \text{if } \alpha_k \text{ and } \beta_k \text{ belong to } V(\Omega_1); \end{cases} \tag{2.26}$$

and in all other cases (there are only finitely many) we set

$$\varphi_k^* = \begin{cases} \varphi_k & \text{if } \varphi_k \notin V(\Omega_1); \\ -\varphi_k & \text{if } \varphi_k \in V(\Omega_1). \end{cases} \tag{2.26$'$}$$

Let us denote by $\Phi = \{\varphi_k^*\}$ and $A = \{\alpha_k^*\}$ the sets of all φ_k^* and of all α_k^*. It is essential that with this choice of the φ_k^* and α_k^* the sets

$$\Omega_1 \cup \Phi, \quad V(\Omega_1) \cup V(\Phi), \quad \Omega_2^s, \quad \Omega_2^a$$

be at positive distance from one another.

Lemma 2.4 (factorization). *The function* $-i(z - z^{-1})N(z)^{-1}$ *can be factored out as follows:*

$$-i(z - z^{-1})N(z)^{-1} = R(z)R(z^{-1}), \tag{2.27}$$

$$R(z) = R_0(z)R_1(z)R_2(z), \tag{2.28}$$

where

$$R_0(z) = ie^{\frac{i}{2}\alpha_0^*}(z^{-1} - e^{-i\alpha_0^*}) \prod_k \Big(\frac{z - e^{i\alpha_k^*}}{z - e^{i\varphi_k^*}}\Big) e^{\frac{i}{2}(\varphi_k^* - \alpha_k^*)}, \tag{2.29}$$

$$R_1(z) = P(z, -\varphi_1), \qquad R_2(z) = P(z, -\varphi_2)\,, \tag{2.30}$$

$$\varphi_1(\alpha) = \chi_1(\alpha)\varphi(\alpha) = \chi_1(\alpha)\frac{\alpha}{|\alpha|}\nu(|\alpha|)\,, \tag{2.31}$$

$$\varphi_2(\alpha) = \chi_2^a(\alpha)\varphi(\alpha) = \frac{1}{2}\chi_2^a(\alpha)\frac{\alpha}{|\alpha|}(\nu(|\alpha|) - \chi(|\alpha|)\pi)\,. \tag{2.32}$$

and the numbers α_k^*, φ_k^* *are defined by equations* (2.25), (2.26), (2.26′).

P r o o f. It follows from the formula (2.9), Lemma 2.3, and the equalities (2.4) that

$$\begin{aligned} N(z)^{-1} &= P(z,\nu)^{-1} = P(z,-\nu) \\ &= P(z, -\varphi - \delta + V(\varphi) + V(\delta)) \exp\Big\{\frac{1}{2i}\int_{-\pi}^{0} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\alpha\Big\}, \end{aligned}$$

$$P(z, -\varphi - \delta + V(\varphi) + V(\delta)) = P(z, -\varphi)P(z^{-1}, -\varphi)P(z, -\delta)P(z^{-1}, -\delta),$$

also since we have

$$\frac{1}{2i}\int_{\gamma_1}^{\gamma_2} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\alpha = \frac{i}{2}(\gamma_2 - \gamma_1) + \ln\left(\frac{z - e^{i\gamma_1}}{z - e^{i\gamma_2}}\right),$$

we get

$$\exp\Big\{\frac{1}{2i}\int_{-\pi}^{0} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\alpha\Big\} = i\frac{z+1}{z-1}$$

and

$$N(z)^{-1} = i\frac{z+1}{z-1}P(z,-\varphi)P(z^{-1},-\varphi)P(z,-\delta)P(z^{-1},-\delta). \qquad (2.33)$$

It follows from (2.22′) that $\varphi(\alpha) = \varphi_1(\alpha) + \varphi_2(\alpha)$ if the functions $\varphi_1(\alpha)$, $\varphi_2(\alpha)$ are defined by the equalities (2.31), (2.32). Therefore

$$P(z,-\varphi) = P(z,-\varphi_1-\varphi_2) = P(z,-\varphi_1)P(z,-\varphi_2)\,,$$

that is,

$$P(z,-\varphi) = R_1(z)R_2(z)\,. \qquad (2.34)$$

Further, according to (2.22″), we have

$$P(z,-\delta) = \exp\left\{\frac{1}{2i}\int_0^\pi \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\chi(\alpha)d\alpha\right\},$$

where $\chi(\alpha)$ is the indicator of the set $\Delta^- = (\alpha_0,\pi)\cup_k(\alpha_k,\varphi_k)$. Hence

$$\begin{aligned}\frac{1}{2i}\int_0^\pi \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\chi(\alpha)d\alpha &= \frac{1}{2i}\int_{\alpha_0}^\pi \frac{e^{i\alpha}+z}{e^{i\alpha}-z}d\alpha + \sum_k \frac{1}{2i}\int_{\alpha_k}^{\varphi_k} \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\chi(\alpha)d\alpha \\ &= \frac{i}{2}(\pi-\alpha_0) + \ln\frac{z-e^{i\alpha_0}}{z+1} \\ &\quad + \sum_k\left\{\frac{i}{2}(\varphi_k-\alpha_k) + \ln\frac{z-e^{i\alpha_k}}{z-e^{i\varphi_k}}\right\},\end{aligned}$$

$$P(z,-\delta) = e^{\frac{i}{2}(\pi-\alpha_0)}\frac{z-e^{i\alpha_0}}{z+1}\prod_k e^{\frac{i}{2}(\varphi_k-\alpha_k)}\left(\frac{z-e^{i\alpha_k}}{z-e^{i\varphi_k}}\right),$$

and

$$\begin{aligned}&P(z,-\delta)P(z^{-1},-\delta) \\ &= e^{i(\pi-\alpha_0)}\left(\frac{z-e^{i\alpha_0}}{z+1}\right)\left(\frac{z^{-1}-e^{i\alpha_0}}{z^{-1}+1}\right)\prod_k e^{\frac{i}{2}(\varphi_k-\alpha_k)}\left(\frac{z-e^{i\alpha_k}}{z-e^{i\varphi_k}}\right)\left(\frac{z^{-1}-e^{i\alpha_k}}{z^{-1}-e^{i\varphi_k}}\right) \\ &= \left(\frac{z-e^{i\alpha_0}}{z+1}\right)\left(\frac{z-e^{-i\alpha_0}}{z+1}\right)\prod_k\left(\frac{z-e^{i\alpha_k}}{z-e^{i\varphi_k}}\right)\left(\frac{z-e^{-i\alpha_k}}{z-e^{-i\varphi_k}}\right).\end{aligned}$$

It is obvious that the right-hand side of the last equality will not change if we replace α_k, φ_k by α_k^*, φ_k^*. Therefore we have

$$P(z,-\delta)P(z^{-1},-\delta) = \left(\frac{z-e^{i\alpha_0^*}}{z+1}\right)\left(\frac{z-e^{-i\alpha_0^*}}{z+1}\right)\prod_k\left(\frac{z-e^{i\alpha_k^*}}{z-e^{i\varphi_k^*}}\right)\left(\frac{z-e^{-i\alpha_k^*}}{z-e^{-i\varphi_k^*}}\right)$$

$$= \frac{1}{(z^{-1}+1)(z+1)}\Big\{ie^{\frac{i}{2}\alpha_0^*}(z^{-1}-e^{-i\alpha_0^*})\prod_k e^{\frac{i}{2}(\varphi_k^*-\alpha_k^*)}\Big(\frac{z-e^{i\alpha_k^*}}{z-e^{i\varphi_k^*}}\Big)\Big\}$$
$$\times\Big\{ie^{\frac{i}{2}\alpha_0^*}(z-e^{-i\alpha_0^*})\prod_k e^{\frac{i}{2}(\varphi_k^*-\alpha_k^*)}\Big(\frac{z^{-1}-e^{i\alpha_k^*}}{z^{-1}-e^{i\varphi_k^*}}\Big)\Big\},$$

that is,

$$P(z,-\delta)P(z^{-1},-\delta) = \frac{R_0(z)R_0(z^{-1})}{(z^{-1}+1)(z+1)}, \tag{2.35}$$

where the function $R_0(z)$ is defined by equality (2.29). Substituting the right-hand sides of (2.34), (2.35) in formula (2.33), we obtain the factorization (2.27), (2.28). ■

Let us denote by $\tilde{A}$ the image of the set A by the map $\alpha \mapsto e^{i\alpha}$:

$$\tilde{A} = \{z \mid z = e^{i\alpha}, \alpha \in A\}.$$

It follows from this and (2.29)–(2.32) that the function $R(z)$ is analytic outside of $\tilde{\Omega}_1 \cup \tilde{\Phi} \cup \tilde{\Omega}_2^a \subset T$ and of the origin. It tends to the finite limit $R(\infty) = -e^{i\theta}$ when $|z| \to \infty$, and has a simple pole with residue $R_{-1}(0) = \overline{R(\infty)} = -e^{-i\theta}$, at the origin, with

$$\theta = \frac{1}{2}\Big\{\pi - \alpha_0^* + \sum_k(\varphi_k^* - \alpha_k^*) + \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\alpha)d\alpha\Big\}. \tag{2.36}$$

According to (1.13), (1.14) it follows from this that the vector-function

$$\begin{aligned} g(z,x) &= e^{\frac{i}{2}k(z)x}R(z)\Phi(z,x) \\ &= e^{\frac{i}{2}k(z)x}R(z)\Big\{F\Big(\frac{z+z^{-1}}{2},x\Big) + n(z)G\Big(\frac{z+z^{-1}}{2},x\Big)\Big\} \end{aligned} \tag{2.37}$$

satisfies the equality

$$g(z,x) = -2(e^{i\theta}, ie^{-i\theta}z^{-1})^T + \frac{1}{2\pi i}\int_\Gamma \frac{g(\xi,x)}{\xi - z}d\xi, \tag{2.38}$$

where Γ is a contour formed by two concentric circles bounding an annulus containing the unit circle T. In this annulus the functions $e^{\frac{i}{2}k(z)x}$, $F(\frac{z+z^{-1}}{2},x)$, $G(\frac{z+z^{-1}}{2},x)$ are analytic, the singularities of the function $n(z)$ lie in the set $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2^s \cup \tilde{\Omega}_2^a$, and the singularities of the function $R(z)$ lie in the set $\tilde{\Omega}_1 \cup \tilde{\Phi} \cup \tilde{\Omega}_2^a$. Hence all the singularities of the vector-function $g(z,x)$ in this annulus lie in the union of the three sets $\tilde{\Omega}_1 \cup \tilde{\Phi}$, $\tilde{\Omega}_2^s$, $\tilde{\Omega}_2^a$, which lie at positive distances one from other and from the set $\widetilde{V(\Omega_1)} \cup \widetilde{V\Phi}$. This makes it possible to deform the contour Γ into a finite system of simple closed contours Γ_1, Γ_2^s, Γ_2^a which enclose the sets $\tilde{\Omega}_1 \cup \tilde{\Phi}$, $\tilde{\Omega}_2^s$, $\tilde{\Omega}_2^a$, and which are so close to them that their interiors lie at positive distances from one another and from the set $V\widetilde{(\Omega_1)} \cup \widetilde{V\Phi}$. Thus formula (2.38) is reduced to

$$\begin{aligned} g(z,x) &= -2(e^{i\theta}, ie^{-i\theta}z^{-1})^T \\ &+\frac{1}{2\pi i}\Big\{\int\limits_{\Gamma_1}\frac{g(\xi,x)}{\xi-z}d\xi+\int\limits_{\Gamma_2^s}\frac{g(\xi,x)}{\xi-z}d\xi+\int\limits_{\Gamma_2^a}\frac{g(\xi,x)}{\xi-z}d\xi\Big\}. \end{aligned} \tag{2.39}$$

3. Deduction of the fundamental equation for the inverse problem

Let us first calculate the integrals in the right-hand side of (2.39). We denote by O_1, O_2^s, O_2^a the domains which lie inside the contours Γ_1, Γ_2^s, Γ_2^a, which are close to the sets $\widetilde{\Omega}_1 \cup \widetilde{\Phi}$, $\widetilde{\Omega}_2^s$, $\widetilde{\Omega}_2^a$, so that the distances between any two of them and between any one of them and the set $V(\widetilde{\Phi \cup \Omega_1})$ are positive.

Calculation of the integrals along the contours belonging to Γ_1

According to (2.37) we have

$$g(\xi,x)=g(\xi^{-1},x)e^{ik(\xi)x}R(\xi^{-1})^{-1}R(\xi)+2G\Big(\frac{\xi+\xi^{-1}}{2},x\Big)e^{\frac{ik(\xi)x}{2}}R(\xi)N(\xi)$$

and from the factorization $R(\xi)R(\xi^{-1})N(\xi) = -i(\xi-\xi^{-1})$ obtained in Lemma 2.4 we get

$$g(\xi,x)=f_1(\xi,x)(-N(\xi))^{-1}+f_2(\xi,x)\,,$$

where

$$\begin{aligned} f_1(\xi,x) &= ig(\xi^{-1},x)e^{ik(\xi)x}R(\xi^{-1})^{-2}(\xi-\xi^{-1})\,, \\ f_2(\xi,x) &= -2iG\Big(\frac{\xi+\xi^{-1}}{2},x\Big)e^{\frac{ik(\xi)x}{2}}R(\xi^{-1})^{-1}(\xi-\xi^{-1}). \end{aligned} \tag{3.1}$$

The functions $R(\xi^{-1})$, $R(\xi^{-1})^{-1}$, $n(\xi^{-1})$ so as the vector-functions $g(\xi^{-1},x)$, $f_1(\xi,x)$, $f_2(\xi,x)$ are holomorphic outside of the compact set

$$V(\widetilde{\Phi\cup\Omega_1})\cup\tilde{\Omega}_2.$$

In particular, all these functions are holomorphic in the domain O_1 and according to the Cauchy theorem we have

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{g(\xi,x)}{\xi-z}d\xi=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f_1(\xi,x)}{\xi-z}(-N(\xi))^{-1}d\xi\,,$$

if $z \notin O_1$. As the function $(-N(\xi))^{-1}$ satisfies the conditions of the F. Riesz–Herglotz theorem and it is continuous and real on the set $T\backslash\{\tilde{\Phi} \cup \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \widetilde{V(\Phi \cup \Omega_1)}\}$ then

$$(-N(\xi))^{-1} = i \int_{-\pi}^{\pi} \frac{e^{i\alpha} + \xi}{e^{i\alpha} - \xi} d\rho(\alpha) \tag{3.2}$$

and the support of the probability measure $d\rho(\alpha)$ is contained in the union of the sets $\Phi \cup \Omega_1$ and $\Omega_2 \cup V(\Phi \cup \Omega_1)$ which lie at positive distance from one another. Therefore we have

$$(-N(\xi))^{-1} = i \int_{\Phi\cup\Omega_1} \frac{e^{i\alpha} + \xi}{e^{i\alpha} - \xi} d\rho(\alpha) + N_1(\xi)\,,$$

where the function

$$N_1(\xi) = \int_{V(\Phi\cup\Omega_1)\cup\Omega_2} i\frac{e^{i\alpha} + \xi}{e^{i\alpha} - \xi} d\rho(\alpha)$$

is holomorphic in the domain O_1. Because the function $f_1(\xi, x)(\xi - z)^{-1}$ is also holomorphic in this domain when $z \notin O_1$, we have

$$\int_{\Gamma_1} \frac{f_1(\xi, x) N_1(\xi)}{\xi - z} d\xi = 0$$

and

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\xi, x)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_1(\xi, x)}{\xi - z} \left\{ i \int_{\Phi\cup\Omega_1} \frac{e^{i\alpha} + \xi}{e^{i\alpha} - \xi} d\rho(\alpha) \right\} d\xi.$$

The family of contours Γ_1 enclose the set $\Phi \cup \Omega_1$ and are at positive distance from it, which allows to change the order of integration. If we do this and use theorem of residues to calculate the inner integral we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\xi, x)}{\xi - z} d\xi = i \int_{\Phi\cup\Omega_1} \frac{2e^{i\alpha} f_1(e^{i\alpha}, x)}{e^{i\alpha} - z} d\rho(\alpha).$$

(Here we remind that the contour Γ_1 is oriented clockwise.) According to (3.1) it follows from this that

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\xi, x)}{\xi - z} d\xi = -4i \int_{\Phi\cup\Omega_1} \frac{e^{i\alpha} \sin\alpha}{e^{i\alpha} - z} R^+(-\alpha)^{-2} e^{-x\sin\alpha} g^+(-\alpha, x) d\rho(\alpha), \tag{3.3}$$

where

$$\begin{aligned} R^+(-\alpha) &= R^-(-\alpha) = R(e^{-i\alpha})\,, \\ g^+(-\alpha,x) &= g^-(-\alpha,x) = g(e^{-i\alpha},x) \end{aligned}$$

on the set $\Phi \cup \Omega_1$ on which integration is performed.

Calculation of the integrals along the contours belonging to Γ_2^s

According to condition A) the part Ω_2^s of the support of the measure $d\mu(\alpha)$ consists of a finite number of pairs of points $(\alpha_l, -\alpha_l)$ at which positive masses $\mu(\alpha_l)$, $\mu(-\alpha_l)$ are concentrated. Let us clarify the behavior of the function $g(\xi,x)$ near the points $\xi_l = e^{i\alpha_l}$ and $\xi_l^{-1} = e^{-i\alpha_l}$. It follows From (1.6), (2.2) that

$$n(\xi) = i\Big\{\frac{\xi_l+\xi}{\xi_l-\xi}\mu(\alpha_l) + \frac{\xi_l^{-1}+\xi}{\xi_l^{-1}-\xi}\mu(-\alpha_l)\Big\} + n_l(\xi)\,, \tag{3.4}$$

$$N(\xi) = i\Big\{\frac{\xi_l+\xi}{\xi_l-\xi} + \frac{\xi_l^{-1}+\xi}{\xi_l^{-1}-\xi}\Big\}\frac{\mu(\alpha_l)+\mu(-\alpha_l)}{2} + N_l(\xi)\,, \tag{3.5}$$

where the functions $n_l(\xi), N_l(\xi)$ are holomorphic near the points ξ_l and ξ_l^{-1}. According to (2.25) and to Lemma 2.4, we have $\alpha_l^* = \alpha_l$, the function $R(\xi)$ is holomorphic near the points ξ_l, ξ_l^{-1} and $R(\xi_l) = 0$, $R(\xi_l^{-1}) \neq 0$. Hence the two functions $R(\xi)$, $R(\xi)n(\xi)$ are holomorphic near the point ξ_l, and so is the vector-function $g(\xi,x)$. The function $R(\xi)$ is holomorphic in a neighborhood of the point ξ_l^{-1} and the function $R(\xi)n(\xi)$ has a simple pole at ξ_l^{-1} with residue $-2i\xi_l^{-1}\mu(-\alpha_l)R(\xi_l^{-1})$. Thus if the contours Γ_l, Γ_l^* bounding the set Ω_2^s enclose the points ξ_l, ξ_l^{-1} and are close enough to them, we have

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{\Gamma_l}\frac{g(\xi,x)}{\xi-z}d\xi &= 0\,, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_l^*}\frac{g(\xi,x)}{\xi-z}d\xi &= \frac{2i\xi_l^{-1}\mu(-\alpha_l)R(\xi_l^{-1})e^{\frac{ik(\xi_l^{-1})x}{2}}G(\frac{\xi_l+\xi_l^{-1}}{2},x)}{\xi_l^{-1}-z}\,. \end{aligned}$$

Hence

$$\frac{1}{2\pi i}\int\limits_{\Gamma_2^s}\frac{g(\xi,x)}{\xi-z}d\xi = \sum_{\alpha_l\in\Omega_2^s}\frac{2i\xi_l^{-1}R(\xi_l^{-1})e^{\frac{ik(\xi_l^{-1})x}{2}}}{\xi_l^{-1}-z}\mu(-\alpha_l)G\Big(\frac{\xi_l+\xi_l^{-1}}{2},x\Big). \tag{3.6}$$

Since $R(\xi_l) = 0$, and according to (2.37), (3.4) we get

$$g(\xi_l,x) = 2i\xi_l\mu(\alpha_l)e^{\frac{ik(\xi_l)x}{2}}G\Big(\frac{\xi_l+\xi_l^{-1}}{2},x\Big)\lim_{\xi\to\xi_l}\{(\xi_l-\xi)^{-1}R(\xi)\},$$

and it follows from Lemma 2.4 and decomposition (3.5) that

$$\begin{aligned}\lim_{\xi\to\xi_l} R(\xi)N(\xi) &= \frac{-i(\xi_l-\xi_l^{-1})}{R(\xi_l^{-1})},\\ \lim_{\xi\to\xi_l} R(\xi)N(\xi) &= i\xi_l(\mu(\alpha_l)+\mu(-\alpha_l))\lim_{\xi\to\xi_l}\{(\xi_l-\xi)^{-1}R(\xi)\}.\end{aligned}$$

Thus

$$\lim_{\xi\to\xi_l}(\xi_l-\xi)^{-1}R(\xi) = -\frac{(\xi_l-\xi_l^{-1})}{\xi_l R(\xi_l^{-1})}(\mu(\alpha_l)+\mu(-\alpha_l))^{-1},$$

$$g(\xi_l,x) = -2i\Big(\frac{\mu(\alpha_l)}{\mu(\alpha_l)+\mu(-\alpha_l)}\Big)\Big(\frac{\xi_l-\xi_l^{-1}}{R(\xi_l^{-1})}\Big)e^{\frac{ik(\xi_l)x}{2}}G\Big(\frac{\xi_l+\xi_l^{-1}}{2},x\Big),$$

and together with the equality $k(\xi_l)=-k(\xi_l^{-1})$ this implies that

$$G\Big(\frac{\xi_l+\xi_l^{-1}}{2},x\Big) = i\frac{(\mu(\alpha_l)+\mu(-\alpha_l))R(\xi_l^{-1})}{\mu(\alpha_l)(\xi_l-\xi_l^{-1})}e^{\frac{ik(\xi_l^{-1})x}{2}}g(\xi_l,x).$$

Substituting this in the right-hand part of (3.6), we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\Gamma_2^s}\frac{g(\xi,x)}{\xi-z}d\xi\\ &= -\sum_{\alpha_l\in\Omega_2^s}\frac{\xi_l^{-1}R(\xi_l^{-1})^2e^{ik(\xi_l^{-1})x}}{(\xi_l-\xi_l^{-1})(\xi_l^{-1}-z)}\frac{\mu(\alpha_l)}{\mu(-\alpha_l)}(\mu(\alpha_l)+\mu(-\alpha_l))g(\xi_l,x).\end{aligned}$$

Let $d\rho_1(\alpha)$ be the measure defined on Ω_2^s by

$$\rho_1(\alpha_l)=0,\quad \rho_1(-\alpha_l)=\frac{\mu(\alpha_l)}{\mu(-\alpha_l)}(\mu(\alpha_l)+\mu(-\alpha_l))$$

the last formula can be rewritten as follows:

$$\frac{1}{2\pi i}\int_{\Gamma_2^s}\frac{g(\xi,x)}{\xi-z}d\xi = -\frac{i}{2}\int_{\Omega_2^s}\frac{e^{i\alpha}R^+(\alpha)^2e^{-x\sin\alpha}}{(e^{i\alpha}-z)\sin\alpha}g^+(-\alpha,x)d\rho_1(\alpha), \qquad (3.7)$$

where the integration is actually performed along the set $(-\pi,0)\cap\Omega_2^s$ on which

$$R^+(\alpha)=R^-(\alpha)=R(e^{i\alpha}),\quad g^+(-\alpha,x)=g^-(-\alpha,x)=g(e^{-i\alpha},x).$$

It follows from the equalities (2.28)–(2.32) defining the function $R(z)$ that we have

$$R^\pm(\alpha) = 2\sin\frac{\alpha-\alpha_0^*}{2}\prod_k\Big(\frac{\sin\frac{\alpha-\alpha_k^*}{2}}{\sin\frac{\alpha-\varphi_k^*}{2}}\Big)|R_1(\alpha)||R_2(\alpha)|e^{-i\{\frac{\alpha}{2}\pm\varphi_1(\alpha)\pm\varphi_2(\alpha)\}},$$

and $\varphi_1(-\alpha) = \varphi_2(-\alpha) = 0$ if $\alpha \in \Phi \cup \Omega_1$, $\varphi_1(\alpha) = \varphi_2(\alpha) = 0$ if $\alpha \in \Omega_2^s$. Hence

$$R^+(-\alpha)^2 = e^{i\alpha}|R(-\alpha)|^2; \quad \alpha \in \Phi \cup \Omega_1, \tag{3.8}$$

$$R^+(\alpha)^2 = e^{i\alpha}|R(\alpha)|^2; \quad \alpha \in \Omega_2^s, \tag{3.9}$$

so equalities (3.3), (3.7) reduce to

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{g(\xi,x)}{\xi - z}d\xi = -i \int\limits_{\Phi\cup\Omega_1} \frac{\tan\frac{\alpha}{2}e^{-x\sin\alpha}}{e^{i\alpha}-z} g^+(-\alpha,x)2\left(\frac{2\cos\frac{\alpha}{2}}{|R(-\alpha)|}\right)^2 d\rho(\alpha)\,, \tag{3.10}$$

$$\frac{1}{2\pi i}\int\limits_{\Gamma_2^s} \frac{g(\xi,x)}{\xi - z}d\xi = -i \int\limits_{\Omega_2^s} \frac{\tan\frac{\alpha}{2}e^{-x\sin\alpha}}{e^{i\alpha}-z} g^+(-\alpha,x)\left(\frac{|R(\alpha)|}{2\cos\frac{\alpha}{2}}\right)^2 d\rho_1(\alpha)\,. \tag{3.11}$$

Calculation of the integrals along the contours Γ_2^a

The set $\tilde\Omega_2^a$ lies in a finite union of arcs $\tilde\Delta_k^-$, $\tilde\Delta_k^+$, $V\widetilde{(\Delta_k^-)}$, $V\widetilde{(\Delta_k^+)}$ of the unit circle T and Γ_2^a consists of a finite number of contours each of which encloses the subsets $\tilde\Delta_k^- \cap \tilde\Omega_2^a$, $\tilde\Delta_k^+ \cap \tilde\Omega_2^a, \ldots$ of $\tilde\Omega_2^a$ and lies close enough to it. The calculations of the integrals along each of these contours are completely similar and we shall examine only one of them, for example, the contour which encloses $\tilde\Delta_k^- \cap \tilde\Omega_2^a$. Let

$$\Delta_k^- = (\alpha_k, \varphi_k), \quad \overline{\alpha}_k = \inf(\Delta_k^- \cap \Omega_2^a), \quad \overline{\varphi}_k = \sup(\Delta_k^- \cap \Omega_2^a).$$

Since the compact set $\Delta_k^- \cap \Omega_2^a$ is contained in the interval Δ_k^- we have $\alpha_k < \overline{\alpha}_k < \overline{\varphi}_k < \varphi_k$ and, without changing the value of integral, the corresponding contour can be replaced by a contour $\gamma(h)$ consisting of two concentric arcs

$$\gamma^\pm(h) = \{\xi \mid \xi = (1 \pm h)e^{i\alpha},\ \alpha_k' < \alpha < \varphi_k'\}$$

and of two linear segments connecting their endpoints. Here α_k', φ_k' denote numbers chosen so that they satisfy the inequalities

$$\alpha_k < \alpha_k' < \overline{\alpha}_k, \quad \overline{\varphi}_k < \varphi_k' < \varphi_k,$$

and h is a small number. On the linear segments of the contour $\gamma(h)$ the vector-function $g(\xi,x)(\xi - z)^{-1}$ is continuous. So the integrals along these segments tend to zero when $h \to 0$, and

$$\frac{1}{2\pi i}\int\limits_{\gamma(h)} \frac{g(\xi,x)}{\xi - z}d\xi = \frac{1}{2\pi i}\lim_{h\to 0}\Bigg(\int\limits_{\gamma^+(h)\cup\gamma^-(h)} \frac{g(\xi,x)}{\xi - z}d\xi\Bigg).$$

There exists, almost everywhere on the interval Δ_k^-, a finite limit

$$\lim_{h\to 0} g\left((1\pm h)e^{i\alpha}, x\right) = g^{\pm}(\alpha, x)$$

and if the vector-function $g\left((1\pm h)e^{i\alpha}, x\right)$ converges to this in L^p-norm on the segment $[\alpha'_k, \varphi'_k]$ $(p \geq 1)$ then

$$\lim_{h\to 0} \int\limits_{\gamma^{\pm}(h)} \frac{g(\xi,x)}{\xi - z} d\xi = \int\limits_{\alpha'_k}^{\varphi'_k} \frac{\pm g^{\pm}(\alpha,x)}{e^{i\alpha} - z} de^{i\alpha}$$

and

$$\frac{1}{2\pi i} \int\limits_{\gamma(h)} \frac{g(\xi,x)}{\xi - z} d\xi = \frac{1}{2\pi} \int\limits_{\alpha'_k}^{\varphi'_k} \frac{g^+(\alpha,x) - g^-(\alpha,x)}{e^{i\alpha} - z} e^{i\alpha} d\alpha.$$

Since the vector-function $g(\xi, x)$ is holomorphic in a neighborhood of $\tilde{\Delta}_k^- \backslash \tilde{\Omega}_2^a$, we have

$$g^+(\alpha,x) - g^-(\alpha,x) = 0, \quad \alpha \in \Delta_k^- \backslash \Omega_2^a$$

so in view of the arguments above we get

$$\frac{1}{2\pi i} \int\limits_{\gamma(h)} \frac{g(\xi,x)}{\xi - z} d\xi = \frac{1}{2\pi} \int\limits_{\Delta_k^- \cap \Omega_2^a} \frac{g^+(\alpha,x) - g^-(\alpha,x)}{e^{i\alpha} - z} e^{i\alpha} d\alpha, \tag{3.12}$$

if

$$\lim_{h\to 0} \| g\left((1\mp h)e^{i\alpha}, x\right) - g^{\pm}(\alpha,x)\|_{L^p[\alpha'_k,\varphi'_k]} = 0. \tag{3.13}$$

It follows from the definition of the functions $R(\xi)$, $n(\xi)$ that we have factorizations:

$$R(\xi) = R_k^{(1)}(\xi) R_k^{(2)}(\xi), \quad n(\xi) = n_k^{(1)}(\xi) n_k^{(2)}(\xi),$$

with

$$R_k^{(1)}(\xi) = \exp\Big(\frac{1}{2\pi i} \int\limits_{\Delta_k^-} \Big(\frac{e^{i\alpha} + \xi}{e^{i\alpha} - \xi}\Big) \frac{\chi_2(\alpha)\varphi(\alpha)}{2} d\alpha\Big), \tag{3.14}$$

$$n_k^{(1)}(\xi) = \exp\Big(-\frac{1}{2\pi i} \int\limits_{\Delta_k^-} \Big(\frac{e^{i\alpha} + \xi}{e^{i\alpha} - \xi}\Big) \eta(\alpha) d\alpha\Big), \tag{3.15}$$

where the functions $R_k^{(2)}(\xi), n_k^{(2)}(\xi)$ are holomorphic near the set $\tilde{\Delta}_k^-$. Hence

$$g(\xi,x) = g_k^{(1)}(\xi,x) R_k^{(1)}(\xi) + g_k^{(2)}(\xi,x) R_k^{(1)}(\xi) n_k^{(1)}(\xi)\,,$$

where the vector-functions $g_k^{(1)}(\xi,x)$, $g_k^{(2)}(\xi,x)$ are holomorphic in a neighborhood of $\tilde{\Delta}_k^-$. So the vector-functions $g_k^{(j)}\left((1\mp h)e^{i\alpha},x\right)$ $(j=1,2)$ converge uniformly to their limits on the interval $[\alpha_k',\varphi_k']\subset\Delta_k^-$ when $h\to=0$. To prove equality (3.12) it is thus sufficient to check that the functions $R_k^{(1)}\left((1\mp h)e^{i\alpha}\right)$, $R_k^{(2)}\left((1\mp h)e^{i\alpha}\right)$, $n_k^{(1)}\left((1\mp h)e^{i\alpha}\right)$ converge to their limits in L^p norm when $h\to 0$ on this segment. This is the consequence of condition B) and of the following Lemma.

Lemma 3.1. *Let $\delta(\alpha)=\delta_1(\alpha)+\delta_2(\alpha)$ and suppose that the function $\delta_2(\alpha)$ satisfies a Hölder condition on an interval $(\alpha_1,\alpha_2)\subset(-\pi,\pi)$ and that on the same interval the function $\delta_1(\alpha)$ satisfies the inequality*

$$\omega=\operatorname{ess}\sup_{\alpha_1<\alpha<\alpha_2}\delta_1(\alpha)-\operatorname{ess}\inf_{\alpha_1<\alpha<\alpha_2}\delta(\alpha)<\pi.$$

Then for any $p<\pi\omega^{-1}$ the function $P\left((1\mp h)e^{i\alpha},\delta\right)$ converges in L^p-norm to a limit $P^{\pm}(\alpha,\delta)$ on all compact subsets of (α_1,α_2).

P r o o f. Let $J(\alpha)$ be the indicator of the interval (α_1,α_2), and let

$$\begin{aligned} d_1(\alpha) &= J(\alpha)\left(\delta_1(\alpha)-C\right),\\ d_2(\alpha) &= \delta(\alpha)-d_1(\alpha)=\left(1-J(\alpha)\right)\delta_1(\alpha)+\delta_2(\alpha)+J(\alpha)C,\end{aligned}$$

where

$$C=\frac{1}{2}\left(\operatorname{ess}\max_{\alpha_1<\alpha<\alpha_2}\delta_1(\alpha)+\operatorname{ess}\min_{\alpha_1<\alpha<\alpha_2}\delta_1(\alpha)\right).$$

Then $\delta(\alpha)=d_1(\alpha)+d_2(\alpha)$,

$$P(z,\delta)=P(z,d_1)P(z,d_2), \tag{3.16}$$

the function $d_1(\alpha)$ satisfies the inequality

$$\operatorname{ess}\max_{-\pi<\alpha<\pi}|d_1(\alpha)|=\frac{\omega}{2}<\frac{\pi}{2}, \tag{3.17}$$

and the function $d_2(\alpha)$ satisfies the Hölder condition on the interval (α_1,α_2). So the function $P(z,d_2)$ converges uniformly to its limit on each segment $[\alpha_1',\alpha_2']\subset(\alpha_1,\alpha_2)$ and it follows from inequality (3.17) that the function $P(z,d_1)$ belongs to the Hardy space H^p for any $p<\pi\omega^{-1}$ (see [4]). Hence

$$\begin{aligned} &\lim_{h\to 0}\max_{\alpha_1'<\alpha<\alpha_2'}\left|P\left((1\mp h)e^{i\alpha},d_2\right)-P^{\pm}(\alpha,d_2)\right|=0,\\ &\lim_{h\to 0}\int_{-\pi}^{\pi}\left|P\left((1\mp h)e^{i\alpha},d_1\right)-P^{\pm}(\alpha,d_1)\right|^p d\alpha=0\end{aligned}$$

and this, together with (3.16) implies (3.15). ■

In view of condition B) and Lemma 2.4 the function $\eta(\alpha)+\eta(-\alpha)$ satisfies a Hölder condition, and inequality (2.13), on the interval Δ_k^-, so it follows that the function $1-P^+(\alpha,-\eta-V(\eta))$ also satisfies a Hölder condition on this interval and that it vanishes nowhere there. Therefore the function

$$\beta(\alpha)=\arg\Big(1-P^+(\alpha,-\eta-V(\eta))\Big)-\pi$$

satisfies a Hölder condition on the interval Δ_k^-. According to (2.16) we have

$$\nu(\alpha)=\eta(\alpha)+\beta(\alpha)+\pi.$$

From the definition of the function $\varphi(\alpha)$ we get the following equalities on the interval Δ_k^-:

$$\begin{aligned}-\frac{\chi_2(\alpha)}{2}\frac{\alpha}{|\alpha|}\varphi(\alpha) &= -\frac{1}{2}\left(\nu(\alpha)-\pi\right)=-\frac{1}{2}\eta(\alpha)-\frac{1}{2}\beta(\alpha)\,,\\ \eta(\alpha)-\frac{\chi_2(\alpha)}{2}\frac{\alpha}{|\alpha|}\varphi(\alpha) &= \frac{1}{2}\eta(\alpha)-\frac{1}{2}\beta(\alpha)\,,\end{aligned}$$

and in view of (3.13), (3.14) it follows that

$$R_k^{(1)}(\xi)=P(\xi,-\delta_1+\delta_2);\quad R_k^{(1)}(\xi)n_k^{(1)}(\xi)=P(\xi,\delta_1+\delta_2) \tag{3.18}$$

with

$$\delta_1(\alpha)=\frac{J(\alpha)}{2}\eta(\alpha),\quad \delta_2(\alpha)=-\frac{J(\alpha)}{2}\beta(\alpha)\,,$$

where $J(\alpha)$ is the indicator of the interval Δ_k^-. The function $\delta_2(\alpha)$ obviously satisfies a Hölder condition on the interval Δ_k^- and according to (2.12) the functions $\pm\delta_1(\alpha)$ satisfy the inequality

$$\operatorname{ess}\max_{\alpha\in\Delta_k^-}\left(\pm\delta_1(\alpha)\right)-\operatorname{ess}\min_{\alpha\in\Delta_k^-}\left(\pm\delta_1(\alpha)\right)=\frac{\omega}{2}<\frac{\pi}{2}\,.$$

In view of Lemma 3.1 this implies that the functions (3.18) converge in L^p-norm to their limits on the segment $[\alpha_k',\varphi_k']\subset\Delta_k^-$, for all $p<2\pi\omega^{-1}$. Since $\omega<\pi$ they also converge in $L^2(\alpha_k',\varphi_k')$.

This ends the proof of equality (3.12) and of formula (3.11). Similar formulas hold for all other components of Γ_2^a so we have

$$\frac{1}{2\pi i}\int_{\Gamma_2^a}\frac{g(\xi,x)}{\xi-z}d\xi=\frac{1}{2\pi}\int_{\Omega_2^a}\frac{g^+(\alpha,x)-g^-(\alpha,x)}{e^{i\alpha}-z}e^{i\alpha}d\alpha. \tag{3.19}$$

Remark. *It follows from the proof above that the functions* $\chi_2(\alpha)g^+(\alpha,x)$ *and* $\chi_2(\alpha)g^-(\alpha,x)$ *belong to the Hilbert space* $L^2(-\pi,\pi)$.

Lemma 3.2. *The equalities*

$$\frac{\alpha}{|\alpha|}\Big(g^+(-\alpha,x)+g^-(-\alpha,x)\Big) \tag{3.20}$$

$$=\frac{2iq(\alpha)e^{x\sin\alpha+i\alpha}\left(g^+(\alpha,x)-g^-(\alpha,x)\right)+im(\alpha)\left(g^+(-\alpha,x)-g^-(-\alpha,x)\right)}{p(\alpha)},$$

with

$$p(\alpha) = \left(\sqrt{\frac{\mu'(-\alpha)}{\mu'(\alpha)}}+\sqrt{\frac{\mu'(\alpha)}{\mu'(-\alpha)}}\right)\tan|\varphi_2(\alpha)|, \tag{3.21}$$

$$q(\alpha) = \frac{|R(-\alpha)|}{|R(\alpha)|}\sqrt{\frac{\mu'(-\alpha)}{\mu'(\alpha)}}, \tag{3.22}$$

$$m(\alpha) = \left(\sqrt{\frac{\mu'(-\alpha)}{\mu'(\alpha)}}-\sqrt{\frac{\mu'(\alpha)}{\mu'(-\alpha)}}\right)\cos\chi(|\alpha|)\pi \tag{3.23}$$

hold almost everywhere on the set Ω_2^a.

P r o o f. Since the functions $k(z)$, $F(\frac{z+z^{-1}}{2},x)$, $G(\frac{z+z^{-1}}{2},x)$ are holomorphic everywhere except at 0 and ∞,the limit-value of the function $g(z,x)$ is equal to

$$g^\pm(\alpha,x)=e^{-\frac{x\sin\alpha}{2}}\left(R^\pm(\alpha)F(\cos\alpha,x)+R^\pm(\alpha)n^\pm(\alpha)G(\cos\alpha,x)\right).$$

Therefore

$$\begin{aligned} g^+(\alpha,x)-g^-(\alpha,x) &= e^{-\frac{x\sin\alpha}{2}}\times\{(R^+(\alpha)-R^-(\alpha))F(\cos\alpha,x) \\ &\quad +(R^+(\alpha)n^+(\alpha)-R^-(\alpha)n^-(\alpha))G(\cos\alpha,x)\} \end{aligned} \tag{3.24}$$

and

$$g^\pm(-\alpha,x)=e^{\frac{x\sin\alpha}{2}}\{R^\pm(-\alpha)F(\cos\alpha,x)+R^\pm(-\alpha)n^\pm(-\alpha)G(\cos\alpha,x)\}.$$

The last two equalities provide us with a system of equations for the components of the vector-functions $F(\cos\alpha,x)$, $G(\cos\alpha,x)$, with determinant

$$D(\alpha)=R^+(-\alpha)R^-(-\alpha)\Big(n^-(-\alpha)-n^+(-\alpha)\Big)\ .$$

According to (2.5), (2.8) this determinant is equal to

$$D(\alpha)=-4\pi ie^{i\alpha}|R(-\alpha)|^2\mu'(-\alpha)\neq 0$$

almost everywhere on the set Ω_2^a. So the system of equations is uniquely solvable and

$$\begin{aligned} F(\cos\alpha,x) &= D(\alpha)^{-1}e^{-\frac{x\sin\alpha}{2}}\{R^-(-\alpha)n^-(-\alpha)g^+(-\alpha,x) \\ &\quad -R^+(-\alpha)n^+(-\alpha)g^-(-\alpha,x)\}\,, \\ G(\cos\alpha,x) &= D(\alpha)^{-1}e^{-\frac{x\sin\alpha}{2}}\{-R^-(-\alpha)g^+(-\alpha,x) \\ &\quad +R^+(-\alpha)g^-(-\alpha,x)\}\,. \end{aligned}$$

Substituting these expressions in (3.24), we obtain after some elementary transformations the equality

$$\begin{aligned} g^+(\alpha,x)-g^-(\alpha,x) &= D(\alpha)^{-1}e^{-x\sin\alpha} \\ &\quad \times\{A(\alpha)g^+(-\alpha,x)+B(\alpha)g^-(-\alpha,x)\}, \end{aligned} \tag{3.25}$$

with

$$\begin{aligned} A(\alpha) &= R^-(-\alpha)R^-(\alpha)\times\Big\{(n^+(\alpha)-n^-(-\alpha))\frac{R^+(\alpha)}{R^-(\alpha)}\Big(\frac{R^-(\alpha)}{R^+(\alpha)}-1\Big) \\ &\quad -(n^+(\alpha)-n^-(\alpha))\Big\}\,, \\ B(\alpha) &= R^+(-\alpha)R^+(\alpha)\times\Big\{(n^-(\alpha)-n^+(-\alpha))\frac{R^-(\alpha)}{R^+(\alpha)}\Big(\frac{R^+(\alpha)}{R^-(\alpha)}-1\Big) \\ &\quad +(n^+(\alpha)-n^-(\alpha))\Big\} = \overline{A(\alpha)}\,. \end{aligned} \tag{3.26}$$

In view of Lemma 2.4 and formula (2.6) we have the following equalities on Ω_2^a:

$$\frac{R^-(\alpha)}{R^+(\alpha)}=e^{2i\varphi_2(\alpha)};\quad R^+(\alpha)R^-(-\alpha)=\frac{2\sin\alpha}{N^+(\alpha)}=\frac{2\sin\alpha}{|N(\alpha)|}e^{-i\nu(\alpha)},$$

from which it follows that

$$\begin{aligned} &R^-(\alpha)R^-(-\alpha)=R^+(\alpha)R^-(-\alpha)\frac{R^-(\alpha)}{R^+(\alpha)}=\frac{2\sin\alpha}{|N(\alpha)|}e^{-i(\nu(\alpha)-2\varphi_2(\alpha))}, \\ &\Big(n^+(\alpha)-n^-(-\alpha)\Big)\frac{R^+(\alpha)}{R^-(\alpha)}=2N^+(\alpha)e^{-2i\varphi_2(\alpha)}=2|N(\alpha)|e^{i(\nu(\alpha)-2\varphi_2(\alpha))}, \\ &\frac{R^-(\alpha)}{R^+(\alpha)}-1=e^{2i\varphi_2(\alpha)}-1=\sin 2\varphi_2(\alpha)(i-\tan\varphi_2(\alpha))\,, \end{aligned}$$

with

$$2\varphi_2(\alpha)=\frac{\alpha}{|\alpha|}\left(\nu(|\alpha|)-\chi(|\alpha|)\pi\right),\quad \nu(\alpha)=\frac{\alpha}{|\alpha|}\left(\nu(|\alpha|)+\Big(1-\frac{\alpha}{|\alpha|}\Big)\frac{\pi}{2}\right).$$

Since $\nu(\alpha) - 2\varphi_2(\alpha) = \left(1 - \frac{\alpha}{|\alpha|}\right)\frac{\pi}{2} + \frac{\alpha}{|\alpha|}\chi(|\alpha|)\pi$, we have

$$e^{i(\nu(\alpha)-2\varphi_2(\alpha))} = \frac{\alpha}{|\alpha|}\cos\chi(|\alpha|)\pi; \quad \sin 2\varphi_2(\alpha) = \frac{\alpha}{|\alpha|}\cos\chi(|\alpha|)\pi\sin\nu(\alpha)$$

and hence

$$\begin{aligned} R^-(\alpha)R^-(-\alpha) &= \frac{2\frac{\alpha}{|\alpha|}\sin\alpha}{|N(\alpha)|}\cos\chi(|\alpha|)\pi\,, \\ \left(n^+(\alpha) - n^-(-\alpha)\right)\frac{R^+(\alpha)}{R^-(\alpha)} &= 2|N(\alpha)|\frac{\alpha}{|\alpha|}\cos\chi(|\alpha|)\pi\,, \\ \frac{R^-(\alpha)}{R^+(\alpha)} - 1 &= \frac{\alpha}{|\alpha|}\cos\chi(|\alpha|)\pi\sin\nu(\alpha)\left(i - \tan\varphi_2(\alpha)\right). \end{aligned}$$

Therefore

$$R^-(\alpha)R^-(-\alpha) = |R(\alpha)||R(-\alpha)|\cos\chi(|\alpha|)\pi,$$

$$\begin{aligned} \left(n^+(\alpha) - n^-(-\alpha)\right)\frac{R^+(\alpha)}{R^-(\alpha)}\left(\frac{R^-(\alpha)}{R^+(\alpha)} - 1\right) &= 2|N(\alpha)|\sin\nu(\alpha)\left(i - \tan\varphi_2(\alpha)\right) \\ &= 2\left(i - \tan\varphi_2(\alpha)\right)\operatorname{Im}N^+(\alpha). \end{aligned}$$

Substituting these expressions in formula (3.26), we get

$$A(\alpha) = |R(\alpha)||R(-\alpha)|\cos\chi(|\alpha|)\pi\{2\left(i - \tan\varphi_2(\alpha)\right)\operatorname{Im}N^+(\alpha) - 2i\operatorname{Im}n^+(\alpha)\}.$$

Finally, it follows from (2.8), (2.9) that

$$\operatorname{Im}N^+(\alpha) = \pi\left(\mu'(\alpha) + \mu'(-\alpha)\right), \quad \operatorname{Im}n^+(\alpha) = 2\pi\mu'(\alpha)$$

and since

$$\tan\varphi_2(\alpha) = \frac{\alpha}{|\alpha|}\tan\frac{1}{2}(\nu(|\alpha|) - \chi(|\alpha|)\pi) = \frac{\alpha}{|\alpha|}\cos\chi(|\alpha|)\pi\tan|\varphi_2(\alpha)|$$

we get

$$\begin{aligned} A(\alpha) &= |R(\alpha)||R(-\alpha)| \times 2\pi\Big\{-\frac{\alpha}{|\alpha|}(\mu'(\alpha) + \mu'(-\alpha))\tan|\varphi_2(\alpha)| \\ &\quad + i(\mu'(-\alpha) - \mu'(\alpha))\cos\chi(|\alpha|)\Big\} \\ &= |R(\alpha)||R(-\alpha)|2\pi\sqrt{\mu'(\alpha)\mu'(-\alpha)}\Big\{-\frac{\alpha}{|\alpha|}p(\alpha) + im(\alpha)\Big\}, \end{aligned}$$

where the functions $p(\alpha)$ and $m(\alpha)$ are defined by the formulas (3.21), (3.23).

Substituting these expressions of $D(\alpha)$, $A(\alpha)$ and $B(\alpha) = \overline{A(\alpha)}$ in formula (3.25), we obtain the equality

$$\begin{aligned} 2iq(\alpha)e^{i\alpha+x\sin\alpha}(g^+(\alpha,x)-g^-(\alpha,x)) &= p(\alpha)\frac{\alpha}{|\alpha|}(g^+(-\alpha,x)+g^-(-\alpha,x)) \\ &\quad -im(\alpha)(g^+(-\alpha,x)-g^-(-\alpha,x)), \end{aligned}$$

which is equivalent to (3.20). ■

Let us denote by $\chi_1(\alpha)$, $\chi_2^s(\alpha)$, $\chi_2^a(\alpha)$ the indicators of the sets $\Phi\cup\Omega_1$, $(-\pi,0)\cap\Omega_2^s$, Ω_2^a and let us define the measure $d\sigma(\alpha)$ and the vector-function $u(\alpha,x) = (u_1(\alpha,x), u_2(\alpha,x))^T$ on the union

$$\Omega_0 = \Phi\cup\Omega_1\cup((-\pi,0)\cap\Omega_2^s)\cup\Omega_2^a$$

of these sets by the equality

$$\begin{aligned} d\sigma(\alpha) &= 2\chi_1(\alpha)\left(\frac{2\cos\frac{\alpha}{2}}{|R(-\alpha)|}\right)^2 d\rho(\alpha) \\ &\quad +\chi_2^s(\alpha)\left(\frac{|R(-\alpha)|}{2\cos\frac{\alpha}{2}}\right)^2 d\rho_1(\alpha)+\chi_2^a(\alpha)\frac{p(\alpha)\cot|\frac{\alpha}{2}|}{2\pi}d\alpha, \end{aligned} \tag{3.27}$$

$$\begin{aligned} u(\alpha,x) &= (\chi_1(\alpha)+\chi_2^s(\alpha))\tan\frac{\alpha}{2}e^{-x\sin\alpha}g(-\alpha,x) \\ &\quad +\chi_2^a(\alpha)ip(\alpha)^{-1}\tan\left|\frac{\alpha}{2}\right|e^{i\alpha}(g^+(\alpha,x)-g^-(\alpha,x)). \end{aligned} \tag{3.28}$$

According to (2.39), (3.9), (3.10), (3.19)

$$g(z,x) = -2\left(e^{i\theta}, ie^{-i\theta}z^{-1}\right)^T - i\int_{\Omega_0}\frac{u(\alpha,x)}{e^{i\alpha}-z}d\sigma(\alpha) \tag{3.29}$$

from which it follows, when $\beta\in\Omega_0$ and $z\to e^{-i\beta}$, that

$$\begin{aligned} \frac{g^+(-\beta,x)+g^-(-\beta,x)}{2} &= -2(e^{i\theta}, ie^{-i\theta}e^{i\beta})^T - i\int_{\Omega_0\setminus\Omega_2^a}\frac{u(\alpha,x)}{e^{i\alpha}-e^{-i\beta}}d\sigma(\alpha) \\ &\quad -i\,\text{v.p.}\int_{\Omega_2^a}\frac{u(\alpha,x)}{e^{i\alpha}-e^{-i\beta}}d\sigma(\alpha), \end{aligned} \tag{3.30}$$

where v.p.$\int$ denotes the principal value of the integral. Let us also denote, for the sake of brevity, the sum of the integrals in the right-hand side of this formula by v.p. $\int$ (although they are not all singular for $\beta \in \Omega_0 \backslash \Omega_2^a$). If $\beta \in \Omega_0 \backslash \Omega_2^a$ then according to (3.28)

$$\frac{g^+(-\beta,x)+g^-(-\beta,x)}{2} = g(-\beta,x)(\chi_1(\beta)+\chi_2^s(\beta))\cot\frac{\beta}{2}e^{x\sin\beta}u(\beta,x),$$

and if $\beta \in \Omega_2^a$ then according to (3.28) and to Lemma 3.2

$$\begin{aligned}\frac{g^+(-\beta,x)+g^-(-\beta,x)}{2} &= \frac{\beta}{|\beta|}\cot\left|\frac{\beta}{2}\right|e^{x\sin\beta}q(\beta)u(\beta,x)\\ &+\frac{\beta}{|\beta|}\cot\left|\frac{\beta}{2}\right|m(\beta)e^{i\beta}u(-\beta,x),\end{aligned}$$

and since $\frac{\beta}{|\beta|}\cot|\frac{\beta}{2}| = \cot\frac{\beta}{2} \quad (-\pi < \beta < \pi)$ we have

$$\frac{g^+(-\beta,x)+g^-(-\beta,x)}{2} = \cot\frac{\beta}{2}\{e^{x\sin\beta}r^2(\beta)u(\beta,x)+m(\beta)e^{i\beta}u(-\beta,x)\}.$$

Hence for all $\beta \in \Omega_0$

$$\frac{g^+(-\beta,x)+g^-(-\beta,x)}{2} = \cot\frac{\beta}{2}e^{x\sin\beta}r^2(\beta)u(\beta,x)+m(\beta)\cot\frac{\beta}{2}e^{i\beta}\chi_2^a(\beta)u(-\beta,x)\,,$$

where

$$r(\beta) = \chi_1(\beta)+\chi_2^s(\beta)+\chi_2^a(\beta).\sqrt{q(\beta)}\,. \tag{3.31}$$

In view of (3.30) it follows from this that the vector-function $u(\alpha,x)$ satisfies the equation

$$\begin{aligned}\cot\frac{\beta}{2}e^{x\sin\beta}r^2(\beta)u(\beta,x) &+ m(\beta)\cot\frac{\beta}{2}e^{i\beta}\chi_2^a(\beta)u(-\beta,x)\\ &+ i\,\text{v.p.}\int\limits_{\Omega_0}\frac{u(\alpha,x)}{e^{i\alpha}-e^{-i\beta}}d\sigma(\alpha) = -2\left(e^{i\theta}, ie^{-i\theta}e^{i\beta}\right)^T\,.\end{aligned}$$

The unique solvability of this equation will be proved in the next section. Now we note that it splits in two independent equations for the components of the vector-function $u(\alpha,x) = (u_1(\alpha,x), u_2(\alpha,x))^T$ which only differ in their right-hand sides. Therefore we may restrict to examining only one of these scalar equations, for example, the equation for the function $u_2(\alpha,x)$:

$$\begin{aligned}\cot\frac{\beta}{2}e^{x\sin\beta}r^2(\beta)u_2(\beta,x)+m(\beta)\cot\frac{\beta}{2}e^{i\beta}\chi_2^a(\beta)u_2(-\beta,x)\\ +i\,\text{v.p.}\int\limits_{\Omega_0}\frac{u_2(\alpha,x)}{e^{i\alpha}-e^{-i\beta}}d\sigma(\alpha) = -2ie^{i(\beta-\theta)}.\end{aligned} \tag{3.32}$$

Substituting the solution $u_2(\alpha, x)$ of this equation in (3.29), multiplying both sides of the resulting equality by z and letting $z \to \infty$, we obtain the equality

$$\lim_{|z|\to\infty} z g_2(z,x) = -2ie^{-i\theta} + i \int\limits_{\Omega_0} u_2(\alpha, x) d\sigma(\alpha).$$

Since $R(\infty) = -e^{i\theta}$ it follows from the last equality and (1.15) that

$$\psi(x) = \frac{e^{-2i\theta}}{2}\Big(1 - \frac{1}{2}\int\limits_{\Omega_0} e^{i\theta} u_2(\alpha, x) d\sigma(\alpha)\Big).$$

Furthermore, according to (3.32) the function

$$y(\alpha, x) = -\frac{e^{i\theta}}{2} u_2(\alpha, x) \tag{3.33}$$

is a solution of the equation

$$\cot\frac{\beta}{2} e^{x\sin\beta} r^2(\beta) y(\beta, x) + m(\beta)\cot\frac{\beta}{2} e^{i\beta}\chi_2^a(\beta) y(-\beta, x) + i\,\text{v.p.}\int\limits_{\Omega_0} \frac{y(\alpha, x)}{e^{i\alpha} - e^{-i\beta}} d\sigma(\alpha) = ie^{i\beta} \tag{3.34}$$

and

$$\psi(x) = \frac{e^{-2i\theta}}{2}\Big(1 + \int\limits_{\Omega_0} y(\beta, x) d\sigma(\beta)\Big). \tag{3.35}$$

Conclusion. *Thus for the reconstruction of the Dirac operator from its spectral data* $(1, n(z))$ *one must find the functions* $r(\beta)$, $m(\beta)$ *and the measure* $d\sigma(\alpha)$ *corresponding to the Weyl function* $n(z)$, *then solve equation* (3.34) *for all* $x \in (-\infty, \infty)$ *and define the potential according to formula* (3.35).

4. Solvability of the fundamental equation

Let us first prove that the boundedness of the functions $r(\alpha)$, $m(\alpha)$, $p(\alpha)$ and $p(\alpha)^{-1}$ is a consequence of condition B). According to (2.8), (2.6), (2.5) the equalities

$$n^+(\alpha) = |P(\alpha, \eta)| e^{i\eta(\alpha)}, \quad \operatorname{Im} n^+(\alpha) = 2\pi\mu'(\alpha),$$

are fulfilled almost everywhere, so

$$\frac{\mu'(-\alpha)}{\mu'(\alpha)} = \frac{|P(-\alpha, \eta)|}{|P(\alpha, \eta)|}\frac{\sin\eta(-\alpha)}{\sin\eta(\alpha)},$$

and since

$$P(z^{-1},\eta)=P(z,-V(\eta))=P(z,\eta)P(z,-\eta-V(\eta))$$

we have

$$|P(-\alpha,\eta)|=|P(\alpha,\eta)||P(\alpha,-\eta-V(\eta))|$$

and

$$\frac{\mu'(-\alpha)}{\mu'(\alpha)}=|P(\alpha,-(\eta+V(\eta)))|\frac{\sin\eta(-\alpha)}{\sin\eta(\alpha)}.$$

In view of condition B) we have almost everywhere on the set Ω_2^a

$$\varepsilon\le\eta(\alpha)\le\pi-\varepsilon$$

and in some neighborhood of it the function $\eta(\alpha)+V(\eta)(\alpha)=\eta(\alpha)+\eta(-\alpha)$ satisfies the Hölder condition. Hence

$$\operatorname*{ess\,max}_{\alpha\in\Omega_2^a}\left|\frac{\sin\eta(-\alpha)}{\sin\eta(\alpha)}\right|=\operatorname*{ess\,max}_{\alpha\in\Omega_2^a}\left|\frac{\sin\eta(\alpha)}{\sin\eta(-\alpha)}\right|\le\frac{1}{\sin\varepsilon},$$

and according to (2.7)

$$\sup_{\alpha\in\Omega_2^a}|P(\alpha,-(\eta+V(\eta)))|=\sup_{\alpha\in\Omega_2^a}|P(-\alpha,-(\eta+V(\eta)))|=M<\infty,$$

from which it follows that

$$\operatorname*{ess\,max}_{\alpha\in\Omega_2^a}\left(\frac{\mu'(-\alpha)}{\mu'(\alpha)}\right)=\operatorname*{ess\,max}_{\alpha\in\Omega_2^a}\left(\frac{\mu'(\alpha)}{\mu'(-\alpha)}\right)\le\frac{M}{\sin\varepsilon}<\infty. \tag{4.1}$$

From the geometric point of view it is clear that if the arguments of complex numbers lie inside some convex angle, then the argument of their sum lies inside this angle too. According to condition B) for almost all $\alpha\in\Omega_2^a$ the arguments $\eta(\alpha)$ and $\pi-\eta(-\alpha)$ of the functions $n^+(\alpha)$ and $-n^-(-\alpha)$ lie between ε and $\pi-\varepsilon$. Hence the argument $\nu(\alpha)$ of the function $N^+(\alpha)=\frac{1}{2}(n^+(\alpha)-n^-(-\alpha))$ lies in the segment $[\varepsilon,\pi-\varepsilon]$ and the function

$$|\varphi_2(\alpha)|=\frac{1}{2}|\nu(|\alpha|)-\chi(|\alpha|)\pi|$$

satisfies the inequalities

$$\frac{\varepsilon}{2}\le|\varphi_2(\alpha)|\le\frac{\pi}{2}-\frac{\varepsilon}{2}.$$

Therefore the inequalities

$$\tan\frac{\varepsilon}{2}\le\tan|\varphi_2(\alpha)|\le\cot\frac{\varepsilon}{2} \tag{4.2}$$

hold almost everywhere on Ω_2^a.

Furthermore, since $V(\varphi_2)(\alpha) = -\varphi_2(\alpha)$ we have

$$R_2(z^{-1}) = P(z^{-1}, -\varphi_2) = P(z, V(\varphi_2)) = P(z, -\varphi_2) = R_2(z)$$

and in view of (2.28) it follows that

$$\frac{|R(-\alpha)|}{|R(\alpha)|} = \frac{|R_0(-\alpha)|}{|R_0(\alpha)|}\frac{|R_1(-\alpha)|}{|R_1(\alpha)|}.$$

On the set $\tilde\Omega_2^a$ and in its neighborhood the functions $R_0(z)$ and $R_1(z)$ are holomorphic and do not vanish. So on the set Ω_2^a the functions $|R_0(\alpha)|$, $|R_1(\alpha)|$ are continuous and are separated away from zero. It follows that

$$\sup_{\alpha\in\Omega_2^a}\left|\frac{R(-\alpha)}{R(\alpha)}\right| = \sup_{\alpha\in\Omega_2^a}\left|\frac{R(\alpha)}{R(-\alpha)}\right| = M_1 < \infty. \tag{4.3}$$

The boundedness of the functions $r(\alpha)$, $m(\alpha)$, $p(\alpha)$ and $p(\alpha)^{-1}$ is the direct consequence of the estimates (4.1)–(4.3) and of the formulas (3.21)–(3.23), (3.31). From the boundedness of these functions it follows that the left-hand side of equality (3.34) defines a bounded linear operator in the Hilbert space $L^2(\Omega_0, d\sigma(\alpha))$. From the results of the previous section it follows that the components of the vector-functions $\chi_2^a(\alpha)g^{\pm}(\alpha, x)$ belong to the space $L^2(-\pi, \pi)$. In view of (3.27), (3.28), this implies that the components of the vector-function $u(\alpha, x)$ belong to $L^2(\Omega_0; d\sigma(\alpha))$. Thus the unique solvability of this equation is equivalent to the invertibility of the corresponding bounded operator which acts in the Hilbert space $L^2(\Omega_0, d\sigma(\alpha))$.

Let us reduce the equation (3.34) to a more convenient form. The function $r(\beta)$ is real and satisfies the equality $r(\beta)r(-\beta) = 1$. So the function

$$E(\beta) = r(\beta)\exp\Big\{\frac{xe^{i\beta} + \beta}{2i}\Big\} \tag{4.4}$$

satisfies the equalities

$$\overline{E(\beta)}E(\beta) = r^2(\beta)e^{x\sin\beta}, \quad \overline{E(\beta)}E(-\beta) = e^{i\beta},$$

which allows to replace equation (3.34) by the equivalent equation

$$\begin{aligned} E(\beta)y(\beta) \;&+\; m(\beta)\chi_2^a(\beta)E(-\beta)y(-\beta) \\ &+\; (\overline{E(\beta)})^{-1} i\,\text{v.p.}\int_{\Omega_0} \frac{\tan\frac{\beta}{2}}{e^{i\alpha} - e^{-i\beta}} y(\alpha)\,d\sigma(\alpha) = i\,\tan\frac{\beta}{2}E(-\beta)\,. \end{aligned} \tag{4.5}$$

Here and further on we will omit the argument x in the functions $E(\beta)$ and $y(\beta)$, for the sake of brevity.

In operator form this equation can be written

$$\Gamma(y)(\beta) = i \tan \frac{\beta}{2} E(-\beta), \tag{4.5'}$$

$$\Gamma = E + mV\chi_2^a E + (\overline{E})^{-1}\Lambda, \tag{4.6}$$

where the operator V, introduced above, and the integral operator Λ are defined by the equalities

$$V(f)(\beta) = f(-\beta); \qquad \Lambda(f)(\beta) = \text{v.p.} \int\limits_{\Omega_0} = \frac{i \tan \frac{\beta}{2}}{e^{i\alpha} - e^{-i\beta}} f(\alpha) d\sigma(\alpha), \tag{4.7}$$

and E, m, $\chi_2^a, \ldots$ denote the operators of multiplication by the functions $E(\beta)$, $m(\beta)$, $\chi_2^a(\beta), \ldots$.

Let us examine the operators of more common type

$$\Gamma = a + bV\chi_2^a a + (\overline{a})^{-1}\Lambda,$$

where $a(\beta)$, $b(\beta)$ are functions satisfying the conditions

$$\begin{aligned} &b(\beta) = -\overline{b(-\beta)}\,, \\ &\operatorname*{ess\,max}_{\beta\in\Omega_0}\{|a(\beta)| + |a(\beta)|^{-1} + |b(\beta)|\} < \infty \end{aligned} \tag{4.8}$$

and $d\sigma(\alpha)$ is a Borel measure satisfying the condition

$$\chi_2^a(\alpha) d\sigma(\alpha) = c(\alpha) d\alpha, \quad \operatorname*{ess\,max}_{\alpha\in\Omega_2^a} |c(\alpha)| < \infty\,, \quad c(\alpha) = c(-\alpha)\,. \tag{4.9}$$

Lemma 4.1. *If the conditions* (4.8), (4.9) *are fulfilled then* Γ *is a bounded invertible operator in the space* $L^2(\Omega_0; d\sigma(\alpha))$ *and*

$$\| \Gamma^{-1} \| \leq \operatorname*{ess\,max}_{\alpha\in\Omega_0} |a(\alpha)|^{-1}\,.$$

P r o o f. The boundedness of the operators a, $bV\chi_2^a$ and $(\overline{a})^{-1}$ is obvious. The boundedness of the operator Λ is the consequence of the boundedness of the function $c(\alpha)$ and of the inequality

$$\operatorname{dist}(\Omega_2^a,\, \Omega_0 \backslash \Omega_2^a) > 0.$$

To prove the invertibility of the operator Γ we write it in the form $\Gamma = Ta$, where

$$T = I + bV\chi_2^a + (\overline{a})^{-1}\Lambda a^{-1}.$$

If the operator T is invertible then so is the operator Γ, and we have $\Gamma^{-1} = a^{-1}T^{-1}$ and

$$\| \Gamma^{-1} \| \leq \| T^{-1} \| \operatorname{ess} \max_{\alpha = \in \Omega_0} |a(\alpha)|^{-1}.$$

Thus it is sufficient to prove the invertibility of the operator T and the inequality $\| T^{-1} \| \leq 1$.

The elementary identity

$$\frac{i \tan \frac{\beta}{2}}{e^{i\alpha} - e^{-i\beta}} \equiv \frac{e^{i\frac{\beta}{2}}}{4 \cos \frac{\beta}{2}} \left(1 - \frac{\sin \frac{\alpha - \beta}{2}}{\sin \frac{\alpha + \beta}{2}} \right) \frac{e^{-\frac{i\alpha}{2}}}{\cos \frac{\alpha}{2}}$$

makes it possible to write the operator $(\overline{a})^{-1}\Lambda a^{-1}$ in the form $(\overline{a})^{-1}\Lambda a^{-1} = \Lambda_1 + i\Lambda_2$, where Λ_1 and Λ_2 are integral operators with kernels

$$\Lambda_1(\beta, \alpha) = \overline{b(\beta)} b(\alpha), \qquad \Lambda_2(\beta, \alpha) = i\, \overline{b(\beta)} \frac{\sin \frac{\alpha - \beta}{2}}{\sin \frac{\alpha + \beta}{2}} b(\alpha)\,,$$

and

$$b(\alpha) = \frac{e^{-i\frac{\alpha}{2}}}{2a(\alpha) \cos \frac{\alpha}{2}}\,.$$

Since $\Lambda_1(\beta, \alpha) = \overline{\Lambda_1(\alpha, \beta)}$, $\Lambda_2(\beta, \alpha) = \overline{\Lambda_2(\alpha, \beta)}$, Λ_1 and Λ_2 are selfadjoint operators and Λ_1 is obviously non-negative. Hence

$$T = I + \Lambda_1 + i(-ibV\chi_2^a + \Lambda_2)\,,$$

where $\Lambda_1^* = \Lambda_1 \geq 0$, $\Lambda_2^* = \Lambda_2$. We now show that the operator $ibV\chi_2^a$ is also self-adjoint. We denote by (f, g) the scalar product in the space $L^2(\Omega_0, d\sigma(\alpha))$. Then

$$\begin{aligned}
(ibV\chi_2^a f, g) &= \int_{\Omega_0} ib(\alpha)\chi_2^a(\alpha) f(-\alpha) \overline{g(\alpha)} d\sigma(\alpha) \\
&= \int_{\Omega_2^a} ib(\alpha)\chi_2^a(\alpha) f(-\alpha) \overline{g(\alpha)} d\sigma(\alpha) \\
&= \int_{\Omega_2^a} ib(\alpha)\chi_2^a(\alpha) f(-\alpha) \overline{g(\alpha)} c(\alpha) d\alpha \\
&= \int_{\Omega_2^a} ib(-\alpha)\chi_2^a(\alpha) f(\alpha) \overline{g(-\alpha)} c(-\alpha) d\alpha \\
&= \int_{\Omega_2^a} f(\alpha) \overline{ib(\alpha)\chi_2^a(\alpha) g(-\alpha)} c(\alpha) d\alpha \\
&= (f, ibV\chi_2^a g),
\end{aligned}$$

from which follows the self-adjointness of the operator $ibV\chi_2^a$. It follows from the statements above that

$$T = T_1 + iT_2$$

with

$$T_1 = I + \Lambda_1 \geq I, \quad T_2 = T_2^* = -ibV\chi_2^a + \Lambda_2,$$

and from the inequality $T_1 \geq I$ we get $T_1 = A^2$, with $A = A^* \geq I$. Therefore

$$T = A(I + iA^{-1}T_2A^{-1})A$$

and since $A^{-1}T_2A^{-1}$ is a self-adjoint operator, the operator $(I + iA^{-1}T_2A^{-1})$ is invertible and $\| (I + iA^{-1}T_2A^{-1})^{-1} \| \leq 1$. Hence the operator T is invertible:

$$T^{-1} = A^{-1}(I + iA^{-1}T_2A^{-1})^{-1}A^{-1}$$

and

$$\| T^{-1} \| \leq \| A^{-1} \|^2 \| (I + iA^{-1}T_2A^{-1})^{-1} \| \leq 1.$$

■

In particular, it follows from the last lemma that the operators (4.6) are invertible and the equations (4.5) have a unique solution in the space $L^2(\Omega_0; d\sigma(\alpha))$.

The collection $\{r(\alpha), m(\alpha), d\sigma(\alpha)\}$ consisting of the functions $r(\alpha)$, $m(\alpha)$ and the measure $d\sigma(\alpha)$ is called *reduced spectral data* of the Dirac operator. The map

$$\psi(x) \mapsto n(z) \mapsto \{r(\alpha), m(\alpha), d\sigma(\alpha)\}$$

described in the previous section is the solution of the direct spectral problem, and the map

$$\{r(\alpha), m(\alpha), d\sigma(\alpha)\} \mapsto \psi(x)$$

solves the inverse spectral problem.

Summing up the results obtained in the preceeding paragraphs, we arrive at the following theorem:

Theorem 4.1. *The Dirac operators satisfying the conditions* A) *and* B) *are uniquely defined by their reduced spectral data. The equations* (4.5) *constructed according to this data have a unique solution in the space* $L^2(\Omega_0; d\sigma(\alpha))$ *for all* $x \in (-\infty, \infty)$. *To reconstruct the potential* $\psi(x)$ *according to a given spectral data it is necessary to solve equation* (4.5) *and then use formula* (3.35).

5. Cauchy problem for nonlinear Schrödinger equation

Let us consider the Cauchy problem

$$\begin{cases} i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\partial^2}{\partial x^2}\psi(x,t) + 2\left(|\psi(x,t)|^2 - \frac{1}{4}\right)\psi(x,t)\,, \\ \psi(x,0) = \psi_0(x) \end{cases} \tag{5.1}$$

with initial data $\psi_0(x)$ equal to the potential of the Dirac operator D_0 satisfying conditions A) and B). To solve this problem we find the reduced spectral data $r(\beta)$, $m(\beta)$, $d\sigma(\beta)$ of the operator D_0, and consider the equations (4.5) and the operators (4.6), with

$$E(\beta) = E(\beta; x, t) = r(\beta)\exp\left(\frac{2ixe^{i\beta} + 2\beta - te^{2i\beta}}{4i}\right) \tag{5.2}$$

coinciding with the function (4.4) for $t = 0$.

According to Lemma 4.1 the operators $\Gamma = \Gamma(x,t)$ are invertible for all $x, t \in R^2$ and the equations (4.5) have a unique solution $y(\beta) = y(\beta; x, t)$.

Theorem 5.1. *The function*

$$\psi(x,t) = \frac{e^{-2i\theta}}{2}\left(1 + \int y(\beta; x, t)d\sigma(\beta)\right) \tag{5.3}$$

is the solution of the Cauchy problem (5.1).

P r o o f. Since for $t = 0$ the functions (4.4) and (5.2) coincide, we have, according to Theorem 4.1, $\psi(x,0) = \psi_0(x)$. It remains to prove that the function $\psi(x,t)$, defined by formula (5.3), satisfies the nonlinear Schrödinger equation. To this end we will use the method developed in [5]. It follows from definition (5.2) of the function $E(\beta)$ that it satisfies the equations

$$\left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)E(\beta) = 0, \qquad \left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\overline{E(\beta)}\right)^{-1} = 0\,,$$

$$\frac{\partial}{\partial x}E(\beta) = E(\beta)\frac{e^{i\beta}}{2i}, \qquad \frac{\partial}{\partial x}\left(\left(\overline{E(\beta)}\right)^{-1}\right) = \left(\overline{E(\beta)}\right)^{-1}\frac{e^{-i\beta}}{2i}\,,$$

and the equalities

$$\left(\overline{E(\beta)}\right)^{-1} = E(-\beta)E^{-i\beta}, \qquad E(\beta)\overline{E(-\beta)} = e^{-i\beta}. \tag{5.4}$$

Thus the operator-valued function $\Gamma = \Gamma(x,t)$ satisfies the equation

$$\left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\Gamma = 0 \tag{5.5}$$

and the equality

$$\frac{\partial}{\partial x}(\Gamma) = \Gamma\frac{e}{2i} + \frac{\overline{E}^{-1}}{2i}\{\overline{e}\Lambda - \Lambda e\}\,, \tag{5.6}$$

where e denotes the operator of multiplication by $e^{i\beta}$. According to definition of the operator Λ (4.7), we have

$$\begin{aligned}\{\overline{e}\Lambda - \Lambda e\}(f)(\beta) &= i\tan\frac{\beta}{2}\text{v.p.}\int\frac{e^{-i\beta} - e^{i\alpha}}{e^{i\alpha} - e^{-i\beta}} f(\alpha)d\sigma(\alpha)\\ &= -i\tan\frac{\beta}{2}\int f(\alpha)d\sigma(\alpha).\end{aligned}$$

Let us introduce the the orthogonal projection P onto the one-dimensional subspace of constant functions in the Hilbert space $L^2(\Omega_0; d\sigma(\alpha))$:

$$P(f)(\beta) = \frac{1}{\sigma}\int\limits_{\Omega_0} f(\alpha)d\sigma(\alpha), \qquad \sigma = \int\limits_{\Omega_0} d\sigma(\alpha) \tag{5.7}$$

and let t denote the operator of muliplication by the function $\tan\frac{\beta}{2}$. Then the last equality take the form

$$\overline{e}\Lambda - \Lambda e = -it\sigma P,$$

and according to (5.6) we have

$$\frac{\partial}{\partial x}(\Gamma) = \Gamma\frac{e}{2i} - \frac{i\sigma t\overline{E}^{-1}}{2i}P. \tag{5.8}$$

Following [5], we introduce the logarithmic derivative

$$\gamma = \Gamma^{-1}\frac{\partial}{\partial x}(\Gamma)$$

of the operator-valued function Γ and find the equation which it satisfies. Differentiating both sides of equality (5.5) with respect to x we obtain

$$\begin{aligned}0 &= \left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}(\Gamma) = \left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(\Gamma\gamma)\\ &= \left(i\frac{\partial}{\partial t}\Gamma\right)\gamma + \Gamma i\frac{\partial}{\partial t}(\gamma) - \left(\frac{\partial^2}{\partial x^2}\Gamma\right)\gamma - 2\left(\frac{\partial}{\partial x}\Gamma\right)\frac{\partial}{\partial x}(\gamma) - \Gamma\frac{\partial^2}{\partial x^2}\gamma\\ &= \left(\left(i\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\Gamma\right)\gamma + \Gamma\left(i\frac{\partial}{\partial t}(\gamma) - \frac{\partial^2}{\partial x^2}\gamma - 2\gamma\frac{\partial}{\partial x}(\gamma)\right).\end{aligned}$$

Since the operator Γ is invertible and in view of (5.5), it follows that

$$i\frac{\partial}{\partial t}(\gamma) - \frac{\partial^2}{\partial x^2}(\gamma) - 2\gamma\frac{\partial}{\partial x}(\gamma) = 0. \tag{5.9}$$

On the other hand, it follows from equality (5.7) that

$$\gamma = \frac{1}{2i}\left(e - \sigma\Gamma^{-1}it(\overline{E})^{-1}P\right),$$

and since P is the projector on the subspace of constants, we have

$$\gamma = \frac{1}{2i}(e - \sigma vP), \tag{5.10}$$

where v denotes the operator of multiplication by the solution $v(\beta)$ of the equation

$$\Gamma(v)(\beta) = i\tan\frac{\beta}{2}(\overline{E(\beta)})^{-1}, \tag{5.11}$$

or in expanded form

$$\begin{aligned} E(\beta)v(\beta) \quad &+ \quad m(\beta)\chi_2^a(\beta)E(-\beta)v(-\beta) \\ &+ \quad (\overline{E(\beta)})^{-1}\text{v.p.}\int_{\Omega_0} \frac{i\tan\frac{\beta}{2}}{e^{i\alpha} - e^{-i\beta}}v(\alpha)d\sigma(\alpha) = i\tan\frac{\beta}{2}(\overline{E(\beta)})^{-1}. \end{aligned}$$

After complex conjugation and multiplication by $e^{i\beta}E(\beta)(\overline{E(\beta)})^{-1}$ we obtain the equation

$$\begin{aligned} E(\beta)e^{i\beta}\overline{v(\beta)} \quad &+ \quad m(\beta)\chi_2^a(\beta)E(-\beta)e^{-i\beta}\overline{v(-\beta)} \\ &+ \quad (\overline{E(\beta)})^{-1}\text{v.p.}\int \frac{i\tan\frac{\beta}{2}}{e^{i\alpha} - e^{-i\beta}}e^{i\alpha}\overline{v(\alpha)}d\sigma(\alpha) = -i\tan\frac{\beta}{2}E(-\beta). \end{aligned}$$

(Here we must take into account that $\overline{(e^{i\alpha} - e^{-i\beta})} = -e^{-i\alpha}e^{i\beta}(e^{i\alpha} - e^{-i\beta})$ and, in view of (5.4), $(\overline{E(\beta)})^{-1}e^{i\beta} = E(-\beta)$, $\overline{E(-\beta)}E(\beta)(\overline{E(\beta)})^{-1}e^{i\beta} = E(-\beta)e^{-i\beta}$.)

Hence the function $-e^{i\beta}\overline{v(\beta)}$ and the solution $y(\beta)$ of the equation (4.5) satisfy the same equation. By the uniqueness property for the solution of this equation we get $v(\beta) = -e^{i\beta}\overline{y(\beta)}$ and according to (5.9)

$$\gamma = \frac{e}{2i}(I + \sigma\overline{y}P) = -ie\left(\frac{I + \sigma\overline{y}P}{2}\right). \tag{5.12}$$

Let us now express the derivative $\gamma_x = \frac{\partial}{\partial x}(\gamma)$ of the operator-valued function γ in terms of the operators e, y and P. Differentiating the equalities (5.12) and (4.5′) with respect to x, we get

$$\gamma_x = \frac{e\sigma}{2i}\overline{y}_xP \tag{5.13}$$

and

$$\Gamma_x(y)+\Gamma(y_x)=i\tan\frac{\beta}{2}E(-\beta)\frac{e^{-i\beta}}{2i}=\frac{i\tan\frac{\beta}{2}(\overline{E(\beta)})^{-1}}{2i},$$

or

$$\Gamma(\gamma(y)+y_x)=\frac{i\tan\frac{\beta}{2}(\overline{E(\beta)})^{-1}}{2i}.$$

In view of (5.11) it follows that

$$\gamma(y)(\beta)+y_x(\beta)=\frac{1}{2i}v(\beta)=-\frac{e^{i\beta}}{2i}\overline{y(\beta)}.$$

Furthermore, since we have according to (5.12)

$$\gamma(y)(\beta)=\frac{e^{i\beta}}{2i}(I+\sigma\overline{y}P)(y)(\beta),$$

we have

$$\begin{aligned} y_x(\beta) &= -\frac{e^{i\beta}}{2i}\{(I+\sigma\overline{y}P)(y)(\beta)+\overline{y(\beta)}\},\\ \overline{y_x(\beta)} &= \frac{e^{-i\beta}}{2i}\{(I+\sigma yP)(\overline{y})(\beta)+y(\beta)\}, \end{aligned}$$

and in view of (5.13)

$$\gamma_x=-\frac{\sigma}{4}\{(I+\sigma yP)\overline{y}P+yP\}=-\frac{1}{4}\{(I+\sigma yP)(I+\sigma\overline{y}P)-I\}. \tag{5.14}$$

Substituting the right-hand sides of (5.12), (5.14) in (5.9), we obtain

$$-ie\Big\{\left(i\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{I+\sigma\overline{y}P}{2}\right)+2\left(\frac{I+\sigma\overline{y}P}{2}\right)\left(\frac{I+\sigma yP}{2}\right)\left(\frac{I+\sigma\overline{y}P}{2}\right)$$
$$-\frac{1}{2}\left(\frac{I+\sigma\overline{y}P}{2}\right)\Big\}=0. \tag{5.15}$$

Finally, we note that it follows from formula (5.3) and from the definition of the projector P that

$$P\left(\frac{I+\sigma\overline{y}P}{2}\right)=\frac{1}{2}\left(1+\int\overline{y(\beta)}d\sigma(\beta)\right)P=\overline{\psi(x,t)}e^{-2i\theta}P.$$

Hence multiplying from the left both sides of equality (5.15) by $iP\overline{e}$, we obtain

$$\left(\left(i\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\overline{\psi}+2|\psi|^2\overline{\psi}-\frac{1}{2}\overline{\psi}\right)e^{-2i\theta}P=0,$$

and from this follows that the scalar function $\psi(x,t) = \psi$ indeed satisfies the equation

$$i\frac{\partial}{\partial t}\psi = -\frac{\partial^2}{\partial x^2}\psi + 2\left(|\psi|^2 - \frac{1}{4}\right)\psi.$$

■

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Задача Коши для нелинейного уравнения Шредингера с ограниченными начальными данными

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Для операторов, являющихся сильными пределами безотражательных операторов Дирака, вводится аналог данных рассеяния и решается соответствующая обратная задача. Развивается основанный на этом метод решения задач Коши для нелинейного уравнения Шредингера с начальными данными из широкого множества неисчезающих на бесконечности функций.

Задача Коши для нелінійного рівняння Шредінгера з обмеженими початковими даними

Анна Буте де Монвель, Володимир Марченко

Для операторів, що є сильними границями безвідбивних операторів Дірака, вводиться аналог даних розсіяння та розв'язується відповідна обернена задача. Розвивається побудований на цьому метод розв'язання задач Коши для нелінійного рівняння Шредінгера з обмеженими початковими даними з широкої множини незникаючих на нескінченності функцій.