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The Boltzmann–Grad limit for equilibrium states of systems of hard spheres in framework of canonical ensemble

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The existence of the Boltzmann–Grad limit is proved for equilibrium states of infinite systems of hard spheres for different normalization of distribution functions.

1. Introduction

The problem of derivation and mathematical justification of the Boltzmann equation permanently attracts attention of mathematical physicists. The problem is as follows: to derive the Boltzmann equation from the fundamental equations of classical statistical mechanics – the BBGKY hierarchy (Bogolubov–Born–Green–Kirkwood–Yvon). In recent time, significant progress connected with solution of this problem has been achieved, namely, the rigorous justification of derivation of the Boltzmann equation has been given for a system of hard-spheres in the Boltzmann–Grad limit [1–3]. It has been shown that for nonequilibrium distribution functions the Boltzmann–Grad limit exists and these functions are equal to the product of one-particle distribution functions which satisfy the Boltzmann equations.

In this paper, we consider a simpler problem, namely, a problem of the existence of the Boltzmann–Grad limit for the equilibrium distribution functions of

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a system of hard spheres in the framework of the canonical ensemble. (Note that the same problem was solved earlier in the framework of grand canonical ensemble [4].) It is proved in the paper that the equilibrium distribution functions normalized on unity tend to zero if one simultaneously passes to the Boltzmann–Grad limit and to the thermodynamic limit. The nonzero limit distribution functions with this normalization can be obtained only for the systems of particles located in a bounded domain.

The system of particles located in the whole space was also investigated for a case of the distribution functions normalized to the number of particles. In this case, the limit distribution functions are nonzero only if the diameter of spheres tends to zero and their density tends to infinity so that their product is constant.

2. The existence of the Boltzmann–Grad limit for normalized equilibrium distribution functions

Statement of problem. Let us consider N hard spheres in a domain $\Lambda \subset \mathbf{R}^3$ with the volume $V(\Lambda) = V$. Denote by $D^{(N)}(t, x_1, \ldots, x_N)$ a distribution function in the phase space of N hard spheres, normalized to unity, $x = (p, q)$ is a point of phase space, p is momentum, q is position of the center of hard sphere. Let us denote by $F^{(N)}(t)$ the sequence of reduced distribution functions

$$\begin{aligned} F^{(N)}(t) \quad = \quad & \Big(F_1^{(N)}(t, x_1), F_2^{(N)}(t, x_1, x_2), \ldots, \\ & F_s^{(N)}(t, x_1, \ldots, x_s), \ldots, F_N^{(N)}(t, x_1, \ldots, x_N), 0, \ldots\Big) \end{aligned} \tag{2.1}$$

which are defined trough the function $D^{(N)}(t, x_1, \ldots, x_N)$ by the following formula:

$$F_s^{(N)}(t, x_1, \ldots, x_s) = \int_{\Lambda^{N-s}} dx_{s+1} \ldots dx_N D^{(N)}(t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_N), \quad 1 \leq s \leq N, \tag{2.2}$$

where

$$\int_\Lambda dx = \int_\Lambda dq \int_{\mathbf{R}^3} dp\,.$$

The distribution function defined according to (2.2) are called normalized (normalized to unity).

The function $F_s^{(N)}(t, x_1, \ldots, x_s)$ satisfy the following equations [5], known as the BBGKY hierarchy (for $t > 0$):

$$\frac{\partial F_s^{(N)}(t, x_1, \ldots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s^{(N)}(t, x_1, \ldots, x_s)$$

$$+(N-s)a^2\sum_{i=1}^{s}\int dndp_{s+1}\Theta((p_i-p_{s+1})\cdot n)n\cdot(p_i-p_{s+1})$$
$$\times\Big[F_{s+1}^{(N)}(t,x_1,\ldots,q_i,p_i-nn\cdot(p_i-p_{s+1}),\ldots,x_s,q_i-an,$$
$$p_{s+1}+nn\cdot(p_i-p_{s+1}))$$
$$-F_{s+1}^{(N)}(t,x_1,\ldots,x_s,q_i+an,p_{s+1})\Big]\,, \tag{2.3}$$

where Θ is the Hewiside function, a is the diameter of sphere, n is unit vector, $|n|=1$. The integral in (2.3) is carried our over unit sphere $|n|=1$ with respect to n and over whole three-dimensional space with respect to p_{s+1}.

The functions $F_s^{(N)}(t,x_1,\ldots,x_s)$ are symmetrical (invariant under permutations), equal to zero on forbidden configurations

$$\mathbf{W}_s\,(q_1,\ldots,q_s\,|\,|q_i-q_j|<a,\quad (i,j)\in(1,\ldots,s))$$

and satisfy the boundary conditions

$$F_s^{(N)}(t,x_1,\ldots,x_i,\ldots,x_j,\ldots,x_s)$$
$$=F_s^{(N)}(t,x_1,\ldots,x_i^*,\ldots,x_j^*,\ldots,x_s)\,,\quad \text{if}|q_i-q_j|=a.$$

Here $x_i^*=(q_i,p_i-nn\cdot(p_i-p_j))$, $x_j^*=(q_j,p_j+nn\cdot(p_i-p_j))$.

They are also equal to zero outside of Λ, $F_s^{(N)}(t,x_1,\ldots,x_s)=0$, if $q_i\notin\Lambda$, $i=1,2,\ldots,s$.

Let us stress that the multiplier $(N-s)$ appeared in the second term of right-hand side of equations (2.3) due to definition (2.2). If one employs an usual definition of reduced distribution functions

$$F_s^{(N)}(t,x_1,\ldots,x_s)=N(N-1)\ldots(N-s+1)$$
$$\times\int_{\Lambda^{N-s}}dx_{s+1}\ldots dx_N D^{(N)}(t,x_1,\ldots,x_s,x_{s+1},\ldots,x_N)\,, \tag{2.4}$$

then the multiplier $(N-s)$ will be absent in the second term of right-hand side of equations (2.3) and the equations for the distribution functions (2.4) will have the following form:

$$\frac{\partial F_s^{(N)}(t,x_1,\ldots,x_s)}{\partial t}=-\sum_{i=1}^{s}p_i\frac{\partial}{\partial q_i}F_s^{(N)}(t,x_1,\ldots,x_s)$$
$$+a^2\sum_{i=1}^{s}\int dndp_{s+1}\Theta((p_i-p_{s+1})\cdot n)n\cdot(p_i-p_{s+1})$$
$$\times\Big[F_{s+1}^{(N)}(t,x_1,\ldots,q_i,p_i-nn\cdot(p_i-p_{s+1}),\ldots,x_s,q_i-an,$$
$$p_{s+1}+nn\cdot(p_i-p_{s+1}))$$
$$-F_{s+1}^{(N)}(t,x_1,\ldots,x_s,q_i+an,p_{s+1})\Big]\,. \tag{2.5}$$

If we define the reduced distribution functions according to formula [6]

$$F_s^{(N)}(t,x_1,\dots,x_s) = V^s \int_{\Lambda^{N-s}} dx_{s+1}\dots dx_N \times D^{(N)}(t,x_1,\dots,x_s,x_{s+1},\dots,x_N)\,, \tag{2.6}$$

then the multiplier $\frac{N-s}{V}$ will be present in the second term on the right-hand side of (2.3) or(2.5). This multiplier is equal to density in thermodynamic limit. (We use the same notation $F_s^{(N)}(t,x_1,\dots,x_s)$ for distribution functions under all normalizations.)

The equilibrium distribution functions defined according to (2.4) or (2.6) are well investigated in the thermodynamic limit when $\Lambda \nearrow \mathbf{R}^3$, $V(\Lambda)\to\infty$, $N\to\infty$, $\frac{N}{V(\Lambda)} = \frac{1}{\nu} = \text{const}$. For details, we refer readers to article of Bogolubov, Petrina and Khatset [7] or to monograph [8]. In the present paper, we carry out analogous investigations for the equilibrium distribution functions defined according to (2.2).

Definitions (2.2) and (2.4), (2.6) are equivalent for finite N and $V(\Lambda)$, because we can pass from one to the other using different normalizations. But this is impossible in the thermodynamic limit when N and $V(\Lambda)$ tend to infinity. Therefore the case (2.2) requires an independent investigation. It is obvious that definitions (2.4) and (2.6) are also equivalent in the thermodynamic limit because

$$\lim_{\substack{N\to\infty, V(\Lambda)\to\infty \\ \frac{N}{V(\Lambda)}=\frac{1}{v}}} \frac{V^s(\Lambda)}{N(N-1)\dots(n-s+1)} = \frac{1}{v^s}\,.$$

Kirkwood–Salsburg equations. It is well known that in order to investigate the equilibrium states it suffices to restrict oneself to equilibrium distribution functions in configuration space. The equilibrium distribution function of N particles distributed in a domain Λ is defined as follows:

$$D^{(N)}(q_1,\dots,q_N) = \frac{1}{Q(N,\Lambda)} \exp\left\{-\beta \sum_{i<j=1}^{N} \Phi(q_i-q_j)\right\},$$

$$Q(N,\Lambda) = \int_{\Lambda^N} \exp\left\{-\beta \sum_{i<j=1}^{N} \Phi(q_i-q_j)\right\} dq_1,\dots,dq_N\,, \tag{2.7}$$

where Φ is the interaction potential of hard spheres

$$\Phi(q) = \begin{cases} \infty\,, & |q|\le a,\\ 0\,, & |q|>a. \end{cases}$$

The s-particle reduced distribution functions are defined, according to (2.2):

$$F_s^{(N)}(q_1,\ldots,q_s) = \int_{\Lambda^{N-s}} D^{(N)}(q_1,\ldots,q_s,q_{s+1},\ldots,q_N)\,dq_{s+1}\ldots dq_N\,, \qquad 1\le s\le N,$$

$$F^{(N)} = \left(F_1^{(N)}(q_1),\ldots,F_s^{(N)}(q_1,\ldots,q_s),\ldots,F_N^{(N)}(q_1,\ldots,q_N),0,\ldots\right).$$

They satisfy the following Kirkwood–Salsburg relations:

$$\begin{aligned} & F_s^{(N)}(q_1,\ldots,q_s) \\ &= \frac{Q(N-1,\Lambda)}{Q(N,\Lambda)}\exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1-q_i)\right\}\times\Big[F_{s-1}^{(N-1)}(q_2,\ldots,q_s) \\ &+ \sum_{k=1}^{N-s}\frac{1}{k!}(N-s)(N-s-1)\ldots(N-s-k+1) \\ &\times \int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\;F_{s-1+k}^{(N-1)}(q_2,\ldots,q_s,y_1,\ldots,y_k)\,dy_1\ldots dy_k\Big], \\ & (N>s>1) \end{aligned}$$

$$\begin{aligned} F_1^{(N)}(q_1) &= \frac{Q(N-1,\Lambda)}{Q(N,\Lambda)}\left[1+\sum_{k=1}^{N-1}\frac{1}{k!}(N-1)(N-2)\ldots(N-k)\right. \\ &\times \left.\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)F_k^{(N-1)}(y_1,\ldots,y_k)\,dy_1\ldots dy_k\right], \end{aligned}$$

$$\begin{aligned} & F_N^{(N)}(q_1,\ldots,q_N) \\ &= \frac{Q(N-1,\Lambda)}{Q(N,\Lambda)}\exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1-q_i)\right\}\times F_{N-1}^{(N-1)}(q_2,\ldots,q_N), \end{aligned}$$

$$\varphi_{q_1}(y) = \exp\{-\beta\Phi(q_1-y)\}-1\,. \tag{2.8}$$

For details of derivation, we refer to monograph [8]. Note that, for the functions $F_s^{(N)}(q_1,\ldots,q_s)$, defined according to (2.4) the multipliers $(N-s)(N-s-1)\ldots(N-s-k+1)$ are absent in relations (2.8) connecting $F^{(N)}$ with $F^{(N-1)}$ (see § 3).

The Boltzmann–Grad limit means that number of particles tends to infinity $N\to\infty$, diameter of particles tends to zero $a\to 0$, but the product $Na^2=\lambda$ is fixed. Our objective is to investigate the sequence of distribution functions $F^{(N)}$

in the Boltzmann–Grad limit. For this aim the Kirkwood–Salsburg relations (2.8) will be used.

Define the numbers $\lambda = \lambda_0 = Na^2$, $\lambda_1 = (N-1)a^2$,..., $\lambda_i = (N-i)a^2$,..., $i < N$. According to definition of the Boltzmann–Grad limit, the numbers λ_i are bounded and $\lambda_i < \lambda$, $i > 1$. It is easy to transform relations (2.8) to the following form:

$$F_s^{(N)}(q_1, \ldots, q_s) = a(N,\Lambda) \exp\left\{-\beta \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} \times \left[F_{s-1}^{(N-1)}(q_2, \ldots, q_s)\right.$$

$$\left. + \sum_{k=1}^{N-s} \frac{1}{k!} \prod_{j=0}^{k-1} \frac{\lambda_{s+j}}{a^2} \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i) \times F_{s-1+k}^{(N-1)}(q_2, \ldots, q_s, y_1, \ldots, y_k)\, dy_1 \ldots dy_k\right],$$

$$F_1^{(N)}(q_1) = a(N,\Lambda)$$
$$\times \left[1 + \sum_{k=1}^{N-1} \frac{1}{k!} \prod_{j=0}^{k-1} \frac{\lambda_{1+j}}{a^2} \int_{\Lambda^k} \prod_{i=1}^{k} \varphi_{q_1}(y_i) \times F_k^{(N-1)}(y_1, \ldots, y_k)\, dy_1 \ldots dy_k\right],$$

$$F_N^{(N)}(q_1, \ldots, q_N) = a(N,\Lambda)$$
$$\times \exp\left\{-\beta \sum_{i=2}^{N} \Phi(q_1 - q_i)\right\} \times F_{N-1}^{(N-1)}(q_2, \ldots, q_N), \qquad (2.9)$$

$$a(N,\Lambda) = \frac{Q(N-1,\Lambda)}{Q(N,\Lambda)}.$$

Relations (2.9) hold for $N > 2$. For $N = 2$, one has

$$F_1^{(2)}(q_1) = \frac{\int_\Lambda \exp\{-\beta\Phi(q_1 - q_2)\}\, dq_2}{\int_{\Lambda^2} \exp\{-\beta\Phi(q_1 - q_2)\}\, dq_1 dq_2},$$

$$F_2^{(2)}(q_1, g_2) = \frac{\exp\{-\beta\Phi(q_1 - q_2)\}}{\int_{\Lambda^2} \exp\{-\beta\Phi(q_1 - q_2)\}\, dq_1 dq_2}.$$

Kirkwood–Salsburg operator. Consider a Banach space E_ξ of sequences $f = (f_1(q_1), \ldots, f_s((q)_s), \ldots)$ of bounded functions $f_s(q_1, \ldots, q_s) = f_s((q)_s)$, $q_1, \ldots, q_s = (q)_s$ with norm

$$||f|| = \sup_{s \geq 1} \frac{1}{\xi^s} \sup_{(q)_s} |f_s((q)_s)|, \qquad \xi > 0.$$

Define in E_ξ operator $K^{(N)}$ in the following way:

$$\left(K^{(N)}f\right)_s((q)_s) = \chi_\Lambda((q)_s)\exp\left\{-\beta\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\Big[f_{s-1}((q)_s^1) + \sum_{k=1}^{N-s}\frac{1}{k!}\prod_{j=0}^{k-1}\frac{\lambda_{s+j}}{a^2}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)\, f_{s-1+k}((q)_s^1,(y)_k)d(y)_k\Big],$$

$$\left(K^{(N)}f\right)_1((q)_1) = \chi_\Lambda((q)_1)\sum_{k=1}^{N-1}\frac{1}{k!}\prod_{j=0}^{k-1}\frac{\lambda_{1+j}}{a^2}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)f_k((y)_k)\, d(y)_k\,,$$

$$\left(K^{(N)}f\right)_N((q)_N) = \chi_\Lambda((q)_N)\exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1-q_i)\right\}f_{N-1}((q)_N^1)\,,$$

$$\left(K^{(N)}f\right)_s((q)_s) = 0, \qquad s > N\,, \tag{2.10}$$

$$(q)_s^1 = (q_2,\ldots,q_s), \qquad d(y)_k = dy_1\ldots dy_k\,,$$

where $\chi_\Lambda((q)_s)$ is the characteristic function of domain Λ^s.

Taking into account that

$$\varphi_q(y) = \begin{cases} -1, & |q-y|\le a,\\ 0, & |q-y|>a,\end{cases} \tag{2.11}$$

we have the following estimate for the norm of the operator $K^{(N)}$:

$$||K^{(N)}|| \le \sup_{s\ge1}\xi^{s-1}\sum_{k=0}^{N-s}\left(\lambda\frac{4}{3}\pi a\xi\right)^k\frac{1}{k!} \le \xi^{-1}\exp\left(\frac{4}{3}\pi a\xi\lambda\right). \tag{2.12}$$

It follows from (2.12) that the operator $K^{(N)}$ is uniformly bounded with respect to N and a if the parameter a lies in the finite interval $0\le a\le a_0$. In what follows we assume that this condition is satisfied.

By using the operator $K^{(N)}$, relations (2.9) can be represented as single operator relation, namely

$$F^{(N)} = a(N,\Lambda)\left(K^{(N)}F^{(N-1)} + F_0^{(N-1)}\right), \tag{2.13}$$

where $F_0^{(N-1)} = (1,0,\ldots) = F_0$.

Our aim is to prove the existence of the thermodynamic and Boltzmann–Grad limits for $F^{(N)}$. To do this, we first need to estimate the value $a(N,\Lambda)$.

Estimation of the value $a(N,\Lambda)$. To estimate the ratio $\frac{Q(N-1,\Lambda)}{Q(N,\Lambda)}$, we use the following trick. Represent $Q(N,\Lambda)$ in the following way:

$$\begin{aligned} Q(N,\Lambda) &= \int_{\Lambda^N} \exp\left\{-\beta \sum_{i<j=1}^{N} \Phi(q_i-q_j)\right\} dq_1 d(q)_N^1 \\ &= \int_{\Lambda^{N-1}} \left\{ \exp\left\{-\beta \sum_{i<j=2}^{N} \Phi(q_i-q_j)\right\} \right. \\ &\times \left. \int_\Lambda \prod_{i=2}^{N} (\exp\{-\beta\Phi(q_1-q_i)\} - 1 + 1)\, dq_1 \right\} d(q)_N^1 . \end{aligned} \tag{2.14}$$

The value $a_i = \exp\{-\beta\Phi(q_1-q_i)\} - 1$ is nonpositive and $0 \geq a_i \geq -1$. Thus $(a_i+1) \geq 0$. Hence the following inequality holds:

$$\prod_{i=2}^{N}(1+a_i) \geq 1 - \sum_{i=2}^{N}|a_i|$$

or, which is the same,

$$\prod_{i=2}^{N}(\exp\{-\beta\Phi(q_1-q_i)\} - 1 + 1) \geq 1 - \sum_{i=2}^{N} |\exp\{-\beta\Phi(q_1-q_i)\} - 1| . \tag{2.15}$$

Substituting inequality (2.15) in (2.14), one gets

$$\begin{aligned} Q(N,\Lambda) &\geq \int_{\Lambda^{N-1}} \left\{ \exp\left\{-\beta \sum_{i<j=2}^{N} \Phi(q_i-q_j)\right\} \right. \\ &\times \left. \int_\Lambda \left(1 - \sum_{i=2}^{N} |\exp\{-\beta\Phi(q_1-q_i)\} - 1|\right) dq_1 \right\} d(q)_N^1 \\ &\geq \int_{\Lambda^{N-1}} \exp\left\{-\beta \sum_{i<j=2}^{N} \Phi(q_i-q_j)\right\} d(q)_N^1 V(\Lambda) \\ &- \int_{\Lambda^{N-1}} \exp\left\{-\beta \sum_{i<j=2}^{N} \Phi(q_i-q_j)\right\} d(q)_N^1 \\ &\times (N-1) \int_{\mathbf{R}^3} |\exp(-\beta\Phi(q)) - 1| dq \\ &= Q(N-1,\Lambda) \left[V(\Lambda) - (N-1)\frac{4}{3}\pi a^3 \right] . \end{aligned}$$

It is obvious that $V(\Lambda) - (N-1)\frac{4}{3}\pi a^3 > 0$, because N spheres are situated in the volume $V(\Lambda)$. Thus one obtains the desired estimate

$$\begin{aligned} a(N,\Lambda) &= \frac{Q(N-1,\Lambda)}{Q(N,\Lambda)} \\ &\leq \frac{Q(N-1,\Lambda)}{Q(N-1,\Lambda)(V(\Lambda)-(N-1)\frac{4}{3}\pi a^3)} = \frac{1}{V(\Lambda)-(N-1)\frac{4}{3}\pi a^3}. \end{aligned} \tag{2.16}$$

Let us suppose that the thermodynamic and Boltzmann–Grad limits are performed simultaneously. The value $Na^2 = \lambda$ is fixed and therefore $Na^3 \to 0$ in the Boltzmann–Grad limit. It follows from estimate (2.16) that the following lemma is true.

Lemma *The ratio* $a(N,\Lambda) = \frac{Q(N-1,\Lambda)}{Q(N,\Lambda)}$ *tends to zero if the following parameters tend to their limits simultaneously:*

$$N \to \infty, a \to 0, Na^2 = \lambda = \text{const},$$

$$\Lambda \nearrow \mathbf{R}^3, V(\Lambda) \to \infty, \frac{N}{V(\Lambda)} = \frac{1}{v} = \text{const}, \tag{2.17}$$

i.e. if the thermodynamic and Boltzmann–Grad limits are performed simultaneously.

It is easy to see that the ratios $a(N-i,\Lambda) = \frac{Q(N-i-1,\Lambda)}{Q(N-i,\Lambda)}$ also tend to zero in the limit (2.17) uniformly with respect to $0 \leq i \leq N-3$ according to estimate

$$a(N-i,\Lambda) \leq \frac{1}{V(\Lambda)-(N-i-1)\frac{4}{3}\pi a^3} \leq \frac{1}{V(\Lambda)-(N-1)\frac{4}{3}\pi a^3}. \tag{2.16'}$$

The limit of $F^{(N)}$.

Theorem I. *The sequence* $F^{(N)}$ *tends to zero in the norm of the space* E_ξ *if the thermodynamic limit and the Boltzmann–Grad limit (2.17) are performed simultaneously.*

P r o o f. To prove this, we consider relations (2.13). Using them many times, one gets

$$\begin{aligned} F^{(N)} &= \sum_{i=1}^{N-3} a(N,\Lambda)K^{(N)}a(N-1,\Lambda)K^{(N-1)}\ldots a(N-i+1,\Lambda)K^{(N-i+1)} \\ &\times\ a(N-i,\Lambda)F_0 + a(N,\Lambda)K^{(N)} \\ &\times\ a(N-1,\Lambda)K^{(N-1)}\ldots a(3,\Lambda)K^{(3)}F^{(2)} + a(N,\Lambda)F_{(0)}. \end{aligned} \tag{2.18}$$

For the operators $K^{(N-i)}$, $0 \le i \le N-3$, estimates (2.12) hold. The value $a(N-i,\Lambda)$ satisfies estimate (2.16$'$) which implies that this value tends to zero in limit (2.17). For the sake of simplicity, we take $V(\Lambda)$ so large and a so small that, according to (2.12) and (2.16), one can put

$$||K^{(N-i+1)}||a(N-i,\Lambda) \le k < 1, \quad 1 \le i \le N-3\,. \tag{2.19}$$

Let us estimate $F^{(2)}$. It is easy to see that

$$F_1^{(2)}(q_1) < \frac{V(\Lambda)}{-V(\Lambda)\frac{4}{3}\pi a^3 + V(\Lambda)^2}\,, \quad F_2^{(2)}(q_1,q_2) < \frac{1}{-V(\Lambda)\frac{4}{3}\pi a^3 + V(\Lambda)^2}\,. \tag{2.20}$$

It follows from (2.20) that $||K^{(3)}||||F^{(2)}|| \le k < 1$ for sufficiently large $V(\Lambda)$. By using estimates (2.12), (2.19), (2.20), we get from (2.18)

$$||F^{(N)}|| \le a(N,\Lambda)\sum_{i=0}^{N-2} k^i \le a(N,\Lambda)\frac{1}{1-k}\,. \tag{2.21}$$

According to (2.16), $||F^{(N)}||$ tends zero in limit (2.17). Note that, on formal level, this property of $F^{(N)}$ immediately follows from expression for $D^{(N)}(q_1,\dots,q_N)$, because $\exp\left\{-\beta\sum_{i<j=1}^{N}\Phi(q_i-q_j)\right\} \le 1$, and $Q(N,\Lambda) \sim V^N$, thus $D^{(N)}(q_1,\dots,q_N) \sim \frac{1}{V^N}$. For the same reason $F_s^{(N)}(q_1,\dots,q_s) \sim \frac{1}{V^s}$.

Thus, if one performs simultaneously, according to (2.17), the thermodynamic limit $N\to\infty$, $\Lambda \nearrow \mathbf{R}^3$, $(V(\Lambda)\to\infty)$, $\frac{N}{V} = \frac{1}{v} =$ const, and the Boltzmann–Grad limit $N\to\infty$, $a\to 0$, $Na^2 = \lambda$ =const, then the sequence of the equilibrium distribution functions $F^{(N)}$ defined according to (2.7) tends to zero in the norm of space E_ξ.

Note that the functions $F_s^{(N)}(q_1,\dots,q_s)$ satisfy the compatibility conditions

$$\int_\Lambda F_{s+1}^{(N)}(q_1,\dots,q_s,q_{s+1})\,dq_{s+1} = F_s^{(N)}(q_1,\dots,q_s), \quad \int_\Lambda F_1^{(N)}(q_1)dq_1 = 1\,,$$

for all N and Λ.

To avoid this extremely undesirable phenomenon of tending to zero of the distribution functions, it is necessary to fix bounded domain Λ and only consider the Boltzmann–Grad limit.

The Boltzmann–Grad limit for fixed bounded domain Λ. Let us return to relations (2.9) or, in a operator form, (2.13) and fix the domain Λ in them. In this section, it is useful to introduce the notations

$$\frac{Q(N-i-1,\Lambda)}{Q(N-i,\Lambda)} = a_i(N), \quad N-3 \ge i \ge 0, \quad a_0(N) \equiv a(N)\,, \tag{2.22}$$

where the dependence on Λ is omitted because Λ is fixed.

The sequence $a_i(N)$ is bounded according to (2.16′) (for fixed i). Therefore one can select a convergent subsequence $a_i(N_j)$

$$\lim_{N_j\to\infty} a_i(N_j) = A_i, \quad A_0 \equiv A\,.$$

By using the diagonal procedure, one can do it for all i. We conclude that the following limits exist:

$$\lim_{N_j\to\infty} a_i(N_j) = A_i \tag{2.23}$$

for all $i = 0, 1, 2, \dots$. Of course, this does not mean, that there are no other convergent subsequences with limits $A_i^{(1)}, \dots, A_i^{(k)}, \dots,\ i = 0, 1, 2, \dots$. Let restrict ourselves to the subsequence $a_i(N_j)$ (2.23). By using this notation, relations (2.13) for $F^{(N-i)}$ take the form

$$F^{(N-i)} = a_i(N)\left(K^{(N-i)}F^{(N-i-1)} + F_0\right), \quad i = 0, 1, 2, \dots . \tag{2.24}$$

Let formally pass to the limit as $N_i \to \infty$ (2.24), assuming that all required limits exist. Then we obtain

$$\begin{aligned} F &= A(KF^1 + F_0), \\ F^i &= A_i(KF^{i+1} + F_0), \quad i = 0, 1, 2, \dots , \end{aligned} \tag{2.25}$$

where

$$F^i = \lim_{N_j\to\infty} F^{(N_j-i)}, \quad K = \lim_{N_j\to\infty} K^{(N_j-i)}, \quad F^0 = F\,. \tag{2.26}$$

Determine the operator K. To do this, we represent the operators $K^{(N)}$ in the following form:

$$K^{(N)} = K_1^{(N)} + K_2^{(N)}\,,$$

where

$$\begin{aligned} \left(K_1^{(N)}f\right)_s((q)_s) &= \chi_\Lambda((q)_s)\exp\left\{-\beta\sum_{i=1}^{s}\Phi(q_1-q_i)\right\} f_{s-1}((q)_s^1), \\ \left(K_2^{(N)}f\right)_s((q)_s) &= \chi_\Lambda((q)_s)\exp\left\{-\beta\sum_{i=1}^{s}\Phi(q_1-q_i)\right\} \sum_{k=1}^{N-s}\frac{1}{k!}\prod_{j=0}^{k-1}\frac{\lambda_{s+j}}{a^2} \\ &\times \int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i) f_{s-1+k}((q)_s^1,(y)_k)d(y)_k\,, \\ & f\in E_\xi\,, \quad f_0 = 0\,, \\ \left(K_1^{(N)}f\right)_s((q)_s) &= 0, \qquad s > N\,, \\ \left(K_2^{(N)}f\right)_s((q)_s) &= 0, \qquad s > N-1\,. \end{aligned} \tag{2.27}$$

The following estimates are obvious:

$$\begin{aligned} \|K_1^{(N)}\| &\le 1, \\ \|K_2^{(N)}\| &\le \xi^{-1} \sum_{k=1}^{N-s} \frac{1}{k!} \left(\frac{4}{3}\pi a \xi \lambda\right)^k \le \xi^{-1} \left(\exp\left\{\frac{4}{3}\pi a\xi\lambda\right\} - 1\right), \end{aligned}$$

which implies that the operator $K_2^{(N)}$ tends to zero in norm as $N \to \infty$, $a \to 0$, $Na^2 = \lambda$ =const (in the Boltzmann–Grad limit),

$$\lim_{\substack{N\to\infty, a\to 0 \\ Na^2=\lambda}} \|K_2^{(N)}\| = 0\,.$$

It is obvious that operators $K_2^{(N-i)}$ tends to zero in norm as $N \to \infty$, $a \to 0$, $Na^2 = \lambda$ =const uniformly with respect to $i \ge 0$. Taking into account that for $|q_1 - q_i| > a$, $i = 2, \ldots, s$,

$$\lim_{a\to 0} \exp\left\{-\beta \sum_{i=1}^{s} \Phi(q_1 - q_i)\right\} = 1\,,$$

one obtains that the operator K acts in E_ξ in the following way:

$$(Kf)_s((q)_s) = \chi_\Lambda((q)_s) f_{s-1}((q)_s^1)\,. \tag{2.28}$$

For the sake of simplicity, we require that numbers A_i satisfy the conditions $A_i < 1$. It is easy to see from (2.16) that this condition is satisfied for sufficiently large $V(\Lambda)$ and sufficiently small $N\frac{4}{3}\pi a^3 = \lambda\frac{4}{3}a\pi$.

Now we are able to formulate the problem, which should be solved in order to justify the existence and uniqueness of limit (2.26). For this, it is necessary:

1) *to prove that the limits* A_i, F^i (2.26) *exist in certain sense*,

2) *to prove that they satisfy relations* (2.25),

3) *to show that the limits* A_i, F^i *do not depend on* i:

$$A_i = A, \quad F^i = F, \quad i = 1, 2, \ldots\,.$$

The similar problem arose in the paper [7] devoted to justification of the existence of the thermodynamic limit for equilibrium distribution functions in framework of canonical ensemble. Therefore we will solve these problems by modification of methods developed in [7] (see also monograph [8]).

By using relations (2.24), (2.25), we get

$$\begin{aligned} F^{(N)} &= \sum_{i=1}^{(N-3)} a(N)K^{(N)}a_1(N)K^{(N-1)} \ldots a_{i-1}(N)K^{(N-i+1)}a_i(N)F_0 + a(N)F_0 \\ &+ a(N)K^{(N)}a_1(N)K^{(N-1)} \ldots a_{N-3}(N)K^{(3)}F^{(2)}\,, \\ F &= \sum_{i=1}^{\infty} AKA_1K \ldots A_{i-1}KA_iF_0 + AF_0\,. \end{aligned} \tag{2.29}$$

The following estimate holds:

$$\begin{aligned}||\chi_{a_0}(F-F^{(N)})|| &\leq ||\chi_{a_0}\Bigg(\sum_{i=1}^{n_0} AKA_1K\ldots A_{i-1}KA_iF_0+\chi_{a_0}AF_0\\ &- \sum_{i=1}^{n_0} a(N)K^{(N)}a_1(N)K^{(N-1)}\ldots a_{i-1}(N)K^{(N-i+1)}\\ &\times \quad a_i(N)F_0+a(N)F^0\Bigg)||+\eta\,, \end{aligned} \tag{2.30}$$

where a_0 is arbitrary fixed number and χ_a is an operator in space E_ξ defined according to the formula

$$(\chi_a f)_s((q)_s)=\chi_a((q)_s)f_s((q)_s), \quad f\in E_\xi\,,$$

where $\chi_a((q)_s)$ is characteristic function of admissible configurations, i.e., $|q_i-q_j|\geq a$ for all $(i,j)\in(1,\ldots,s)$. Obviously that $||\chi_a||=1$.

Series (29.2) representing $F^{(N)}$ and F converge in the norm of the space E_ξ if $|a_i^{(N)}|\,||K^{(N-i)}||<k<1$, $|A_i|\,||K||<k<1$ and therefore for sufficiently large n_0, $||\eta||<\frac{\epsilon}{2}$, where ϵ is as small as desired. These inequalities for $||K^{(N-i)}||$, $||K||$ hold if $\xi^{-1}\exp\left\{\frac{4}{3}\pi a\xi\lambda\right\}<k$, i.e., if $\xi^{-1}<k$ and a is sufficiently small.

Substituting $K^{(N)}=K_1^{(N)}+K_2^{(N)}$ into (30.2), taking into account that $a_i^{(N)}\to A_i$, $||K_2^{(N)}||\to 0$ as $N\to\infty$, $a\to 0$, $Na^2=\lambda$, and moreover,

$$\chi_{a_0}AK_1^{(N)}A_1K_1^{(N-1)}\ldots A_{i-1}K_1^{(N-i+1)}A_iF_0=\chi_{a_0}AKA_1K\ldots A_{i-1}KA_iF_0\,,$$

we conclude that the first term in (2.30) can be also made less than $\frac{\epsilon}{2}$ for sufficiently large N. This means that

$$\lim_{\substack{N\to\infty,a\to 0\\ Na^2=\lambda}}||\chi_{a_0}(F-F^{(N)})||=0\,.$$

Note that $F_s((q)_s)=\text{const}\neq 0$ if $q_i\in\Lambda$, $i=1,\ldots,s$ for arbitrary $(q)_s$, but $F_s^{(N)}((q)_s)=0$ on forbidden configurations $\mathbf{W}_s$. This is the reason why we consider the difference $\chi_{a_0}(F-F^{(N)})$ but not $F-F^{(N)}$.

Let us show that $A_i=A$, $F^i=F$. For this purpose, we note that (2.25), (2.28) yield

$$F_1(q_1)=A, \quad F_1^1(q_1)=A_1,\ldots F_1^i(q_1)=A_i,\ldots, \quad q_1\in\Lambda\,.$$

We also have the normalization conditions

$$\int_\Lambda F_1(q_1)dq_1=1,\cdots\int_\Lambda F_1^i(q_1)dq_1=1,\cdots,$$

which yield

$$A = A_1 = \cdots = A_i = \cdots = \frac{1}{V(\Lambda)}\,.$$

This means that relations (2.25) reduce to a single equation

$$F = A(KF + F_0) \tag{2.31}$$

and

$$F = F^1 = \cdots = F^i = \ldots .$$

The proof of equalities $A_i^{(1)} = \cdots = A_i^{(k)} = \cdots = A$ is analogous.

Obtained above results can be summarized in the following theorem.

Theorem II. *Sequences $F^{(N-i)}$, which satisfy relations (2.24), converge in Boltzmann–Grad limit $N \to \infty$, $a \to 0$, $Na^2 = \lambda =$const and with fixed domain Λ to a sequence F, which satisfy equation (2.31), in the following sense*:

$$\lim_{\substack{N\to\infty, a\to 0\\ Na^2=\lambda}} ||\chi_{a_0}(F - F^{(N-i)})|| = 0\,,$$

where a_0 is arbitrary small fixed number.

Remark that sequence F can be represented, according to (2.29) by convergent in norm of space E_ξ series

$$F = \sum_{i=0}^{\infty}(AK)^i AF_0$$

and from this representation it immediatly follows that sequence F satisfies equation (2.31)

3. The existence of the Boltzmann–Grad limit and thermodynamic limit for standartly normalized distribution functions on numbers of particles

The Kirkwood–Salsburg equations. Consider the equilibrium distribution functions with normalization (2.4). They are defined according to the formulae

$$F_s^{(N)}(q_1,\ldots,q_s) = N(N-1)\ldots(N-s+1) \times \int_{\Lambda^{N-s}} D^{(N)}(q_1\ldots q_s, q_{s+1},\ldots,q_N)\,dq_{s+1}\ldots dq_N \tag{3.1}$$

and satisfy the following Kirkwood–Salsburg relations:

$$F_s^{(N)}(q_1,\ldots,q_s) = \frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)}\exp\left\{-\beta\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\Big[F_{s-1}^{(N-1)}(q_2,\ldots,q_s)$$
$$+\sum_{k=1}^{N-s}\frac{1}{k!}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)F_{s-1+k}^{(N-1)}(q_2,\ldots,q_s,y_1,\ldots,y_k)dy_1\ldots dy_k\Big]\,,$$
$$1 < s < N,$$

$$
\begin{aligned}
F_1^{(N)}(q_1) &= \frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)} \\
&\times \left[1+\sum_{k=1}^{N-1}\frac{1}{k!}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)F_k^{(N-1)}(y_1,\dots,y_k)dy_1\dots dy_k\right], \\
F_N^{(N)}(q_1,\dots,q_N) &= \frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)} \\
&\times \exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1-q_i)\right\}\times F_{N-1}^{(N-1)}(q_2,\dots,q_N)\,. \qquad (3.2)
\end{aligned}
$$

The distribution functions are normalized on numbers of particles.

Define by $a(N,\Lambda)$ the ratio

$$
a(N,\Lambda)=\frac{NQ(N-1,\Lambda)}{Q(N,\Lambda)}
$$

and introduce the renormalized distribution functions

$$
\tilde{F}_s^{(N)}(q_1,\dots,q_s)=a^{2s}F_s^{(N)}(q_1,\dots,q_s)\,. \qquad (3.3)
$$

(We stress that value $a(N,\Lambda)$ (3.3) differs by the multiplier N from value $a(N,\Lambda)$, defined according to (2.9) in Section 2.)

They satisfy the following Kirkwood–Salsburg relations (in what follows, for the sake of simplicity, we preserve the previous notation for the renormalized distribution functions):

$$
\begin{aligned}
F_s^{(N)}(q_1,\dots,q_s) &= a(N,\Lambda)a^2\exp\left\{-\beta\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\Big[F_{s-1}^{(N-1)}(q_2,\dots,q_s) \\
+\sum_{k=1}^{N-s}\frac{1}{k!}\frac{1}{a^{2k}} &\times \int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)F_{s-1+k}^{(N-1)}(q_2,\dots,q_s,y_1,\dots,y_k)dy_1\dots dy_k\Big]\,, \\
&N>s>1\,,
\end{aligned}
$$

$$
\begin{aligned}
F_1^{(N)}(q_1) &= a(N,\Lambda)a^2 \\
&\times \left[1+\sum_{k=1}^{N-1}\frac{1}{k!}\frac{1}{a^{2k}}\int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)F_k^{(N-1)}(y_1,\dots,y_k)dy_1\dots dy_k\right]\,,
\end{aligned}
$$

$$
\begin{aligned}
F_N^{(N)}(q_1,\dots,q_N) &= a(N,\Lambda)a^2 \\
&\times \exp\left\{-\beta\sum_{i=2}^{N}\Phi(q_1-q_i)\right\}\times F_{N-1}^{(N-1)}(q_2,\dots,q_N)\,. \qquad (3.4)
\end{aligned}
$$

Let us represent relations (3.4) in an operator form.

It is useful to extract the term with $k = 0$ and, by analogy with (2.27), to define the operators $K_1^{(N)}$ and $K_1^{(N)}$ in E_ξ as follows:

$$\begin{aligned}
\left(K_1^{(N)}f\right)_s((q)_s) &= \chi_\Lambda((q)_s)\exp\left\{-\beta\sum_{i=1}^{s}\Phi(q_1-q_i)\right\}f_{s-1}((q)_s^1),\\
N\ge s\ge 1, &\qquad f_0=0,\\
\left(K_1^{(N)}f\right)_s((q)_s) &= 0, \qquad s>N,\\
\left(K_2^{(N)}f\right)_s((q)_s) &= \chi_\Lambda((q)_s)\exp\left\{-\beta\sum_{i=2}^{s}\Phi(q_1-q_i)\right\}\sum_{k=1}^{N-s}\frac{1}{k!}\frac{1}{a^{2k}}\\
&\times \int_{\Lambda^k}\prod_{i=1}^{k}\varphi_{q_1}(y_i)f_{s-1+k}((q)_s^1,(y)_k)d(y)_k\,,\\
N-1\ge s\ge 1,&\\
\left(K_2^{(N)}f\right)_s((q)_s) &= 0, \qquad s>N-1\,.
\end{aligned} \tag{3.5}$$

The following estimates for the norms of these operators are obvious:

$$||K_1^{(N)}||=\xi^{-1},\quad ||K_2^{(N)}||\le\xi^{-1}\sum_{k=1}^{N-s}\frac{1}{k!}\left(\frac{4}{3}\pi a\xi\right)^k. \tag{3.6}$$

Relations (3.4) can be written as the single operator equation

$$\begin{aligned}
F^{(N)} &= a(N,\Lambda)a^2\left[\left(K_1^{(N)}+K_2^{(N)}\right)F^{(N-1)}+F_0\right]\\
&\vdots\\
F^{(N-i)} &= a(N-i,\Lambda)a^2\left[\left(K_1^{(N-i)}+K_2^{(N-i)}\right)F^{(N-i-1)}+F_0\right]\\
&\vdots\\
F_0 &= (1,0,\ldots)\,.
\end{aligned} \tag{3.7}$$

We are faced with the following problem:

1) *To show that for* $N\to\infty$, $a\to 0$, $Na^2=\lambda$, $V(\Lambda)\to\infty$, $\frac{N}{V(\Lambda)}=\frac{1}{v}$ *(i.e., in the Boltzmann–Grad limit and the thermodynamic limit (2.17) there exist the limits of the sequences* $F^{(N-i)}$, *of the operators* $K_1^{(N-i)}$, $K_2^{(N-i)}$, *and of the numbers* $a(N-i)a^2$.

2) *To show that corresponding limits* F^i, K_1, K_2, A_i, $i=0,1,2,\ldots$, $A_0=A$, $F^0=F$ *satisfy the equations*

$$F^i = A_i[(K_1+K_2)F^{i+1}+F_0] = A_i(KF^{i+1}+F_0),\quad i=0,1,2,\ldots\,. \tag{3.8}$$

The proof of the existence of the limits of $F^{(N)}$, $K_1^{(N)}$, $K_2^{(N)}$ and $a(N,\Lambda)a^2$. It follows from estimate (3.6) that the operator $K_2^{(N)}$ tends to zero in norm in limit (2.17), $K_2 = 0$.

Define the operators K_1 and χ_a, χ_Λ in E_ξ as follows

$$\begin{aligned}(K_1 f)_s((q)_s) &= f_{s-1}((q)_s),\\ (\chi_a f)_s((q)_s) &= \chi_a((q)_s) f_s((q)_s),\\ (\chi_\Lambda f)_s((q)_s) &= \chi_\Lambda((q)_s) f_s((q)_s),\end{aligned} \tag{3.9}$$

where $\chi_a((q)_s)$ is the characteristic function of admissible configurations.

Since for $|q_i - q_j| > 0$, $(i,j) \in (1,\ldots,s)$,

$$\lim_{a\to 0} \chi_a((q)_s) = 1\,,$$

$$\lim_{a\to 0} \exp\left\{-\beta \sum_{i=2}^{s} \Phi(q_1 - q_i)\right\} = 1\,,$$

the operator $K_1^{(N)}$ converges to the operator K_1 in the following sense (in limit (2.17))

$$||\chi_a \chi_\Lambda (K_1 - K_1^{(N)}) f|| \to 0 \tag{3.10}$$

for arbitrary finite sequence $f \in E_\xi$, i.e, $f_s((q)_s) = 0$ for $s > n$, where n is some number. Consider the sequence $a(N-i,\Lambda)a^2$. The following estimate holds:

$$\begin{aligned} a(N-i,\Lambda)a^2 &< \frac{(N-i)a^2}{V(\Lambda) - (N-i-1)\frac{4}{3}\pi a^3} \le \frac{Na^2}{V(\Lambda) - Na^2 \frac{4}{3}\pi a}\\ &= \frac{\lambda}{V(\Lambda) - \lambda\frac{4}{3}\pi a} = \frac{1}{v} a^2 \frac{1}{1 - \frac{1}{v} a^2 \frac{4}{3}\pi a}\,, \end{aligned} \tag{3.11}$$

which implies that, for fixed bounded $Na^2 = \lambda$ and $V(\Lambda) \to \infty$, the value $a(N-i,\Lambda) \to 0$, i.e., $a(N-i,\Lambda) \to 0$ in limit (2.17).

Repeating the proof of the first section, one obtains that $||F^{(N-i)}|| \to 0$ in limit (2.17).

The obtained above results can be summarized is the following theorem.

Theorem III. *The renormalized distribution functions (3.1), (3.3) tend to zero, i.e.,* $||F^{(N-i)}|| \to 0$ *in the Boltzmann–Grad limit and the thermodynamic limit.*

It is obvious that one can consider the fixed bounded domain Λ. In this case, the values $a(N-i,\Lambda)$ are bounded according to (3.11) and one can repeat the reasoning of the previous section and the theorem II is true for sequence of distribution functions defined according to (3.1) and (3.3).

The existence of the limit distribution functions for $\frac{N}{V}a^2 = \frac{1}{v}a^2 =$ const. One can see from estimate (3.11) that the unique possibility to avoid the convergence of the distribution functions to zero consists in the substitution of the condition $\frac{N}{V}a^2 = \frac{1}{v}a^2 = \text{const}$ for the conditions $Na^2 = \lambda = \text{const}$ in the limit transition (2.17).

Now the Boltzmann–Grad limit and the thermodynamic limit mean that

$$N \to \infty,\ \Lambda \nearrow \mathbf{R}^3,\ V(\Lambda) \to \infty,\ \frac{N}{V} = \frac{1}{v} \to \infty,\ a \to 0,\ \frac{1}{v}a^2 = \text{const}\,. \qquad (3.12)$$

We put $\frac{1}{v}a^2 = \frac{N}{V(\Lambda)}a^2 < 1$. Then $a(N-i,\Lambda)a^2 < 1$, $A_i < 1$, for sufficiently small d. The following theorem is true.

Theorem IV. *The sequences $F^{(N-i)}$ and F^i exist and belong to E_ξ. The sequences $F^{(N-i)}$ converge to the sequences F^i in the limit (3.12) in the following sense*:

$$||\chi_a\chi_\Lambda(F^i - F^{(N-i)})|| \to 0, \quad i = 0,1,2,\dots\,.$$

The sequences F^i satisfy relations (3.8).

P r o o f. Using (3.7) and (3.8), we get

$$\begin{aligned} F^{(N)} &= \sum_{i=1}^{N-3} a(N,\Lambda)a^2K^{(N)}a(N-1,\Lambda)a^2K^{(N-1)}\dots a(N-i+1,\Lambda)a^2 \\ &\times\ K^{(N-i+1)}a(N-i,\Lambda)a^2F_0 + a(N,\Lambda)a^2K^{(N)} \\ &\times\ a(N-1,\Lambda)a^2K^{(N-1)}\dots a(N-i+1,\Lambda)a^2 \\ &\times\ K^{(N-i+1)}\dots a(3,\Lambda)a^2K^{(3)}F^{(2)} + a(N,\Lambda)a^2F_0\,, \qquad (3.13) \\ F &= \sum_{i=1}^{\infty} AKA_1K\dots A_{i-1}KA_iF_0 + AF_0\,. \qquad (3.14) \end{aligned}$$

Since $||K^{(N-i)}|| \le \xi^{-1}\exp\left(\frac{4}{3}\pi a\xi\right)$, $||K|| \le \xi^{-1}$, for sufficiently large ξ and sufficiently small a, one has $||K^{(N-i)}|| < 1$ uniformly with respect to N and i (it suffices to consider $\xi > 1$). Moreover, $a(N-i,\Lambda)a^2 < 1$, $A_i < 1$. Therefore series (3.13), (3.14) are convergent in the norm of E_ξ and the sequences $F^{(N)}$ and F exist and belong to E_ξ. The following estimate holds:

$$\begin{aligned} ||\chi_a\chi_\Lambda(F - F^{(N)})|| &\le ||\sum_{i=1}^{n_0}\chi_a\chi_\Lambda AKA_1K\dots A_{i-1}KA_iF_0 \\ &- \sum_{i=1}^{n_0}\chi_a\chi_\Lambda a(N,\Lambda)a^2K^{(N)}a(N-1,\Lambda)a^2K^{(N-1)} \\ &\dots a(N-i+1,\Lambda)a^2K^{(N-i+1)}a(N-i,\Lambda)a^2F_0 \\ &+ \chi_\Lambda(AF^0 - a(N,\Lambda)a^2F_0)|| + \eta\,, \qquad (3.15) \end{aligned}$$

where $n_0 = n_0(\eta)$ is some bounded number and η can be made as small as desired for sufficiently large $n_0(\eta)$. In sum (3.15) with bounded n_0, we perform the limit transition (3.12) (with fixed $\frac{1}{v}a^2$). For this purpose, we represent all the operators $K^{(N-i)}$ in (3.15) as the sum $K^{(N-i)} = K_1^{(N-i)} + K_2^{(N-i)}$ and take into account that, according to (3.6), $||K_2^{(N-i)}|| \to 0$ and the numbers $a(N-i,\Lambda) \to A_i$ in limit (3.12). Since the sequences $K^{(i)}F_0$ and $K^{(N)}K^{(N-1)}\dots K^{(N-i+1)}F_0$ are finite, using (3.10), we get that the first term in (3.15) tends to zero in limit (3.12).

Thus it is proved that $\chi_a\chi_\Lambda F^{(N)} \to \chi_a\chi_\Lambda F$ in norm in limit (3.12). The proof of the theorem for $F^{(N-i)}$ and F^i $i = 1, 2,\dots$ is the same. Show that

$$A = A_1 = \dots = A_i = \dots, \quad F = F^1 = \dots = F^i = \dots .$$

To do this, we use the relations

$$F_1(q_1) = A, \quad F_1^1(q_1) = A_1, \dots F_1^i(q_1) = A_i, \dots \tag{3.16}$$

and the normalization conditions

$$\frac{1}{V(\Lambda)}\int_\Lambda F_1(q_1)\,dq_1 = \frac{1}{v}a^2, \dots \frac{1}{V(\Lambda)}\int_\Lambda F_1^i(q_1)\,dq_1 = \frac{1}{v}a^2, \dots . \tag{3.17}$$

It follows from relation (3.16), (3.17) that

$$A = A_1 = \dots = A_i = \frac{1}{v}a^2$$

and, consequently,

$$F = F^1 = \dots = F^i = \dots .$$

The sequence F satisfies equation

$$F = A(KF + F_0).$$

It follows from representation (3.14) with $A_i = A$, $i > 1$. It remains to consider the case where the distribution functions are not subjected to the scale transformation.

The proof of the existence of the limit for $F^{(N)}$ without the scale transformation. In this case, the proof is almost the same as in the previous subsection. Estimates (3.6) are valid and it follows from them that $||K_2^{(N)}|| \to 0$ in limit (2.17). The differences will arise only for numbers $a(N,\Lambda)$, which satisfy the inequalities

$$a(N-i,\Lambda) < \frac{N}{V(\Lambda) - N\frac{4}{3}\pi a^3} = \frac{1}{v}\frac{1}{1 - \frac{1}{v}\frac{4}{3}\pi a^3}$$

and therefore are bounded for bounded density. According to this, one can select a convergent subsequence from them. All further proof for $\frac{1}{v} < 1$ is analogous to that performed in the previous section. The proof of the estimate (3.15) and of the uniqueness of the limit distribution functions are the same.

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Предел Больцмана–Грэда для равновесных состояний системы упругих шаров в рамках канонического ансамбля

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В работе доказано существование предела Больцмана–Грэда для равновесных состояний бесконечных систем упругих шаров при различных нормировках функций распределения.

Границя Больцмана–Греда для рівноважних станів системи пружних куль в рамках канонічного ансамблю

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В роботі доведено існування границі Больцмана–Греда для рівноважних станів безмежних систем пружних куль при різних нормуваннях функцій розподілу.