Matematicheskaya fizika, analiz, geometriya
2000, v. 4, No. 1/2, p. 212–247

On isometric reflections in Banach spaces

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Received December 25, 1995

We obtain the following characterization of Hilbert spaces. Let E be a Banach space the unit sphere S of which has a hyperplane of symmetry. Then E is a Hilbert space iff any of the following two conditions is fulfilled:
a) the isometry group $\operatorname{Iso} E$ of E has a dense orbit in S;
b) the identity component G_0 of the group $\operatorname{Iso} E$ endowed with the strong operator topology acts topologically irreducible on E. Some related results on infinite dimensional Coxeter groups generated by isometric reflections are given which allow us to analyse the structure of isometry groups containing sufficiently many reflections.

Dedicated to Professor S. G. Krein
on the occasion of his 80-th birthday

Introduction

Let E be a real Banach space, $S = S(E)$ the unit sphere in E, $\operatorname{Iso} E$ the isometry group of E endowed with the strong operator topology, and $G_0 = G_0(E)$ the identity component of $\operatorname{Iso} E$. *A reflection* in E is an operator of the form $s_{e,e^*} = 1_E - 2e^* \otimes e$, where $e \in E, e^* \in E^*$ and $e^*(e) = 1$. If $s = s_{e,e^*} \in \operatorname{Iso} E$, then one may assume also that $||e||_E = ||e^*||_{E^*} = 1$; in this case we will call e *the reflection vector* and e^* *the reflection functional*; regarding as sphere points, e and $-e$ are called *reflection points*. The unit sphere S is symmetric with respect to *the mirror hyperplane* $\operatorname{Ker} e^*$ of s. It turns out that this imposes strong restrictions on the isometry group $\operatorname{Iso} E$.

We say that a proper subspace $H \subset E$ is *biorthogonally complemented in E* if there exists a bicontractive projection p of E onto H, i.e. such that $||p||_E = = ||1_E - p||_E = 1$.

Theorem 1. *Let s_{e,e^*} be an isometric reflection in E. Let $H = \overline{\mathrm{span}}(G_0\, e)$ be the minimal closed subspace of E containing the orbit $G_0\, e$. Then*

a) H is a Hilbert space and H is biorthogonally complemented in E or $H = E$;

b) furthermore, there exists a projection p of E onto H such that

i) $1_E - 2p \in \mathrm{Iso}\, E$,

ii) $(1_E - p) + \bar{u}p \in \mathrm{Iso}\, E$ *for any* $\bar{u} \in \mathrm{O}(H) = \mathrm{Iso} H$, *and*

iii) any $g \in \mathrm{Iso}\, E$ *such that* $g|H = \bar{u} \in \mathrm{O}(H)$ *has the form* $g = \bar{v}(1_E - p) + \bar{u}p$, *where* $\bar{v} = g|\mathrm{Ker}\, p \in \mathrm{Iso}\,\mathrm{Ker}\, p$;

c) either $G_0 e = e$, or the orbit $G_0\, e$ coincides with the unit sphere $S(H)$ of H.

This subject is related to the following Banach–Mazur rotation problem ([3, p. 242]):

Let E be a separable Banach space such that the group $\mathrm{Iso}\, E$ *acts transitively on the unit sphere S. Is it true that E is a Hilbert space?*

Recall (see [23, Ch. IX, §6]) that the group $\mathrm{Iso}\, L_p$, where $L_p = L_p[0;1]$ and $1 \le p \ne 2 < \infty$, has exactly two orbits on the unit sphere $S_p = S(L_p)$. One of them consists of the functions in S_p with the zero set of a positive measure, and the other one contains the rest. Thus, both orbits are dense in S_p. One says that the group $\mathrm{Iso}\, E$ acts *almost transitively* on S if it has a dense orbit in S. This is the case in the above examples and also in the anisotropic spaces L_{pq}. In a non-separable L_p–space the second of the above two orbits is empty, and thus it is a non-Hilbert Banach space with the isometry group acting transitively on the unit sphere. This shows that the assumption of separability in the Banach–Mazur problem is essential.

Observe that $\mathrm{Iso}\, E$ is a Banach–Lie group. If this group is transitive on the unit sphere S, then S is a homogeneuos space of $\mathrm{Iso}\, E$. If in addition S has a hyperplane of symmetry L, it should be a symmetric space. Indeed, L is a mirror hyperplane of an isometric reflection. The unit sphere S having a reflection point, by transitivity each point $x \in S$ should be a reflection point of an isometric reflection $s = s_{x,x^*}$. Furthermore, x is an isolated fixed point of the involution $-s\,|\,S$ which acts as -1 at the supporting hyperplane $x^* = 1$ to S at x (we are grateful to J. Arazy for this remark). From Theorem 2 below it follows that S being a symmetric space of the group $\mathrm{Iso}\, E$, E should be a Hilbert space. In fact, in Theorem 2 more strong criteria for E to be a Hilbert space are done. They hold without the separability assumption.

Theorem 2. *Let the group* $\mathrm{Iso}\, E$ *contains a reflection s_{e,e^*} along the vector $e \in S$. Then E is a Hilbert space iff either of the following two conditions is fulfilled:*

a) $\mathrm{Iso}\,E$ *acts almost transitively on* S*;*

b) e *is a cyclic vector of the strong identity component* G_0 *of* $\mathrm{Iso}\,E$ *(i.e.,* $E = \overline{\mathrm{span}}(G_0\,e)$*).*

The second statement is a corollary of Theorem 1; the first one, being much simpler, is proved along the same lines.

By a theorem of Godement [9], any isometric operator in a Banach space has a non-trivial invariant subspace (see also [28] for a more general fact). From Theorem 1 one obtains the following

Corollary. *Let* E *be a non-Hilbert Banach space. If there is an isometric reflection* s_{e,e^*} *in* E*, then all operators in* $G_0(E)$ *have a common non-trivial invariant Hilbert subspace* H*, biorthogonally complemented in* E*. Moreover, if* $G_0(E) \neq \{e\}$*, then* $\dim H > 1$*. In particular, in this case there is a biorthogonally complemented euclidean plane in* E*.*

Note that, by a theorem of Yu. Lyubich [20], if a finite dimensional Banach space has an infinite isometry group, i.e., if the group $G_0(E)$ is non-trivial, then E has a euclidean plane L with a contractive projection $p : E \to L$ (in this case L is called *orthogonally complemented* in E) (see also [16, 21]). On the other hand, there are Banach spaces of infinite dimension with big isometry groups, but without any orthogonally complemented euclidean subspace of dimension greater than 1. Indeed, $L_p = L_p[0; 1]$, where $1 < p \neq 2 < \infty$, contains no such a subspace, whereas the group G_0 is non-trivial. Furthermore, there is no bicontractive projection of L_p $(p \neq 2)$ onto a hyperplane [13, 14]; in particular, there is no isometric reflection. The same is true in general for rearrangement-invariant (r.i.) ideal Banach latticies, or symmetric spaces, of (classes of) measurable functions different from L_2 [14, Theorem 4.4]. Recall [17, 19] that a r.i. (or symmetric) space E on the interval $[0; 1]$ satisfies the following axioms:

1) $1 \in E$ and $||1||_E = 1$.

2) For any measure preserving transformation α of the interval [0;1] the *shift operator* $T_\alpha : x(t) \to x(\alpha(t))$ acts isometrically in E.

3) If $x(t) \in E$ and $|y(t)| \leq |x(t)|$ a.e., then $y(t) \in E$ and $||y(t)||_E \leq ||x(t)||_E$.

If E is a r.i. space different from L_2, then every $g \in \mathrm{Iso}\,E$ has *a weighted shift representation* $g : x(t) \to h(t)x(\phi(t))$, where $h = g(1) \in E$ and ϕ is a transformation of $[0; 1]$ preserving measurability (see [30, 31] for the complex case and [13, 14] for the real one; see also [1, 18, 22, 29]. As for symmetric sequence spaces, see [23, Ch.IX; 2, 6, 8]). Furthermore, ϕ should be measure–preserving except in the case where E coincides with some of the L_p probably endowed with a new equivalent norm [30] (see also [14, 18, 22]). In particular, this shows that L_p are the only r.i. spaces where the orbits of the isometry group are dense in the unit sphere.

The content of the paper is the following. Section 1 contains a preliminary finite dimensional version of Theorem 1. The proofs of Theorems 1 and 2 are given in Section 2. Besides, Section 2 contains a version of Theorem 1 where no operator topology is prescribed (see Theorem 2.10). In Section 3 we classify the Coxeter groups in infinite dimensional case (probably, this classification is not new). In Sections 4 and 5 we consider Banach spaces possessing total families of isometric reflections. A kind of a structure theorem for isometry groups is proven (Theorem 5.7). It addresses to the notions of Hilbert and Coxeter partial orthogonal subspace decompositions, introduced earlier in this section. In the last section we give an application to isometry groups of the ideal generalized sequence spaces.

Some results of this paper were previously announced in [26, 27].

1. Isometric reflections in finite dimensional Banach spaces

Let A be a set of reflections in a real vector space E and W be the group generated by the reflections in A. Denote by $\Gamma_{W,A}$ the Coxeter graph of W. Recall [5] that $\Gamma_{W,A}$ has A as the set of vertices; two vertices are connected by an edge iff the corresponding reflections do not commute. By Γ_W we denote the full Coxeter graph of W, i.e., $\Gamma_W = \Gamma_{W,R}$, where $R = R(W)$ is the set of all the reflections in W.

1.1. Lemma ([5, Ch. V, 3.7]). *A group W generated by a set A of orthogonal reflections in $\mathbf{R}^n$ is irreducible iff the origin is the only fixed point of W and the Coxeter graph $\Gamma_{W,A}$ is connected. In particular, Γ_W is connected iff its subgraph $\Gamma_{W,A}$ is connected.*

Let E be a finite dimensional Banach space. Then $\operatorname{Iso} E$ is a compact Lie group, and there exists a scalar product in E invariant with respect to $\operatorname{Iso} E$. It can be defined, for instance, by averaging of any given scalar product over the Haar measure on $\operatorname{Iso} E$. In general, such an invariant scalar product is not unique. Being orthogonal, two isometric reflections in E along the vectors $e_1, e_2 \in S(E)$ commute iff either $e_1 = \pm e_2$ or $e_1 \perp e_2$.

The proof of the following lemma is simple and can be omited.

1.2. Lemma. *Let a connected submanifold M of $\mathbf{R}^n$ be invariant under a reflection s_{e,e^*} which fixes a point $x \in M$ and acts identically on the tangent space T_xM. Then M is contained in the mirror hyperplane $\operatorname{Ker} e^*$.*

The main result of this section is the following

1.3. Proposition. *Let E be a real Banach space of dimension n. Let $G \subset \operatorname{Iso} E$ be a closed subgroup of a positive dimension which contains reflections $t_1, \ldots, t_n$ along linearly independent vectors $e_1, \ldots, e_n$. Then there exists a subspace $H \subset E$ such that* *a)* $\dim H \geq 2$, *H is euclidean and biorthogonally complemented in E;*

b) the unit sphere $S(H)$ of H coincides with an orbit of the identity component G_0 of G;

c) there exists a projection p of E onto H such that $1_E - 2p \in G$ and p commutes with any reflection $t \in G$. Furthermore, $(1_E - p) + \bar{u}p \in G$ for any $\bar{u} \in O(H)$.

P r o o f. Fix an invariant scalar product in E and identify E with $\mathbf{R}^n$ in such a way that $\mathrm{Iso}\, E \subset O(n)$. Let $u_1, \ldots, u_n$ be the system of vectors in $\mathbf{R}^n$ biorthogonal to the system $e_1, \ldots, e_n$. Since $\dim G > 0$, the orbit Gu_i has a positive dimension for at least one value of i, say for $i = 1$. We may also assume that $u_1 \in S^{n-1}$, where S^{n-1} is the euclidean unit sphere in $\mathbf{R}^n$. Let M be the connected component of the orbit Gu_1 which contains u_1. Since u_1 is fixed by any of the reflections $t_i, i = 2, \ldots, n$, M is invariant under these reflections, and hence the tangent space $T = T_{u_1}M$ is invariant, too. Thus for each $i = 2, \ldots, n$ either $e_i \in T$ or $e_i \perp T$. Put

$$A = \{i \in \{2, \ldots, n\} \mid e_i \in T\}$$

and

$$B = \{i \in \{2, \ldots, n\} \mid e_i \perp T\} \ .$$

Since $G \subset O(n)$ and $u_1 \in S^{n-1}$, we have $M \subset S^{n-1}$, and so $T \subset T_{u_1}S^{n-1}$. Therefore $T \perp u_1$. It follows that $T \subset L$, where $L = \mathrm{span}(e_2, \ldots, e_n)$, and therefore $T = \mathrm{span}(e_i \mid i \in A)$ (hereafter *span* means *the linear span*).

Thus, $\dim M = \dim T = \mathrm{card} A$. Since M is t_i-invariant for $i \in B$, by Lemma 1.2, M is contained in the subspace $H = \{v \in E \mid v \perp e_i, i \in B\}$. It is easily seen that $k = \dim H = \mathrm{card} A + 1 = \dim M + 1$. Thus M is a closed submanifold of each of the unit spheres $S_r(H) = S_r(E) \cap H$, where $r = ||u_1||_E$, and $S^{k-1} = S^{n-1} \cap H$, of the same dimension $\dim M = \dim H - 1 = k - 1$. Hence M coincides with both of them. At the same time, being connected M coincides with the orbit $G_0 u_1$. Here $k \geq 2$, since $\dim M > 0$. Therefore, H is euclidean and the unit sphere $S(H)$ coincides with the orbit $G_0(u_1/r)$.

Since $T \subset H$ and $e_i \in T$ for each $i \in A$, where $\mathrm{card} A = k - 1 > 0$, there exists $i_0 \in A$ such that $e_{i_0} \in S(H)$, and thus $S(H) = G_0 e_{i_0}$. Let $w_1, \ldots, w_k \in H$ be an orthogonal basis in H with $||w_i||_E = 1, i = 1, \ldots, k$, and $g_1, \ldots, g_k \in G$ be such that $g_j(e_{i_0}) = w_j$. Then $s_j = g_j t_{i_0} g_j^{-1} \in G$ is the orthogonal reflection along the vector $w_j, j = 1, \ldots, k$. By the same reasoning as above, for any vector $w \in S(H)$ the orthogonal reflection s_{w,w^*} along w belongs to G.

The reflections $s_j, j = 1, \ldots, k$, pairwise commute, and so $p = \frac{1}{2}(1_E - \prod_{i=1}^k s_i)$ is the orthogonal projection of E onto H such that $\tau = 1_E - 2p = \prod_{i=1}^k s_i \in G \subset \mathrm{Iso}\, E$. Thus, $||p||_E = \frac{1}{2}||1_E - \tau||_E = 1$ and either $E = H$ or $||1_E - p||_E = \frac{1}{2}||1_E + \tau||_E = 1$. Therefore, H is a biorthogonally complemented subspace of E.

Any orthogonal reflection $\bar{s}$ in H coincides with the restriction to H of some reflection $s \in G$; actually, $s = (1_E - p) + \bar{s}p$. The same is true for any orthogonal operator $\bar{u} \in O(H)$; indeed, the group $O(H)$ is generated by orthogonal reflections.

Let $t \in G$ be a reflection. The mirror hyperplane of t intersects with H in a subspace of H of dimension $k - 1 > 0$. Therefore, t has a fixed point on the sphere $M = S^{k-1} \subset H$, and so $t(M) \cup M$ is connected and contained in the orbit Gu_1. It follows that $t(M) = M$, H is invariant with respect to t and so t and p commute. This completes the proof. ■

1.4. Corollary. *Let W be a group generated by isometric reflections in a finite dimensional Banach space E. If W is irreducible and infinite, then E is euclidean and W is dense in the orthogonal group* $\operatorname{Iso} E \approx O(n)$, $n = \dim E$.

P r o o f. Let G be the closure of W in $\operatorname{Iso} E$ and G_0 be the identity component of G. Since W is irreducible, by Lemma 1.1, it contains n reflections along linearly independent vectors, and the Coxeter graph Γ_W is connected. Let H be the euclidean subspace of E constructed in Proposition 1.3. Since by (c), H is invariant with respect to the reflections from W, for each $s_{e,e^*} \in W$ either $e \in H$ or $e \perp H$. If A resp. B is the set of reflections from W of the first resp. second type, then each element of A commutes with every element of B. By the connectedness of the graph Γ_W, one of the sets A and B should be empty. This shows that $H = E$. By (c), $\bar{u} \in G$ for any $\bar{u} \in O(H)$. Therefore, $G = O(H)$ and we are done. ■

R e m a r k. Related results can be found, e.g., in [6; 10, (1.7); 23, 25].

2. Proofs of Theorems 1 and 2

2.1. Definition. Let E be a real Banach space, and let s_1, s_2 be two isometric reflections in E along linearly independent vectors e_1, $e_2 \in S = S(E)$. Denote by $\alpha(s_1, s_2)$ the minimal positive angle between the lines containing e_1 and e_2, measured with respect to an invariant inner product in the plane $L = \operatorname{span}(e_1, e_2)$. Put $\alpha(s_1, s_2) = 0$ iff $e_1 = \pm e_2$.

2.2. R e m a r k s. a) It is easily seen that the above definition does not depend on the choice of an invariant scalar product in L.
b) An isometric reflection $s = s_{e,e^*}$ in E is uniquely defined by the reflection point $e \in S(E)$. Indeed, this is true for the restriction of s to any finite dimensional subspace F containing e, since the mirror hyperplane $\operatorname{Ker} e^* \cap F$ of $s|F$ is orthogonal to e with respect to an invariant scalar product on F. Thus, this is true for s itself.

c) Two isometric reflections s_1 and s_2 commute iff either $\alpha(s_1, s_2) = 0$, i.e., $e_1 = \pm e_2$ or $\alpha(s_1, s_2) = \frac{\pi}{2}$, i.e., $e_1 \perp e_2$ in L.

2.3. Lemma. *Let $s_i = s_{e_i, e_i{}^*}, i = 1, 2$, be two isometric reflections in E. Then*

$$\cos^2 \alpha(s_1, s_2) = e_1{}^*(e_2) e_2{}^*(e_1).$$

P r o o f. This is evidently true if $e_1 = \pm e_2$. Assume, further, that e_1 and e_2 are linearly independent. Let an invariant scalar product in the plane $L = \mathrm{span}(e_1, e_2)$ be given by the bilinear form $B = \begin{pmatrix} b & a \\ a & c \end{pmatrix}$ with respect to the basis (e_1, e_2) in L. Consider the orthogonal projection $p_i = \frac{1}{2}(1_L + s_i|L)$ of L onto the mirror line l_i of the axial reflection $s_i|L$, $i = 1, 2$. Since $p_i(e_j) \perp e_i$ for $j \neq i$, we have

$$0 = B(p_1(e_2), e_1) = B(e_2 - e_1{}^*(e_2)e_1, e_1) = a - e_1{}^*(e_2) b$$

and

$$0 = B(p_2(e_1), e_2) = a - e_2{}^*(e_1) c \ .$$

Thus

$$a^2 = e_1{}^*(e_2) e_2{}^*(e_1) bc \ ,$$

and so

$$\cos^2 \alpha(s_1, s_2) = \frac{a^2}{bc} = e_1{}^*(e_2) e_2{}^*(e_1) \ .$$

■

2.4. Corollary.

$$\cos \alpha(s_1, s_2) \geq 1 - ||e_1 - e_2||_E.$$

In particular, if $e_1 \neq e_2$ and $||e_1 - e_2||_E < 1$, then s_1 and s_2 do not commute.

P r o o f. Since $s_i \in \mathrm{Iso}\, E$, and so $||e_i||_E = ||e_i^*||_{E^*} = e_i^*(e_i) = 1$, we have

$$|1 - e_1^*(e_2)| = |e_1^*(e_1 - e_2)| \leq ||e_1 - e_2||_E$$

and

$$|1 - e_2^*(e_1)| \leq ||e_1 - e_2||_E \ .$$

We can assume that $||e_1 - e_2||_E < 1$. Then from the above inequalities we obtain

$$|e_1^*(e_2)| \geq 1 - ||e_1 - e_2||_E$$

and

$$|e_2^*(e_1)| \geq 1 - ||e_1 - e_2||_E \ .$$

The desired inequality follows from the latter two by multiplying them and making use of Lemma 2.3. ■

2.5. Lemma. *Let* $s = s_{e,e^*} \in \operatorname{Iso} E$. *Consider the function on* $\operatorname{Iso} E \times \operatorname{Iso} E$

$$\phi_s(g_1, g_2) = \sin^2 \alpha(s_1, s_2) ,$$

where $s_i = g_i s g_i{}^{-1}$, $i = 1, 2$. *Then*

a) ϕ_s *is left invariant, i.e.,*

$\phi_s(g_1, g_2) = \phi_s(gg_1, gg_2) = \phi_s(1_E, g_1{}^{-1} g_2)$

for each $g, g_1, g_2 \in \operatorname{Iso} E$.

b)

$\phi_s(g_1, g_2) = 1 - e^*(g_1{}^{-1} g_2(e)) e^*(g_2{}^{-1} g_1(e))$.

Therefore, ϕ_s *is continuous on* $(\operatorname{Iso} E)^2$ *in the strong operator topology.*

c) For any two elements $g', g'' \in G_0$ *such that* $\phi_s(g', g'') > 0$, *and for any* ϵ, $0 < \epsilon < 1$, *one can find a finite chain of elements* $h_0 = g', h_1, \ldots, h_n = g''$ *with the property* $0 < \phi_s(h_i, h_{i+1}) < \epsilon$, *so that the reflections* $t_i = h_i s h_i{}^{-1}$ *and* t_{i+1} *do not commute for all* $i = 0, 1, \ldots, n-1$.

P r o o f. (a) is evident. The identity in (b) easily follows from the equality

$$\phi_s(g_1, g_2) = 1 - e^*(g_1{}^{-1} g_2(e))(g_1{}^{-1} g_2)^*(e^*)(e) \ ,$$

which follows from (a) and Lemma 2.3. The second statement of (b) is true since $\operatorname{Iso} E$ is a topological group with respect to the strong operator topology. To prove (c), consider the covering of G_0 by the open subsets

$$U_\epsilon(g) = \{h \in G_0 \mid \phi_s(g, h) < \epsilon\} \ .$$

Since G_0 is connected, any two of them $U_\epsilon(g')$ and $U_\epsilon(g'')$ can be connected by a finite chain of such subsets, and the assertion follows. ■

2.6. Proposition. *Let* $s = s_{e,e^*} \in \operatorname{Iso} E$, $g_1, \ldots, g_n \in G_0$ *and* $H' = \operatorname{span}(e_1, \ldots, e_n)$, *where* $e_i = g_i(e)$, $i = 1, \ldots, n$. *Then*

a) H' *is euclidean;*

b) there exists a unique projection p' *of* E *onto* H' *such that* $1_E - 2p \in \operatorname{Iso} E$;

c) the unit sphere $S(H')$ *of* H' *is contained in the orbit** $G_0\, e$, *and for each vector* $v \in S(H')$ *there exists a reflection* $s_{v,v^*} \in \operatorname{Iso} E$ *along* v *commuting with* p'.

P r o o f. First we construct a finite dimensional subspace F containing H' which satisfies all the properties of a), b), c) above.

*Of course, except of the case where $\dim H' = 1$.

Put $g_0 = 1_E$, and for each pair (g_i, g_{i+1}), $i = 0, \dots, n-1$, find a chain $\{h_{ij}\}_{j=0}^{n_i}$ as in Lemma 2.5.c) above. The proposition is evident in the case where $\dim H' = 1$, and so we may assume that $g_i(e) \neq e$ for at least one value i_0 of i. Since the continuous function ϕ_s takes all its intermediate values on G_0, we can also choose the element $h = h_{i_0,1}$ in such a way that the angle $\alpha(s, hsh^{-1})$ is irrational modulo π, and thus the group generated by the reflections s and hsh^{-1} is infinite.

Put $F = \mathrm{span}(h_{ij}(e) \,|\, j = 0, \dots, n_i, i = 0, \dots, n)$. Let W be the group generated by the reflections $\{t_{ij}|F\}$ in F, where $t_{ij} = h_{ij}sh_{ij}^{-1}, j = 0, \dots, n_i, i = 0, \dots, n$. It is clear that the origin is the only fixed point of W in F. Since, by the construction, the Coxeter graph Γ_W of W is connected, by Lemma 1.1, W is irreducible. W being infinite, by Corrolary 1. 4, the subspace F is euclidean and the closure of W coincides with the group $\mathrm{Iso}\, E = O(F)$.

Let $v_1, \dots, v_l$, where $l = \dim F$, be a basis of F chosen from the system $(h_{ij}(e))$, and $t_k = t_{v_k, v_k{}^*}, k = 1, \dots, l$, be the corresponding reflections from the system (t_{ij}). Put $M = \bigcap_{k=1}^{l} \mathrm{Ker}\, v_k{}^*$. It is easily seen that $E = M \oplus F$. Let F' be a finite dimensional subspace of E containing F, endowed with an invariant scalar product. Then for each $k = 1, \dots, l$ the restriction $t_k' = t_k|F'$ is an orthogonal reflection in F', and so $v_k \perp (\mathrm{Ker}\, v_k{}^* \cap F')$. Therefore, $F \perp (M \cap F')$. It follows that each of the vectors $h_{ij}(e) \in F$ is orthogonal to $M \cap F'$, too, so that the restriction $t_{ij}|(M \cap F')$ is the identity mapping. This yields the presentation $t_{ij} = (1_E - p) + t_{ij}p$, where p is the projection of E onto F along M. Thus, each element $\bar{g} \in W$ can be presented as the restriction to F of the isometry $g = (1_E - p) + \bar{g}p \in \mathrm{Iso}\, E$. If a sequence $\bar{g}_i \in W$ converges to an element $\bar{h} \in O(F)$, then the sequence of extensions g_i converges to the extension $h = (1_E - p) + \bar{h}p$ of $\bar{h}$, where $h \in \mathrm{Iso}\, E$. In particular, in this way each orthogonal reflection in F extends to a unique isometric reflection in E, and each element $\bar{u} \in O(F)$ extends to the unique isometry $u = (1_E - p) + \bar{u}p \in \mathrm{Iso}\, E$. It follows that $S(F) \subset G_0(e)$.

Let $f_1, \dots, f_l$ be an orthogonal basis in F and $\bar{s}_1, \dots, \bar{s}_l$ be the orthogonal reflections in F along these vectors. It is easily seen that then $p = \frac{1}{2}(1_E - \prod_{i=1}^{l} s_i)$, and thus $1_E - 2p = \prod_{i=1}^{l} s_i \in \mathrm{Iso}\, E$. If $s' \in \mathrm{Iso}\, E$ is a reflection along a vector $v' \in S(F)$, then as above $s' = (1_E - p) + s'p = (1_E - p) + ps'p$, and so s' and p commute.

It is evident that the subspace $H' \subset F$ has the same properties as F itself, and therefore a), b), c) are fulfilled. ■

2.7. R e m a r k. It is easily seen that if $H' \subset H''$ are two subspaces as in Proposition 2.6, then for the corresponding projections p', p'' we have $p' \prec p''$, i.e., $p'p'' = p'(= p''p')$.

2.8. P r o o f o f T h e o r e m 1.a). Let x, y be two arbitrary vectors in H. Then for any $\epsilon > 0$ in the linear span of the orbit G_0e there exist two vectors x_ϵ, y_ϵ such that $||x - x_\epsilon||_E < \epsilon, ||y - y_\epsilon||_E < \epsilon$. Let $x_\epsilon = \sum_{i=1}^{n} a_i g_i(e)$ and

$y_\epsilon = \sum_{i=1}^n b_i g_i(e)$, where $g_i \in G_0, i = 1, \ldots, n$. Put $H' = \text{span}(g_i(e)|i = 1, \ldots, n)$. By Proposition 2.6, the subspace H' is euclidean, and therefore the norm in H' satisfies the four squares identity. In particular,

$$||x_\epsilon + y_\epsilon||^2 + ||x_\epsilon - y_\epsilon||^2 = 2(||x_\epsilon||^2 + ||y_\epsilon||^2).$$

Passing to the limit, we see that the same identity holds for $x, y \in H$. It follows that H is a Hilbert space (see [7, Ch.7, §3]).

Consider further the family of all finite dimensional subspaces H' which belong to the linear span of the orbit $G_0 e$. Let $\mathcal{P} = \{p'\}$ be the corresponding partially ordered family of finite dimensional projections $E \to H'$ such that $1_E - 2p' \in \text{Iso}\, E$. For a fixed vector $v \in S(E)$ and for each $p' \in P$ consider the subset

$$Y_{p'} = \omega\{p''(v) \mid p'' \in P,\ p' \prec p''\}\ ,$$

where ω denotes the closure with respect to the weak topology in E. The family $\{Y_{p'}\}$ has the property that for each finite system of projections $p'_1, \ldots, p'_n \in P$ the intersection $\cap_{i=1}^n Y_{p'_i}$ is non-empty. Indeed, let $H_0 = \text{span}(\text{Im}\, p'_1, \ldots, \text{Im}\, p'_n)$, and let $p'_0 \in \mathcal{P}$ be the corresponding projection of E onto H_0. Then $p'_i \prec p'_0$, and hence $p'_0(v) \in Y_{p'_i}$ for each $i = 1, \ldots, n$. Since H is a Hilbert space, the unit ball $B(H)$ of H is weakly compact. It follows that the centralized family $\{Y_{p'}\}_{p' \in \mathcal{P}}$ of weakly closed subsets of $B(H)$ has a non-empty intersection. By the Barry theorem [4], the generalized sequence of projections $\mathcal{P} = (p')$ converges in the strong operator topology to its upper bound p which is a projection of E onto H, and which satisfies the condition $i)$ $1_E - 2p \in \text{Iso}\, E$. In particular, H is biorthogonally complemented in E. This proves $a)$.

P r o o f o f T h e o r e m 1.b), c). Let $s' = s_{x,x^*} \in \text{Iso}\, E$, where $v \in G_0 e$. Then s' commutes with any projection $p' \in \mathcal{P}$ such that $p'(x) = x$, which means that $\text{Ker}\, p' \subset \text{Ker}\, x^*$. Passing to the limit, we see that p commutes with s' and $\text{Ker}\, p \subset \text{Ker}\, x^*$, too. It follows that $s' = (1_E - p) + s'p$.

Let $x_0 \in S(H)$ be the limit of a generalized sequence of vectors $x_\alpha \in G_0 e \cap S(H)$. Then the corresponding sequence of isometric reflections $s_\alpha = s_{x_\alpha, x_\alpha^*} = g_\alpha s g_\alpha^{-1}$, where $g_\alpha \in G_0$ and $g_\alpha(e) = x_\alpha$, is strongly convergent to the reflection $s_0 = s_{x_0, x_0^*} \in \text{Iso}\, E$. Indeed, from the representation $s_\alpha = (1_E - p) + s_\alpha p$ it easily follows that the generalized sequence $\{s_\alpha\}$ converges to $s_0 = (1_E - p) + s_0 p$ on each of the complementary subspaces $\text{Ker}\, p$ and H.

Let $\bar{u} \in O(H)$ and $x \in E$. Consider the extension $u = (1_E - p) + \bar{u}p$ of $\bar{u}$ to E. Since $||p(x)||_E = ||up(x)||_E$, there exists an orthogonal reflection $\bar{s}_0$ in H such that $\bar{s}_0 p(x) = up(x)$. Let $s_0 = (1_E - p) + \bar{s}_0 p \in \text{Iso}\, E$. Then we have $||x||_E = ||s_0(x)||_E = ||(1_E - p)(x) + \bar{s}_0 p(x)||_E = ||u(x)||_E$. Therefore, $u \in \text{Iso}\, E$, and thus $ii)$ is fulfilled.

Now it is clear that the orbit $G_0 e \subset S(H)$ contains the orbit of the strong identity component of the orthogonal group $O(H)$, and so it coincides with the unit sphere $S(H)$ if dim $H > 1$. This proves *c*).

Let $g \in \mathrm{Iso}\, E$ be such that $\bar{u} = g|H \in O(H)$. We will show that g leaves the subspace $\mathrm{Ker}\, p$ invariant and thus $\bar{v} = g|\mathrm{Ker}\, p \in \mathrm{Iso}(\mathrm{Ker}\, p)$ and $g = \bar{v}(1_E - p) + \bar{u}p$.

Suppose that $g(\mathrm{Ker}\, p) \not\subset \mathrm{Ker}\, p$. Consider the operator $g_1 = gu^{-1} \in \mathrm{Iso}\, E$. We have $g_1|H = 1_H$ and $g_1|\mathrm{Ker}\, p = g|\mathrm{Ker}\, p$. By our assumption, there exists a vector $x \in \mathrm{Ker}\, p$ such that $g_1(x) \notin \mathrm{Ker}\, p$, and so $pg_1(x) \neq 0$. Denote $x_1 = (1_E - p)g_1(x)$ and $x_2 = pg_1(x)$. Then $g_1(x) = x_1 + x_2$, hence ${g_1}^{-1}(x_1) = {g_1}^{-1}(g_1(x) - x_2) = x - x_2$.

Consider two functions $\phi(t) = ||x_1 + tx_2||_E$ and $\psi(t) = ||x + tx_2||_E$. Since $1_E - 2p \in \mathrm{Iso}\, E, x, x_1 \in \mathrm{Ker}\, p$ and $x_2 \in \mathrm{Im}\, p$, we have $||x_1 + tx_2|| = ||x_1 - tx_2||$ and $||x + tx_2|| = ||x - tx_2||$. Thus, both ϕ and ψ are even functions. From the equalities

$$ {g_1}^{-1}(x_1 + tx_2) = x - x_1 + tx_2 = x + (1-t)x_2 $$

and

$$ g_1(x + tx_2) = x_1 + (1+t)x_2\,, $$

and the fact that $g_1 \in \mathrm{Iso}\, E$ we obtain that $\phi(t) = \psi(1-t)$ and $\psi(t) = \phi(1+t)$. It follows that $\phi(t) = \phi(-t) = \psi(1+t) = \phi(t+2)$. Therefore, being convex and periodic function on $\mathbf{R}$, $\phi(t)$ should be constant. This is possible only if $x_2 = 0$, i.e., $g_1(x) \in \mathrm{Ker}\, p$, which is a contradiction. Thus, *iii*) is fulfilled as well. This completes the proof of Theorem 1. ∎

It has been already noted that the statement of Theorem 2.b) is a direct corollary of Theorem 1. Thus, it is enough to prove Theorem 2.a).

2.9. Proof of Theorem 2.a). It is enough to show that the four squares identity holds in E. For this it is enough, as it was done in the proof of Theorem 1, to approximate an arbitrary pair of vectors $x, y \in E$ by a sequence $\{(x_\alpha, y_\alpha)\}$ of pairs of vectors belonging to finite dimensional euclidean subspaces H_α of E. In turn, it is enough to show that any pair of vectors in the linear span of the orbit Ge of the group $G = \mathrm{Iso}\, E$ belongs to a finite dimensional euclidean subspace H' of E. Indeed, it is easily seen that under our assumptions any orbit of G in the unit sphere $S = S(E)$ is dense in S. In particular, the orbit Ge is dense in S.

Fix such a pair $x, y \in \mathrm{span}(Ge)$ and consider a subspace $H' = \mathrm{span}(g_1(e), \ldots, g_n(e))$, $g_i \in G, i = 1, \ldots, n$, containing this pair. Since the orbit Ge is dense in S, for any two vectors $g'(e)$ and $g''(e) \in G\,e$ one can find a finite chain of vectors $h_j(e) \in G\,e, j = 0, \ldots, k$, such that $h_0(e) = g'(e), h_k(e) = g''(e)$ and $||h_{j+1}(e) - h_j(e)||_E < \frac{1}{10}$, $j = 0, 1, \ldots, k-1$. Find such a chain $\{h_{ij}(e)\}_{j=1,\ldots,k_i}$ for each of the pairs $(g_i(e), g_{i+1}(e)), i = 1, \ldots, n-1$, and put $F = \mathrm{span}(h_{ij}(e), j = 0, \ldots, k_i, i =$

$1, \ldots, k-1$). We may assume that E is infinite dimensional (otherwise the proof is simple), and that $\dim L > 8$. Let $\{t_{ij} = h_{ij} s h_{ij}{}^{-1}\}$ be the system of isometric reflections along the vectors $h_{ij}(e), j = 1, \ldots, k_i, i = 1, \ldots, n-1$, and let W be the group generated by the restrictions $t_{ij}|F$. By Corollary 2.4, the Coxeter graph Γ_W is connected, and since the system of vectors $(h_{ij}(e))$ is complete in F, by Lemma 1.1, the group W is irreducible. For a pair of vectors $(v' = h_{ij}(e), v'' = h_{ij+1}(e))$ we have $0 < ||v' - v''|| < \frac{1}{10}$, and so, by Corollary 2.4,

$$0 < \alpha(t_{ij}, t_{ij+1}) < \arccos \frac{9}{10} .$$

From the classification of Coxeter groups [5, Ch.VI, Sect. 4] it follows that the group W is infinite. Thus, by Corollary 1.4, the subspaces F and $H' \subset F$ are euclidean. The theorem is proved. ∎

A priori, the strong operator topology could be still too strong in order that the identity component G_0 be big enough to apply Theorem 1 in an efficient way. Next we give a version of Theorem 1 which does not involve any operator topology.

Recall that a group G is *locally finite* if every finitely generated subgroup of G is finite.

2.10. Theorem. *Let $s = s_{e,e^*}$ be an isometric reflection in a Banach space E. Denote*

$$U = \{g \in \operatorname{Iso} E \,|\, [s, g^{-1}sg] \neq 1_E\} \ .$$

Let G_1 be the subgroup of $\operatorname{Iso} E$ *generated by U, and $H = \overline{\operatorname{span}}\,(G_1\, e)$. If any of the following two conditions i), ii) is fulfilled, then all the conclusions a), b), c) of Theorem 1 hold:*

i) The group W_1 generated by the set of reflections $\mathrm{IR}_1 = \{g^{-1}sg\}_{g\in G_1}$ *is not locally finite.*

ii) The orbit $G_1\, e$ contains three linearly independent vectors e_1, e_2, e_3, where $||e_1 - e_2||_E < 1 - \cos \pi/5$.

P r o o f. Repeating the arguments used in the proofs of Theorems 1 and 2, it is enough to show that for each finite subset $\sigma \subset \mathrm{IR}_1$ there exists another finite subset $\sigma_1 \subset \mathrm{IR}_1$ such that $\sigma \subset \sigma_1$ and the group generated by the reflections from σ_1, as well as its restriction to the subspace $\operatorname{span}(v \,|\, s_{v,v^*} \in \mathrm{IR}_1)$, is infinite. In other words, each finite subgraph γ of the Coxeter graph Γ_{W_1} should be contained in a finite connected subgraph $\gamma_1 \subset \Gamma_{W_1}$ with the following property: the group $W(\gamma_1)$ generated by the reflections which correspond to the vertices of γ_1, is infinite. The latter holds as soon as the Coxeter graph Γ_{W_1} is connected and contains a finite subgraph γ_0 such that the group $W(\gamma_0)$ is infinite. If the first of these conditions is fulfilled, than the second one follows from each of the assumptions *i*) and *ii*) above. Indeed, as for *i*), it is clear. As for *ii*), by the connectedness of the graph Γ_{W_1}, one can find a finite connected subgraph $\gamma_0 \subset \Gamma_{W_1}$ which contains three vertices corresponding to the reflections $s_1, s_2, s_3 \in \mathrm{IR}_1$ with the reflection vectors e_1, e_2, e_3, resp. From the classification of finite Coxeter groups [5, Ch.VI, Sect. 4] it follows that the group $W(\gamma_0)$ is infinite. Eventually, if $V(\gamma_0)$ is the subspace generated by the reflection vectors of the reflections from $W(\gamma_0)$, then $\dim V(\gamma_0) \geq 3$ and, by Corollary 2.3, the order of the rotation $s_1 s_2 \in W(\gamma_0)$ is greater than 5.

Thus, it remains to check that the graph Γ_{W_1} is connected. Let the vertices v, v' of Γ_{W_1} correspond to the reflections s, $s' = h^{-1}sh$ resp., where $h = g_n g_{n-1} \dots g_1 \in G_1$ is arbitrary and $g_i \in U$, $i = 1, \dots, n$. Put $h_i = g_i g_{i-1} \cdot \dots \cdot g_1$ and $s_i = h_i^{-1} s h_i$, $i = 0, \dots, n$, so that $h_0 = 1_E, h_1 = g_1, h_n = h$ and $s_0 = s$, $s_n = s'$. Since $g_1 \in U$, the reflections $s_0 = s$ and s_1 do not commute, and thus $0 < \phi(1_E, h_1) < 1$. By Lemma 2.4.a), $0 < \phi(1_E, h_1) = \phi(g_1, g_1 h_1) = \phi(h_1, h_2) < 1$, and therefore the reflections s_1 and s_2 do not commute, too. By induction, we see that s_i does not commute with s_{i+1} for all $i = 0, \dots, n-1$, and so the vertices v, v' of the graph Γ_{W_1} are connected by a path. This concludes the proof. ■

3. Infinite Coxeter groups

In Sections 4–6 below we will use the classification of infinite Coxeter groups. Although it should be well known, in view of the lack of references we reproduce it here in all details.

By *an infinite Coxeter group* we mean an infinite locally finite group W generated by reflections in a real vector space V which is algebraically irreducible in V. We fix the following notation and conventions.

3.1. Notation. Denote by $\mathbf{R}^\Delta$ the linear space of all the real functions with finite support defined on a given set Δ, and by $\mathbf{R}_0^\Delta$ the subspace of functions with the zero mean value. Let

A_Δ be the group of finite permutations of elements of Δ acting in $\mathbf{R}_0^\Delta$;

B_Δ be the group of finite permutations of Δ and changes of sign of values at the points of finite subsets of Δ acting in $\mathbf{R}^\Delta$;

D_Δ be the subgroup of B_Δ which consists of finite permutations and changes of signs of even numbers of coordinates acting in $\mathbf{R}^\Delta$.

If Δ is infinite then A_Δ, B_Δ, D_Δ are infinite Coxeter groups. Let ϵ_δ be the characteristic function of the one-point subset $\{\delta\}$ of Δ, so that $(\epsilon_\delta \,|\, \delta \in \Delta)$ is the standard Hamel basis of $\mathbf{R}^\Delta$. Let $\mathbf{R}^\Delta$ be endowed with the standard scalar product. Then A_Δ (resp. B_Δ, D_Δ) is generated by orthogonal reflections along vectors from the infinite root system $(\epsilon_\delta - \epsilon_{\delta'})$ (resp. $(\pm\epsilon_\delta,\ \pm\epsilon_\delta \pm \epsilon_{\delta'})$, $(\pm\epsilon_\delta \pm \epsilon_{\delta'})$) $(\delta, \delta' \in \Delta,\ \delta \neq \delta')$.

In the category of pairs (W, V), where W is a group generated by reflections in a real vector space V, there is a natural notion of isomorphism. We will also use a notion of subpair. Namely, we will say that (W', V') is *a subpair* of (W, V) if W' is the restriction of a subgroup of W generated by reflections to its invariant subspace V'. An embedding of pairs is an isomorphism with a subpair. In the proposition below *isomorphism of Coxeter groups* means isomorphism of pairs, rather then isomorphism of abstract groups.

3.2. Proposition. *Any infinite Coxeter group W is isomorphic to one and only one of the groups A_Δ, B_Δ, D_Δ.*

P r o o f. In the sequal γ denotes a finite connected subgraph of the Coxeter graph Γ_W, $G(\gamma)$ denotes the finite subgroup of W generated by reflections $s_i = s_{e_i, e_i^*} \in \gamma, i = 1, \ldots, \operatorname{card}\gamma$, $V(\gamma) = \operatorname{span}(e_i, i = 1, \ldots, \operatorname{card}\gamma)$ and $n(\gamma) = \dim V(\gamma)$. By Lemma 1.1, the restriction $G(\gamma) \,|\, V(\gamma)$ is irreducible, so it is a finite Coxeter group. The full Coxeter graph $\Gamma_{G(\gamma)}$ can be naturally identified with a finite connected subgraph $\bar\gamma$ of Γ_W containing γ; in fact, $\bar\gamma$ is the maximal subgraph of Γ_W with the properties that $V(\bar\gamma) = V(\gamma)$ and $G(\bar\gamma) = G(\gamma)$ (but the first one alone does not determine $\bar\gamma$). If $n = n(\gamma) > 8$, then $G(\gamma)$ is one of the Coxeter groups A_n, B_n, D_n [5, Ch. VI, Sect. 4].

Let Δ be a set with $\operatorname{card}\Delta = \dim V$, where $\dim V$ is the cardinality of a Hamel basis in V. The proposition follows from the assertions i)–iii) below.

i) $(W, V) \approx (B_\Delta, \mathbf{R}^\Delta)$ if $(G(\gamma_0), V(\gamma_0)) \approx (B_n, \mathbf{R}^n)$ for some $\gamma_0 \subset \Gamma_W$;

ii) $(W, V) \approx (D_\Delta, \mathbf{R}^\Delta)$ if there is no $\gamma \subset \Gamma_W$ such that $(G(\gamma), V(\gamma)) \approx (B_n, \mathbf{R}^n)$ and $(G(\gamma_0), V(\gamma_0)) \approx (D_n, \mathbf{R}^n)$ for some $\gamma_0 \subset \Gamma_W$ with $n = n(\gamma_0) \geq 4$;

iii) $(W, V) \approx (A_\Delta, \mathbf{R}_0^\Delta)$ in the other cases.

From now on we consider Coxeter graphs as weighted graphs. As usual, the weight of an edge (s', s'') is the order of the product $s's''$. Since W is a locally finite group, the weights on Γ_W take only finite values. Recall that the Coxeter graphs of types A_n and D_n have only edges of weight 3, while in any of the Coxeter graphs of type B_n there are edges of weight 4. Thus, if the assumption of i) holds, then $(G(\gamma'), V(\gamma')) \approx (B_{n'}, \mathbf{R}^{n'})$ for any $\gamma' \supset \gamma_0$ with $n' = n(\gamma') > 8$, and thus (W, V) is the inductive limit of some net of pairs $(B_n, \mathbf{R}^n)$.

Next we show that for $\gamma \subset \gamma'$, where $\gamma \supset \gamma_0$ and $n(\gamma) > 8$, the embedding $(G(\gamma), V(\gamma)) \subset (G(\gamma'), V(\gamma'))$ is coordinatewise. This means that under the isomorphisms $(G(\gamma), V(\gamma)) \approx (B_n, \mathbf{R}^n)$ and $(G(\gamma'), V(\gamma')) \approx (B_{n'}, \mathbf{R}^{n'})$ the pair $(B_n, \mathbf{R}^n)$ is a coordinate subpair of $(B_{n'}, \mathbf{R}^{n'})$. Indeed, identify $(G(\gamma'), V(\gamma'))$

with $(B_{n'}, \mathbf{R}^{n'})$. Then $V(\gamma) \subset \mathbf{R}^{n'}$ is spanned by a subsystem of the root system $(\pm\epsilon_i,\ \pm\epsilon_i \pm \epsilon_j)$, $1 \le i < j \le n'$, and $G(\gamma)$ is generated by the corresponding orthogonal reflections. A plane in $\mathbf{R}^{n'}$ spanned by roots may contain reflection root vectors of 2, 3 or 4 different reflections from $B_{n'}$. It is a coordinate plane precisely when it contains 4 reflections. Since $(G(\gamma), V(\gamma)) \approx (B_n, \mathbf{R}^n)$, the subspace $V(\gamma)$ contains $\begin{pmatrix} n \\ 2 \end{pmatrix}$ planes of the latter type which span it. At the same time, these planes should be coordinate planes of $\mathbf{R}^{n'}$. Therefore, $V(\gamma)$ is a coordinate subspace $\mathbf{R}^n \subset \mathbf{R}^{n'}$, and so $G(\gamma)$ coincides with the group B_n generated by reflections along those roots of the above root system which belong to $V(\gamma)$.

For any of the graphs $\gamma \supset \gamma_0$ with $n = n(\gamma) > 8$ all the vertices of the full Coxeter graph $\bar{\gamma}$ (see the notation above) are divided in two types: those which correspond to sign change reflections, i.e., reflections along the roots of the form $\epsilon_i, i = 1, \dots, n$, and others. Being coordinatewise, embeddings of pairs respect this division. Thus, it is well defined in the inductive limit Γ_W. Note that the vertices of change sign type in Γ_W are those which are incident only with edges of weight 4. Denote by Δ the set of all the vertices of Γ_W of change sign type. Fix $\delta_0 \in \Delta \cap \gamma_0$, and let $\epsilon_0 = \epsilon_{\delta_0}$ be one of the two opposite roots in $V(\gamma_0)$ which correspond to the reflection δ_0. It is easily seen that for any $\gamma \supset \gamma_0$ with $n = n(\gamma) > 8$ the orbit $G(\gamma)\,\epsilon_0$ consists of the roots of coordinate type $\pm\epsilon_i$ in $V(\gamma)$, and so the class of conjugates of δ_0 in W coincides with Δ. Choosing one of any two opposite root vectors in the orbit $W(\epsilon_0)$, we obtain a Hamel basis of the W-invariant subspace $\mathrm{span}(W(\epsilon_0))$ which coincides with V since W is assumed to be irreducible in V. Thus, we obtain a Hamel basis of V formed by roots of coordinate type. This yields an isomorphism $V \approx \mathbf{R}^{\Delta}$. The root system of W, which consists of the vectors of the two orbits $W(\epsilon_0)$ and $W\,(\epsilon_0 + \epsilon_1)$, where $\epsilon_1 \neq \epsilon_0$ is another coordinate vector in $V(\gamma_0)$, corresponds under this isomorphism to the root system $(\pm\epsilon_\delta,\ \pm\epsilon_\delta \pm \epsilon_{\delta'} \,|\, \delta, \delta' \in \Delta, \delta \neq \delta')$ of the group B_Δ. Therefore, $(W, V) \approx (B_\Delta, \mathbf{R}^\Delta)$. This proves i).

Next we consider the case (ii), where there is no subgroup $G(\gamma) \subset W$ of type B_n, but at least one of them, say $G(\gamma_0)$, has type D_n for some $n = n(\gamma_0) \ge 4$. First we show that any subgroup $G(\gamma) \supset G(\gamma_0)$ is of type $D_{n(\gamma)}$, and all the embeddings $G(\gamma) \hookrightarrow G(\gamma')$ are coordinatewise.

The group D_4 contains 4 pairwise commuting reflections along the root vectors $\epsilon_1 \pm \epsilon_2, \epsilon_3 \pm \epsilon_4$. If $v_1, \dots, v_4$ are 4 mutually orthogonal root vectors from the root system $(\pm(\epsilon_i - \epsilon_j) \,|\, 1 \le i < j \le n')$ of type $A_{n'}$ and $L = \mathrm{span}\,(v_1, \dots, v_4)$, then the only reflections in $A_{n'} \,|\, L$ are the orthogonal reflections along $v_1, \dots, v_4$, and so $A_{n'} \,|\, L$ does not contain D_4. Therefore, the Coxeter group $G(\gamma) \supset G(\gamma_0)$ is not of type $A_{n(\gamma)}$, and thus it must be of type $D_{n(\gamma)}$.

Let F be a subspace of dimension 4 of $\mathbf{R}^{n'}$ generated by 4 mutually orthogonal root vectors from the root system $(\pm\epsilon_i \pm \epsilon_j)$, $1 \le i < j \le n'$, of type $D_{n'}$, and $G(F) \subset D_{n'}$ be the subgroup generated by the orthogonal reflections along the roots in F. Then $G(F)$ is irreducible (and of type D_4) iff F is a coordinate subspace of $\mathbf{R}^{n'}$. Thus, if $(G(\gamma), V(\gamma)) \subset (D_{n'}, \mathbf{R}^{n'})$ is of type D_n, where $n = n(\gamma) \ge 4$, then $V(\gamma)$ is a coordinate subspace of $\mathbf{R}^{n'}$.

Fix a reflection vector v_0 of a reflection from $G(\gamma_0)$. If $G(\gamma) \supset G(\gamma_0)$, then the orbit $G(\gamma)\, v_0$ is a root system of type $D_{n(\gamma)}$ of the Coxeter group $G(\gamma)$. Consider the infinite root system $W(v_0)$ in V. Since W is irreducible, this system is complete in V. Note that two roots v', v'' of D_n are contained in the same coordinate plane in $\mathbf{R}^n$ iff the sets of their neigborhooding vertices in the full Coxeter graph Γ_{D_n} coincide. In this case the vectors $(\frac{\pm v' \pm v''}{2}) = (\pm\epsilon_i, \pm\epsilon_j)$, $i \ne j$, are contained in the coordinate axes which are the intersections of coordinate planes. The same pairing is defined on the above root system of W. In this way, fixing one of any two opposite vectors $(\frac{\pm v' \pm v''}{2})$ arbitrarily, we obtain a Hamel basis Δ in V, which in turn provides us with an isomorphism $(W, V) \approx (D_\Delta, \mathbf{R}^\Delta)$. This proves (ii).

Assume further that any subgroup $G(\gamma') \subset W$ with $n' = n(\gamma') > 8$ is of type $A_{n'}$, where $A_{n'}$ acts by permutations in $\mathbf{R}_0^{n'+1}$. Let $(G(\gamma), V(\gamma)) \subset (A_{n'}, \mathbf{R}_0^{n'+1})$ be of type A_n. We will show that $V(\gamma)$ is a coordinate subspace of $\mathbf{R}_0^{n'+1}$. Let s_{ij} be the orthogonal reflections (transpositions) along the roots $\pm(\epsilon_i - \epsilon_j)$, $1 \le i < j \le n' + 1$. Put

$$I = \{i \in \{1, \ldots, n'+1\} \,|\, s_{ij} \in G(\gamma) \text{ for some } j \in \{1, \ldots, n'+1\}\} \ .$$

Thus, if $s_{kl} \in G(\gamma)$, then $k, l \in I$. Vice versa, $s_{kl} \in G(\gamma)$ for any pair $k, l \in I, k \ne l$. This follows from the connectedness of the Coxeter graph $\Gamma_{G(\gamma)}$ and the following remark: if $s_{ij} \in G(\gamma)$ and $s_{jk} \in G(\gamma)$, then $s_{ik} \in G(\gamma)$. Indeed, $s_{jk}(\epsilon_i - \epsilon_j) = \epsilon_i - \epsilon_k$ and thus $s_{ik} = s_{jk} s_{ij} s_{jk}$. Now we see that $V(\gamma) = \mathbf{R}_0^I = \text{span}\,(\epsilon_i - \epsilon_j \,|\, i, j \in I)$ is a coordinate subspace of $\mathbf{R}_0^{n'+1}$, and $G(\gamma) \subset A_{n'}$ is a subgroup of permutations of the set I.

On the set of edges of the full Coxeter graph Γ_{A_n} consider the following equivalence relation: $(s_{i_1,j_1}, s_{i_2,j_2}) \sim (s_{k_1,l_1}, s_{k_2,l_2})$ iff these four transpositions have an index in common. Then this index is the same for the whole equivalence class, so that the set of classes is $\{1, \ldots, n\}$. Since this equivalence relation is compatible with the embedding of pairs $(G(\gamma), V(\gamma)) \hookrightarrow (G(\gamma'), V(\gamma'))$, it can be defined as well on the whole graph Γ_W. Let Δ be the set of the equivalence classes. Let $v \in \Gamma_W$ be a vertex. Then all the edges incident with v belong to two different classes $\delta, \delta' \in \Delta$, where each class $\delta \in \Delta$ consists of the edges of a complete subgraph of Γ_W, and each pair of these complete subgraphs which correspond to some $\delta, \delta' \in \Delta$, $\delta \ne \delta'$, has exactly one vertex $v(\delta, \delta')$ in common. It is easily seen that the action of W on Γ_W by inner automorphisms is locally finite and compatible with the equivalence relation, and so it induces the action of W on Δ by finite permutations, such that the reflections in W act as transpositions.

Fix a reflection $s_0 = s(\delta, \delta') \in W$ which corresponds to the transposition (δ, δ'), with the reflection vector $e_0 = e(\delta, \delta')$. Then the orbit $W(e_0)$ is a root system of W which spans V. Fixing one of the each two opposite roots, we obtain a Hamel basis of V which corresponds to the basis of $\mathbf{R}_0^\Delta$ consisting of the root vectors $\epsilon_\delta - \epsilon_{\delta'}$ $(\delta, \delta' \in \Delta, \delta \neq \delta')$ of A_Δ. This gives an isomorphism $(W, V) \approx (A_\Delta, \mathbf{R}_0^\Delta)$. The proof is complete. ■

4. Total families of isometric reflections

Denote by $\mathrm{IR}(E)$ the set of all the reflections in $\operatorname{Iso} E$, and let $W = W(E)$ be the subgroup of $\operatorname{Iso} E$ generated by the reflections from $\mathrm{IR}(E)$. In this section we assume that $\mathrm{IR}(E)$ contains a total subset of reflections $\{s_\alpha = s_{e_\alpha, e_\alpha^*}\}_{\alpha \in A}$, which means that the family of linear functionals $T = \{e_\alpha^*\}_{\alpha \in A} \subset E^*$ is a total family.

4.1. Lemma. *Let $g_1, g_2 \in \operatorname{Iso} E$. In the notation as above assume that $g_1(e_\alpha) = g_2(e_\alpha)$ for all $\alpha \in A$. Then $g_1 = g_2$.*

P r o o f. Put $g_0 = g_1^{-1} g_2$. Then $g_0(e_\alpha) = e_\alpha$ for all $\alpha \in A$. Since s_α is the only isometric reflection in the direction of e_α (see Remark 2.2.b)), it coincides with $s'_\alpha = g_0 s_\alpha g_0^{-1} = s_{e_\alpha, g_0^{*-1}(e_\alpha^*)}$, and so $g_0^{*-1}(e_\alpha^*) = e_\alpha^*$, i.e., $g_0^*(e_\alpha^*) = e_\alpha^*$ or, in other words, $e_\alpha^*(g_0(v) - v) = 0$ for all $\alpha \in A$. Since T is total, it follows that $g_0 = 1_E$. ■

Let, as before, G_0 be the strong identity component of $\operatorname{Iso} E$, and let W be the group generated by the reflections in $\operatorname{Iso} E$.

4.2. Lemma. *W is locally finite iff G_0 is trivial.*

P r o o f. Suppose that G_0 is trivial. To prove that W is locally finite it is enough to show that each subgroup W' of W generated by a finite number of reflections $\{s_i = s_{e_i, e_i^*}\}_{i=1,\dots,n} \subset \mathrm{IR}(E)$ is finite. Suppose that W' is an infinite group. Put $F' = \operatorname{span}(e_i \mid i = 1, \dots, n)$. Let G' be the closure of W' in $\operatorname{Iso} E$ in the strong operator topology. It is easily seen that the closed subspace $M' = \bigcap_{i=1}^n \operatorname{Ker} e_i^*$ is a complementary subspace of F', i.e., $E = M' \oplus F'$, and it coincides with the fixed point subspace of W'. Hence, it also coincides with the fixed point subspace of G'. It follows that $G' = 1_{M'} \oplus \bar{G}'$, where $\bar{G}' \subset O(F')$ is the closure of $W' \mid F'$ in $\operatorname{Iso} F'$. Thus, G' is a compact Lie group, and being infinite it has a non-trivial identity component. This is a contradiction.

Assume further that G_0 is non-trivial. Then, as it was shown in the proof of Proposition 2.6, there exist reflections $s', s'' \in \mathrm{IR}(E)$ such that the angle $\alpha(s', s'')$ is irrational modulo π, and so the subgroup of W generated by these two reflections is infinite. ■

Recall that a group G of operators in a Banach space E is called *topologically irreducible* if it has no non-trivial closed invariant subspace.

4.3. Lemma. *Let W' be a group generated by a set of reflections $\{s_\alpha = s_{e_\alpha, e^*_\alpha}\}_{\alpha \in A'} \subset \mathrm{IR}(E)$. Then W' is topologically irreducible iff the following two conditions are fulfilled:*

i) The system of vectors $(e_\alpha \,|\, \alpha \in A')$ is complete, i.e., $E = \overline{\mathrm{span}}(e_\alpha \,|\alpha \in A')$.

ii) The Coxeter graph $\Gamma_{W',A'}$ is connected.

P r o o f. Since the closed subspace $E' = \overline{\mathrm{span}}(e_\alpha \,|\, \alpha \in A')$ is invariant with respect to W', the first condition i) is necessary for W' being irreducible. Let Γ' be a connected component of $\Gamma_{W',A'}$. It is easily seen that the closed subspace $F' = \overline{\mathrm{span}}(e \,|\, s_{e,e^*} \in \Gamma')$ is invariant, too. Thus, the second condition ii) is also necessary.

Suppose further that i) and ii) are fulfilled. Let F' be a closed invariant subspace of W'. Put $A = \{\alpha \in A' \,|\, e_\alpha \in F'\}$ and $B = \{\alpha \in A' \,|\, e_\alpha \notin F'\}$. Being invariant F' is contained in $\mathrm{Ker}\, e^*_\beta$ for each $\beta \in B$. It follows that $\alpha(s_\alpha, s_\beta) = \pi/2$, and so $[s_\alpha, s_\beta] = 1_E$ for any $\alpha \in A, \beta \in B$. By ii) this implies that either $A = \emptyset$ or $B = \emptyset$. From i) it easily follows that the system of linear functionals $(e^*_\alpha \,|\, \alpha \in A) \subset E^*$ is total. Thus, if $A = \emptyset$, then $F' \subset \bigcap\{\mathrm{Ker}\, e^*_\alpha \,|\, \alpha \in A'\} = \{\bar{0}\}$, and if $B = \emptyset$, then $F' \supset \overline{\mathrm{span}}\,(e_\alpha \,|\, \alpha \in A') = E$. In any case, F' is not a proper subspace. This shows that W' is topologically irreducible. ■

Let things be as in Lemma 4.3. Consider the algebraic linear subspace $V' = \mathrm{span}\,(e_\alpha \,|\, \alpha \in A')$. The group W' is algebraically irreducible in V' iff the Coxeter graph $\Gamma_{W',A'}$ is connected. In this case, W' is topologically irreducible in the closed subspace $E' = \overline{V}' = \overline{\mathrm{span}}\,(e_\alpha \,|\, \alpha \in A')$. If W' is finite and the Coxeter graph $\Gamma_{W'}$ is connected, then $\dim V' = n < \infty$ and $W' \,|\, V'$ is a finite Coxeter group, i.e., a finite irreducible group generated by orthogonal reflections in $\mathbf{R}^n$ (here we identify V' with $\mathbf{R}^n$ by choosing an orthonormal basis with respect to an invariant scalar product in V'). Let the group $\mathrm{Iso}E$ be discrete in the strong operator topology, i.e., $G_0 = \{1_E\}$. Then by Lemma 4.3, W' is a locally finite group. If $\dim V' = \infty$ and the Coxeter graph $\Gamma_{W'}$ is connected, then W' is an infinite Coxeter group, and by Proposition 3.2, it is isomorphic to one of the groups A_Δ, B_Δ, D_Δ. The next proposition shows that if the pair (W', V') is maximal, it can not be of type D_Δ.

4.4. Proposition. *Let the notation be as above. If $\dim V' = \infty$ and $(W', V') \approx (D_\Delta, \mathbf{R}^\Delta)$, then the group W' can be extended to a subgroup $W'' \subset \mathrm{Iso}E$ generated by reflections along vectors in V' and such that $(W'', V') \approx (B_\Delta, \mathbf{R}^\Delta)$.*

For the proof we need the following lemma on partial orthogonal decompositions in Banach spaces.

4.5. Lemma. *Let* $\{p_i\}_{i=1,2,\ldots}$ *be a sequence of projections in a Banach space* E *such that*

a) $1_E - 2p_i \in \mathrm{Iso}E$ for all $i = 1, 2, \ldots$;

b) the projections p_i *are mutually orthogonal, i.e.,* $p_i p_j = 0$ for all $i \neq j$.

Then $\limsup_{i\to\infty} ||(1_E - p_i)(x)||_E = ||x||_E$ for all $x \in E$.

P r o o f. By a), we have $||p_i||_E = ||1_E - p_i||_E = 1$ for all $i = 1, 2, \ldots$. From a) and b) it follows that $\prod_{i=1}^{k}(1_E - 2p_i) = 1_E - 2\sum_{i=1}^{k} p_i \in \mathrm{Iso}E$, and so $||\sum_{i=1}^{k} p_i||_E = ||1_E - \sum_{i=1}^{k} p_i||_E = 1$, as well.

Suppose that there exist $x_0 \in E$ and $\epsilon_0 > 0$ such that

$$||(1_E - p_i)(x_0)||_E \leq ||x_0||_E - \epsilon_0 \text{ for all } i = 1, 2, \ldots \ .$$

Then

$$||\frac{1}{k}\sum_{i=1}^{k}(1_E - p_i)(x_0)||_E \leq \frac{1}{k}\sum_{i=1}^{k} ||(1_E - p_i)(x_0)||_E \leq ||x_0||_E - \epsilon_0 \ .$$

Therefore

$$\epsilon_0 \leq ||x_0||_E - ||\frac{1}{k}\sum_{i=1}^{k}(1_E - p_i)(x_0)||_E \leq ||x_0 - \frac{1}{k}\sum_{i=1}^{k}(1_E - p_i)(x_0)||_E$$

$$= \frac{1}{k}||(\sum_{i=1}^{k} p_i)(x_0)||_E \leq \frac{1}{k}||x_0||_E \ .$$

This is a contradiction. ■

P r o o f o f P r o p o s i t i o n 4.4. Identify V' with $\mathbf{R}^\Delta$ via an isomorphism $(W', V') \approx (D_\Delta, \mathbf{R}^\Delta)$, and consider in V' the root system $\{\pm\epsilon_{\delta'} \pm \epsilon_{\delta''}\}$ of type D_Δ. Denote by $s^+_{\delta',\delta''}$ the isometric reflection along the vector $v^+_{\delta',\delta''} = \epsilon_{\delta'} + \epsilon_{\delta''}$, $\delta', \delta'' \in \Delta$, $\delta' \neq \delta''$, and by $s^-_{\delta',\delta''}$ the isometric reflection along the vector $v^-_{\delta',\delta''} = \epsilon_{\delta'} - \epsilon_{\delta''}$. Put $d_{\delta',\delta''} = s^+_{\delta',\delta''} s^-_{\delta',\delta''} \in W'$, so that $d_{\delta',\delta''}$ is the operator of change of signs of the coordinates δ' and δ''.

Choose a countable subset $\{\delta_i\}_{i=1,2,\ldots} \subset \Delta$ and put $d_{i,j} = d_{\delta_i,\delta_j}$. Then the involutions $d_{i,j}$ pairwise commute and $d_{n,k}d_{k,m} = d_{n,m}$. The orthogonal projections $p_{i,j} = \frac{1}{2}(1_E - d_{i,j})$ onto planes also pairwise commute. For each triple of different indices n, m, k consider the one-dimensional projection $p_n^{k,m} = p_{n,k}p_{n,m}$. Since $p_n^{k,m}$ and $p_n^{i,j}$ commute and have the same image, they coincide; indeed, $p_n^{k,m} = p_n^{i,j} p_n^{k,m} = p_n^{k,m} p_n^{i,j} = p_n^{i,j}$. Denote by p_n their common value, and consider the corresponding reflection $s_n = 1_E - 2p_n$ along the coordinate vector ϵ_{δ_n}. It is easily seen that $s_n s_m = d_{n,m}$ and $s_m(1_E - p_{m,k}) = 1_E - p_{m,k}$. By Lemma 4.5, for a fixed $n \in \mathbf{N}$ and for any $x \in E, \epsilon > 0$ there exist $k, m \in \mathbf{N}$ such that

$$\|s_n(x)\|_E \le \|(1_E - p_{k,m})s_n(x)\|_E + \epsilon = \|s_n(1_E - p_{k,m})(x)\|_E + \epsilon$$
$$= \|s_n s_m(1_E - p_{k,m})(x)\|_E + \epsilon = \|d_{n,m}(1_E - p_{k,m})(x)\|_E + \epsilon$$
$$= \|(1_E - p_{k,m})(x)\|_E + \epsilon \le \|x\|_E + \epsilon \ .$$

It follows that $s_n \in \mathrm{Iso}E$. Since $\delta_n \in \Delta$ is taken as arbitrary, this implies that for any $\delta \in \Delta$ there exists an isometric reflection along the vector ϵ_δ. Thus, the group $\mathrm{Iso}E$ contains the subgroup W'' generated by reflections along vectors of the root system $\{\pm\epsilon_\delta, \pm\epsilon_{\delta'} \pm \epsilon_{\delta''}\}$ of type B_Δ. ■

4.6. Corollary. *Let* $\dim E = \infty$, $G_0 = G_0(E) = \{1_E\}$, *and let the group* $W = W(E)$ *generated by all the isometric reflections in* E *be topologically irreducible. Then* W *is an infinite Coxeter group of type* A_Δ *or* B_Δ.

4.7. R e m a r k. Let $\dim E = n < \infty$. Then Proposition 4.4 still holds in the case when n is odd. Indeed, in this case $B_n = W''$ is the subgroup of the group $\mathrm{Iso}E$ generated by the subgroup $W' = D_n$ and the element -1_E. But for n even the statement of Proposition 4.4 is not true, in general. As an example, consider $E = \mathbf{R}^n$, where $n = 2k \ge 4$, with the unit ball $B(E)$ being the convex hull of the D_n-orbit of the point $v_0 = (1, 2, \dots, n)$. Then the image $s_n(v_0) = (1, 2, \dots, n-1, -n)$ of v_0 by the reflection $s_n = s_{\epsilon_n, \epsilon_n^*}$ does not belong to $B(E)$ (indeed, it is separated from $B(E)$ by the hyperplane $-x_n + \sum_{i=1}^{n-1} x_i = \frac{n(n+1)}{2}$). Hence $B(E)$ is not invariant with respect to the action of the Coxeter group B_n on $\mathbf{R}^n = E$, and so B_n is not a subgroup of $\mathrm{Iso}\, E$.

5. Hilbert and Coxeter decompositions

Let, as before, E be a Banach space with a total family of isometric reflections. In this section we construct a partial orthogonal decomposition of E which consists of two parts: the *Hilbert decomposition* into a direct sum of biorthogonally complemented Hilbert subspaces, and the *Coxeter decomposition* into a direct sum of closed subspaces endowed with topologically irreducible Coxeter groups generated by isometric reflections. In a sense, this decomposition is orthogonal (see Lemma 5.4 and Proposition 5.6). Both of these decompositions are stable under the action of the isometry group $\mathrm{Iso}E$, and the second one is fixed under the action of the identity component G_0. The main result of the section, Theorem 5.7, is a kind of a structure theorem for the isometry group $\mathrm{Iso}E$.

5.1. N o t a t i o n. As above, by $\mathrm{IR}(E)$ we denote the set of all the isometric reflections in E which is assumed to be total. To each subspace V of E we attach two closed subspaces, *the kernel*

$$V_0 = \overline{\mathrm{span}}\,(e \in V \mid s_{e,e^*} \in \mathrm{IR}(E)\,)$$

and *the hull*

$$\hat{V} = \bigcap \{\mathrm{Ker}\, e^* \mid s_{e,e^*} \in \mathrm{IR}(E)\,,\, V \subset \mathrm{Ker}\, e^*\}\ ;$$

we put $\hat{V} = E$ if there is no $s_{e,e^*} \in \mathrm{IR}(E)$ such that $V \subset \mathrm{Ker}\, e^*$. It is easily seen that

i) $V_0 \subset \overline{V} \subset \hat{V}$,

ii) $V_{00} = V_0\,,\ \hat{\hat{V}} = \hat{V}$, *and*

iii) *if* $V \subset V'$, *then* $V_0 \subset V_0'$ *and* $\hat{V} \subset \hat{V'}$.

Observe that possibly V_0 resp. $\overline{V}$ is a proper subspace of $\overline{V}$ resp. $\hat{V}$. For instance, this is the case when $E = \mathrm{l}_\infty$ and $V = \mathrm{c}$ (the subspace of convergent sequences); indeed, then $\hat{V} = E$ and $V_0 = \mathrm{c}_0$ (the subspace of sequences convergent to zero).

Denote also $\mathrm{IR}_V = \{s_{e,e^*} \in \mathrm{IR}(E) \mid e \in V\}$. Let W_V be the group generated by the reflections from IR_V.

5.2. Coxeter decomposition. This is a partial subspace decomposition defined on the fixed point subspace $F = \mathrm{Fix}\, G_0$ of the group $G_0 = G_0(E)$. Let $\Gamma_F = \Gamma_{W_F}$ be the full Coxeter graph of the group W_F, and let $\mathcal{A}$ be the set of the connected components of Γ_F. For $\alpha \in \mathcal{A}$ denote by IR_α the set of reflections in IR_F which correspond to vertices of the component α of Γ_F. Put $V_\alpha = \overline{\mathrm{span}}\,(e \mid s_{e,e^*} \in \mathrm{IR}_\alpha)$; so, $\mathrm{IR}_\alpha = \mathrm{IR}_{V_\alpha}$. Put also $W_\alpha = W_{V_\alpha}$. Then V_α is a closed subspace of the kernel F_0, and the group $W_\alpha \mid V_\alpha$ is topologically irreducible. By the discussion after Lemma 4.3, W_α is a Coxeter group. If $\dim V_\alpha = \infty$, then by Corollary 4.6, W_α has type A_Δ or B_Δ.

The set $\mathcal{A}$ can be devided into equivalence classes which correspond to the isomorphism types of the Coxeter pairs (W_α, V_α). Since G_0 is a normal subgroup of the group $\mathrm{Iso}\, E$, its fixed point subspace F is invariant with respect to $\mathrm{Iso}\, E$; the same is true for the kernel F_0 and the hull $\hat{F}$. Each isometry $g \in \mathrm{Iso}\, E$ acts (by conjugation) on the set IR_F and also on the graph Γ_F, and so on the set $\mathcal{A}$. It is clear that the above partition of $\mathcal{A}$ is stable under this action and its equivalence classes are invariant.

5.3. Hilbert decomposition. Consider the following equivalence relation defined on the set $\mathrm{IR}(E) \setminus \mathrm{IR}_F$:

$$s_{e,e^*} \sim s_{e',e'^*} \ \text{ iff } \ e' \in G_0\, e \ \text{ or } \ -e' \in G_0\, e\,.$$

Let $\mathcal{B}$ be the set of its equivalence classes. By Theorem 1, to each $\beta \in \mathcal{B}$ there corresponds the unique Hilbert subspace $H_\beta = \overline{\mathrm{span}}\,(G_0\, e \,|\, s_{e,e^*} \in \beta)$ and the unique bicontractive projection $p_\beta : E \to H_\beta$ satisfying all the properties of Theorem 1.b). An isometry $g \in \mathrm{Iso}\, E$ induces the action g_* on the set $\mathcal{B}$ which is defined as follows: $g_*\beta = \beta'$ iff $g(H_\beta) = H_{\beta'}$. In particular, the orthogonal bases in H_β and in $H_{\beta'}$ are of the same cardinality. The following lemma shows that this partial decomposition into Hilbert subspaces is orthogonal; moreover, all of the subspaces H_β are orthogonal to the fixed point subspace F.

Let $\mathrm{IR}_\beta = \mathrm{IR}_{H_\beta}$. Note that $\mathrm{IR}(E) = \mathrm{IR}_{\mathcal{A}} \cup \mathrm{IR}_{\mathcal{B}}$, where $\mathrm{IR}_{\mathcal{A}} = \mathrm{IR}_F = \bigcup_{\alpha\in\mathcal{A}} \mathrm{IR}_\alpha$ and $\mathrm{IR}_{\mathcal{B}} = \bigcup_{\beta\in\mathcal{B}} \mathrm{IR}_\beta$.

5.4. Lemma. *a) Let $s, s' \in \mathrm{IR}(E)$. If $[s, s'] \neq 1_E$, then s, s' belong either to the same subset IR_α, where $\alpha \in \mathcal{A}$, or to the same subset IR_β, where $\beta \in \mathcal{B}$.*
b) The projection p_β commutes with any reflection $s \in \mathrm{IR}(E)$ for any $\beta \in \mathcal{B}$.
c) Furthermore, $p_\beta p_{\beta'} = 0$ for any $\beta, \beta' \in \mathcal{B}, \beta \neq \beta'$, and $p_\beta \,|\, F = 0$ for any $\beta \in \mathcal{B}$.

P r o o f. a). Let $s_i = s_{e_i,e_i^*}$, $i = 1, 2$, be two arbitrary distinct reflections from IR_β, where $\beta \in \mathcal{B}$. Being restricted to the Hilbert subspace H_β the rotation $r = s_1 s_2 \in \mathrm{Iso}\, E$ in the plane $L = \mathrm{span}\,(e_1, e_2)$ belongs to the connected component $G_0(H_\beta)$ of the orthogonal group, and so by Theorem 1.c), $r \in G_0$. Since $F = \mathrm{Fix}\, G_0 \subset \mathrm{Fix}\, r = \mathrm{Ker}\, e_1^* \cap \mathrm{Ker}\, e_2^*$, we have that $e_i^*(e) = 0$ for each $e \in F$. Therefore, if $s = s_{e,e^*} \in \mathrm{IR}_F$ then by Lemma 2.3, $\alpha(s_i, s) = \frac{\pi}{2}$, and thus $[s_i, s] = 1_E$, $i = 1, 2$ (see Remark 2.2.c)).

If $\beta' \in \mathcal{B}$ and $\beta' \neq \beta$, then the subspace $H_{\beta'}$ is invariant with respect to the rotation $r = s_1 s_2 \in G_0$. One may assume that $r \,|\, L \neq -1_L$, and so either $H_{\beta'} \subset \mathrm{Fix}\, r$ or $L \subset H_{\beta'}$. The second case is impossible (indeed, otherwise by the construction, we would have $H_\beta = H_{\beta'}$, and so $\beta = \beta'$). Thus, $H_{\beta'} \subset \mathrm{Fix}\, r = \mathrm{Ker}\, e_1^* \cap \mathrm{Ker}\, e_2^*$. As above, it follows that $[s_i, s] = 1_E$ for each $s \in \mathrm{IR}_{\beta}$.

To prove a) it remains to note that the definition of the set $\mathcal{A}$ (5.2) yields that $[s, s'] = 1_E$ if $s \in \mathrm{IR}_\alpha$, $s' \in \mathrm{IR}_{\alpha'}$, where $\alpha, \alpha' \in \mathcal{A}$ and $\alpha \neq \alpha'$.

Now b) and c) easily follow from a) by the construction of the projections p_β as in 2.8. ■

5.5. Lemma. *a) $\hat{F}_0 = \hat{F} = F$.*
b) $(H_\beta)_0 = \hat{H}_\beta = H_\beta$ for any $\beta \in \mathcal{B}$.

P r o o f. a). Let $s_{e,e^*} \in \mathrm{IR}(E)$ be such that $F_0 \subset \mathrm{Ker}\, e^*$. Then $e \notin F_0$, and hence $e \in H_\beta$ for some $\beta \in \mathcal{B}$. As in the proof of Lemma 5.4, it follows that $F \subset \mathrm{Ker}\, e^*$, and so $F \subset \hat{F}_0 = \bigcap_{\beta\in\mathcal{B}} \mathrm{Ker}\, p_\beta$. Since the subspace $\bigcup_{\beta\in\mathcal{B}} H_\beta$ is G_0-invariant, it is clear that $\hat{F}_0$ is G_0-invariant, too.

If $\hat{F}_0 \neq F$, then there exists $g_0 \in G_0$ such that $g_0 \,|\, \hat{F}_0 \neq 1_{F_0}$, and so $g_0(x) \neq x$ for some $x \in \hat{F}_0$. Note that both $g_0(x)$ and x belong to $\mathrm{Ker}\, e^*$ for each e^* such

that $s_{e,e^*} \in \mathrm{IR}(E) \setminus \mathrm{IR}_F$ (indeed, in this case $g_0(e) = e$, and thus $g_0^*(e^*) = e^*$). Therefore, $e^*(g_0(x) - x) = 0$ for each e^* as above, and also for each e^* such that $s_{e,e^*} \in \mathrm{IR}(E)$. Since the system of functionals $(e^* \,|\, s_{e,e^*} \in \mathrm{IR}(E))$ is total, we have $g_0(x) - x = 0$, which is a contradiction. This proves (a).
b). If $\hat{H}_\beta \neq H_\beta$ for some $\beta \in \mathcal{B}$, then $(1_E - p_\beta)(x) \neq 0$ for some vector $x \in \hat{H}_\beta$. By Lemma 5.4.b), the projection p_β commutes with any reflection $s = s_{e,e^*} \in \mathrm{IR}(E)$. Thus, if $H_\beta \subset \mathrm{Ker}\, e^*$, then also $\hat{H}_\beta \subset \mathrm{Ker}\, e^*$. Therefore, $s(y) = y$ for all $y \in \hat{H}_\beta$, $p_\beta s(y) = s p_\beta(y) = p_\beta(y)$ and $s(1_E - p_\beta)(y) = (1_E - p_\beta)(y)$. The latter means that $(1_E - p_\beta)(y) \in \mathrm{Ker}\, e^*$. Hence, $(1_E - p_\beta)(\hat{H}_\beta) \subset \hat{H}_\beta$.

Now we have $(1_E - p_\beta)(x) \in \mathrm{Ker}\, e^*$ for any e^* such that $s_{e,e^*} \in \mathrm{IR}(E)$. This contradicts to the assumption that the system $\mathrm{IR}(E)$ is total, since $(1_E - p_\beta)(x) \neq 0$. ■

Put $R_0 = \overline{\mathrm{span}}\,(\bigcup_{\beta\in\mathcal{B}} H_\beta)$ and $\hat{R} = \hat{R}_0$.

5.6. Proposition. *a) The subspace $R_0 \dot{+} F$ is closed, and if $p_{R_0,F} : R_0 \dot{+} F \to R_0$ is the first projection, then $1_{R_0 \dot{+} F} - 2p_{R_0,F} \in \mathrm{Iso}\,(R_0 \dot{+} F)$. Therefore, the projection $p_{R_0,F}$ is bicontractive.*

b) For any $\alpha \in \mathcal{A}$ there exists a projection $p_\alpha : F_0 \dot{+} \hat{R} \to V_\alpha$ such that

- *i) p_α commutes with any reflection $s \in \mathrm{IR}\,(E)$ and $p_\alpha p_{\alpha'} = p_{\alpha'} p_\alpha = 0$ resp. $p_\alpha p_\beta = p_\beta p_\alpha = 0$ for all $\alpha' \in \mathcal{A}$, $\alpha' \neq \alpha$, $\beta \in \mathcal{B}$;*
- *ii) $||p_\alpha||_{F_0 \dot{+} \hat{R}} \leq 2$ and $||1_{F_0 \dot{+} \hat{R}} - p_\alpha||_{F_0 \dot{+} \hat{R}} = 1$, if the latter projection is non-zero;*
- *iii) moreover, if the Coxeter group W_α is a group of type B_Δ, then $1_{F_0 \dot{+} \hat{R}} - 2p_\alpha \in \mathrm{Iso}\,(F_0 \dot{+} \hat{R})$, and so $||p_\alpha||_{F_0 \dot{+} \hat{R}=1}$, too.*

c) The subspace $F_0 \dot{+} \hat{R}$ is closed, and if both subspaces F_0 and $\hat{R}$ are non-trivial and $p_{F_0,\hat{R}} : F_0 \dot{+} \hat{R} \to F_0$ is the first projection, then $||1_{F_0 \dot{+} \hat{R}} - p_{F_0,\hat{R}}||_{F_0 \dot{+} \hat{R}} = 1$ and $||p_{F_0,\hat{R}}||_{F_0 \dot{+} \hat{R}} \leq 2$.

P r o o f. a). Let $x = x_1 + x_2$, where $x_1 \in R_0$ and $x_2 \in F$. For any $\epsilon > 0$ there exists a finite subset $\sigma \subset \mathcal{B}$ and a vector $x_1^\sigma \in \bigoplus_{\beta\in\sigma} H_\beta$ such that $||x_1 - x_1^\sigma||_E < \epsilon$. Since $u_\sigma = \prod_{\beta\in\sigma}(1_E - 2p_\beta) \in \mathrm{Iso}\, E$ and $u_\sigma(x_1^\sigma) = -x_1^\sigma$, $u_\sigma(x_2) = x_2$, we have $||x_1^\sigma + x_2||_E = ||-x_1^\sigma + x_2||_E$. Thus, if $R_0 \neq \{0\}$ and $F \neq \{0\}$, then $||1_{R_0 \dot{+} F} - 2p_{R_0,F}||_{R_0 \dot{+} F} = 1$, and therefore also $||p_{R_0,F}||_{R_0 \dot{+} F} = ||1_{R_0 \dot{+} F} - p_{R_0,F}||_{R_0 \dot{+} F} = 1$, if both subspaces R_0 and F are non-trivial. By the closed graph theorem, this implies that the subspace $R_0 \dot{+} F$ is closed.
b). If $\dim V_\alpha < \infty$, put $p'_\alpha = (1/\mathrm{card}\, W_\alpha) \sum_{g\in W_\alpha} g$. Then p'_α is a projection on the fixed point subspace F_α of the group W_α, which coincides with $\cap_{s_{e,e^*}\in \mathrm{IR}_\alpha} \mathrm{Ker}\, e^*$,

and thus it is a complementary subspace to $V_\alpha = \operatorname{Ker} p'_\alpha$. It is clear that $||p'_\alpha||_E = 1$, and so $||1_E - p'_\alpha||_E \le 2$. From Lemma 5.4 and the definition of p'_α it follows that the projection $p_\alpha = (1_E - p'_\alpha) \mid (R_0 \dot{+} F)$ satisfies i); by the above inequalities, it also satisfies ii).

Next consider the case when $\dim V_\alpha = \infty$ and the Coxeter group W_α is of type A_Δ. For a finite subset $\sigma \subset \Delta$ denote by V_σ the subspace generated by the root vectors $\epsilon_\delta - \epsilon_{\delta'}$, where $\delta,\ \delta' \in \sigma,\ \delta \ne \delta'$, and by W_σ the Coxeter group of type A_n, where $n = \dim V_\sigma$, generated by the isometric reflections along these vectors. Define the projections p'_σ resp. p_σ in the same way as p'_α resp. p_α above. It is clear that p_σ commutes with any reflection $s \in \mathrm{IR}\,(E) \setminus \mathrm{IR}_\alpha$ and satisfies all the other properties in i), ii). It is easily seen that the net (p_σ) is strongly convergent to the identity on the subspace V_α, and that all the projections p_σ vanish on the subspace $\hat{R} \dot{+} V'_\alpha$, where $V'_\alpha = \overline{\operatorname{span}}\,(\cup_{\alpha' \in \mathcal{A} \setminus \{\alpha\}} V_{\alpha'})$. As in (a) above it follows that $F_0 = V_\alpha \dot{+} V'_\alpha$. Therefore, this net is strongly convergent on the subspace $F_0 \dot{+} \hat{R}$ to the projection p_α which has the properties i) and ii).

Finally, suppose that W_α is a Coxeter group of type B_Δ. Then for any finite subset $\sigma \subset \Delta$ the product $u_\sigma = \prod_{\delta \in \sigma} s_\delta$ of pairwise commuting reflections $s_\delta = s_{\epsilon_\delta, \epsilon^*_\delta} \in \mathrm{IR}_\alpha$ is an isometric involution with the fixed point subspace $\cap_{\delta \in \sigma} \operatorname{Ker} \epsilon^*_\delta \supset \hat{R} \dot{+} V'_\alpha$. Similarly as above, the net of the restrictions $(u_\sigma \mid (F_0 \dot{+} \hat{R}))$ is strongly convergent to an isometric involution u_α which has V_α and $\hat{R} \dot{+} V'_\alpha$ as its spectral subspaces. It is easily seen that the projection $p_\alpha = (1_{F_0 \dot{+} \hat{R}} + u_\alpha)/2$ possesses all the properties mentioned in i), ii) and iii).

c). By the closed graph theorem, it is enough to check the second statement. For a finite subset $\sigma \subset \mathrm{IR}_{\mathcal{A}}$ let W_σ be a finite group generated by reflections from σ, and let V_σ be the linear span of the reflection vectors of these reflections. Then the action of W_σ in V_σ is fixed point free, and so the projection $p'_\sigma = (1/\operatorname{card} W_\sigma) \sum_{g \in W_\sigma} g$ onto the fixed point subspace $F_\sigma \supset \hat{R}$ of W_σ vanishes on V_σ. Consider the net of finite dimensional projections $(p_\sigma = 1_{F_0 \dot{+} \hat{R}} - p'_\sigma \mid (F_0 \dot{+} \hat{R}))$ onto the subspaces V_σ. Observe that $\bigcup_\sigma V_\sigma$ is dense in the subspace F_0. Since all of p_σ vanish on $\hat{R}$ and satisfy the norm inequalities of ii), this net is strongly convergent to the projection $p_{F_0, \hat{R}}$, which also satisfies these inequalities. This completes the proof. ■

R e m a r k. For further information on the Hilbert decomposition, see Proposition 6.2 and Examples 6.8 below.

5.7. Theorem. *The subspaces* $F, F_0, \hat{R}$ *and* R_0 *are invariant with respect to the group* $\operatorname{Iso} E$, *and there are natural monomorphisms* $\operatorname{Iso} E \hookrightarrow \operatorname{Iso} R_0 \times \operatorname{Iso} F_0$, $G_0(E) \hookrightarrow \prod_{\beta \in \mathcal{B}} G_0(H_\beta)$ *and* $\prod_{\beta \in \mathcal{B}} \mathrm{O}(H_\beta) \hookrightarrow \operatorname{Iso}(F \dot{+} R_0)$.

P r o o f. The invariance of the subspaces F and F_0 was already established in 5.2; the invariance of R_0 follows from the remark in 5.3. Similar arguments applied to the conjugate action of $\mathrm{Iso}\, E$ on E^* provide the invariance of $\hat{R}$.

Since the set $\{e \in S(E) \mid s_{e,e^*} \in IR(E)\}$ is contained in $F_0 \cup (\bigcup_{\beta\in\mathcal{B}} H_\beta) \subset R \dot{+} F_0$, the latter summands being invariant, it follows from Lemma 4.1 that the restriction mappings $g \longmapsto g \mid R_0$, $g \longmapsto g \mid F_0$, $g \longmapsto g \mid H_\beta$ induce the monomorphisms $\mathrm{Iso}\, E \hookrightarrow \mathrm{Iso}\, R_0 \times \mathrm{Iso}\, F_0$ and $G_0(E) \hookrightarrow \prod_{\beta\in\mathcal{B}} G_0(H_\beta)$.

As for the last statement, fix arbitrary $g = \prod_{\beta\in\mathcal{B}} \bar{u}_\beta \in \prod_{\beta\in\mathcal{B}} \mathrm{O}(H_\beta)$. For any finite subset $\sigma \subset \mathcal{B}$ put $u_\sigma = \prod_{\beta\in\sigma} u_\beta$, where $u_\beta = \bar{u}_\beta p_\beta + (1_E - p_\beta) \in \mathrm{Iso}\, E$ (see Theorem 1.b)). We will show that the net $\{u_\sigma \mid (F \dot{+} R_0)\} \subset \mathrm{Iso}\,(F \dot{+} R_0)$ strongly converges to an element $u \in \mathrm{Iso}\,(F \dot{+} R_0)$ such that $u \mid H_\beta = \bar{u}_\beta$. Therefore, the correspondence $\prod_{\beta\in\mathcal{B}} \mathrm{O}(H_\beta) \ni g \longmapsto u \in \mathrm{Iso}\,(F \dot{+} R_0)$ yields the desired monomorphism.

By the Banach–Steinhaus theorem, it is enough to show that for any $x \in F \dot{+} R_0$ the generalized sequence $(u_\sigma(x))$ is convergent. Let $x = x_1 + x_2$, where $x_1 \in F$ and $x_2 \in R_0$. For any $\epsilon > 0$ there exists a finite subset $\sigma \subset \mathcal{B}$ such that $||(1_E - \sum_{\beta\in\sigma} p_\beta)(x_2)||_E < \epsilon/2$. If σ' and σ'' are two finite subsets of $\mathcal{B}$ containing σ, then $u_{\sigma'} - u_{\sigma''} = (u_{\sigma'} - u_{\sigma''})(1_E - \sum_{\beta\in\sigma} p_\beta)$, and so

$$||(u_{\sigma'} - u_{\sigma''})(x_2)||_E \le ||u_{\sigma'}(1_E - \sum_{\beta\in\sigma} p_\beta)(x_2)||_E + ||u_{\sigma''}(1_E - \sum_{\beta\in\sigma} p_\beta)(x_2)||_E < \epsilon \ .$$

Thus, $(u_\sigma(x))$ is a generalized Cauchy sequence, and hence it is convergent. This proves the theorem. ■

R e m a r k. In general, the monomorphisms in Theorem 5.7 are not surjective; see Example 6.8.2 below.

6. An application: Isometry groups of ideal generalized sequence spaces

6.1. D e f i n i t i o n s. Recall the following notions (see, e.g., [17, 19]). Let $(e_\alpha)_{\alpha\in\Delta}$ be a system of vectors in a Banach space E_0. It is called *a generalized Shauder basis* of E_0 if each vector $e \in E_0$ has a unique, up to permutations, decomposition $e = \sum_{i=1}^\infty a_i e_{\alpha_i}$, where $(\alpha_i)_{i=1,\ldots}$ is a sequence of pairwise distinct indicies from Δ. If this series is still convergent to e after any permutation of its members, then this basis is called *unconditional*. In this case for any choices of signes $\theta = (\theta_\alpha)_{\alpha\in\Delta}$, where $\theta_\alpha = \pm 1$, the linear operators $M_\theta(e) = \sum_{i=1}^\infty \theta_{\alpha_i} a_i e_{\alpha_i}$ are uniformly bounded. The number $\sup_\theta ||M_\theta||_{E_0}$ is called *the unconditional constant* of the basis $(e_\alpha)_{\alpha\in\Delta}$. For instance, any complete orthonormal system in a Hilbert space is an unconditional basis with the unconditional constant 1. If the index set Δ is countable, we have the usual notion of an unconditional basis.

The generalized unconditional basis $(e_\alpha)_{\alpha\in\Delta}$ is called *symmetric* if for any bijection $\pi : \Delta \to \Delta$ the linear operator

$$\pi^* : E_0 \ni e = \sum_{i=1}^{\infty} a_i e_{\alpha_i} \mapsto \sum_{i=1}^{\infty} a_i e_{\pi(\alpha_i)} = \pi^*(e) \in E_0$$

is bounded, and so the infinite symmetric group $S_\Delta = \mathrm{Biject}(\Delta)$ acts in E_0, being uniformly bounded there. The constant $\sup_{\theta,\pi}||M_\theta \pi^*||_{E_0}$ is called *the symmetric constant* of the basis $(e_\alpha)_{\alpha\in\Delta}$.

For instance, in the classical Banach space $c_0(\Delta)$ of generalized sequences convergent to zero, with Δ as a set of indicies, the system of the standard basis vectors $(\epsilon_\delta)_{\delta\in\Delta}$ form a symmetric basis with the symmetric constant 1 (observe that each vector in $c_0(\Delta)$ has a countable support). Fixing a generalized unconditional basis $(e_\alpha)_{\alpha\in\Delta}$ in E_0, we obtain a representation of E_0 as a generalized sequence space contained in $c_0(\Delta)$. If the unconditional constant of this basis is 1, then E_0 is an ideal Banach lattice.

Recall that *an ideal generalized sequence space* E is a Banach space of sequences defined on an index set Δ such that if $x = (x_\alpha)_{\alpha\in\Delta} \in E$, then for any sequence $y = (y_\alpha)_{\alpha\in\Delta}$ with $|y_\alpha| \le |x_\alpha|$ for all $\alpha \in \Delta$ one has $y \in E$ and $||y||_E \le ||x||_E$. It is called *a symmetric generalized sequence space* if E is an ideal generalized sequence space, where the symmetric group S_Δ of all bijections of Δ acts isometrically.

The next simple lemma should be well known; by the lack of references we give a proof. We say that a family of reflections is *orthogonal* if the reflections from the family pairwise commute.

6.2. Lemma. *Let E be a Banach space with a total orthogonal family of isometric reflections $(s_\delta = s_{\epsilon_\delta,\epsilon^*_\delta})_{\delta\in\Delta}$. Identify E with a generalized sequence space with the index set Δ by posing $\bar{x} = (\epsilon^*_\delta(x))_{\delta\in\Delta}$ for $x \in E$. Let $E_0 = \overline{\mathrm{span}}\,(\epsilon_\delta \,|\, \delta \in \Delta)$. Then we have*

a) The system $(\epsilon_\delta)_{\delta\in\Delta}$ is a generalized unconditional basis in E_0 with the unconditional constant 1, and so E_0 is an ideal generalized sequence space.

b) If the Coxeter group B_Δ of permutations and sign changes of finite number of coordinates acts isometrically in E_0, then $(\epsilon_\delta)_{\delta\in\Delta}$ is a symmetric basis in E_0 with the symmetric constant 1, and so E_0 is a symmetric generalized sequence space.

P r o o f. a). Let σ be a finite subset of Δ. Consider the coordinate subspace

$$E_\sigma = \{x = (x_\delta)_{\delta\in\Delta} \in E_0 \,|\, x_\delta = 0 \text{ for all } \delta \notin \sigma\} \ .$$

Let $p_\sigma = \frac{1}{2}(1_{E_0} - u_\sigma)$, where $u_\sigma = \prod_{\delta\in\sigma} s_\delta$, be the coordinate projection $E_0 \to E_\sigma$. Since $u_\sigma \in \operatorname{Iso} E_0$, we have $||p_\sigma||_{E_0} = ||1_{E_0} - p_\sigma||_{E_0} = 1$.

Fix an arbitrary vector $x \in E_0$. For any $n \in \mathbf{N}$ there exists a finite subset σ_n of Δ and $y_n \in E_{\sigma_n}$ such that $||x - y_n||_E < 1/n$. Then also $||p_{\sigma_n}(x) - y||_{E_0} < 1/n$, and so $||(1_{E_0} - p_{\sigma_n})(x)||_{E_0} < 2/n$. It follows that x has at most countable support contained in $\Omega = \bigcup_{i=1}^{\infty} \sigma_i = \{\delta_1, \ldots, \delta_k, \ldots\}$, and $||x - \sum_{i=1}^{k} x_i \epsilon_{\delta_i}||_{E_0} \to 0$. Thus the system $(\epsilon_\delta)_{\delta\in\Delta}$ is a generalized Shauder basis in E_0. It is easily seen that for any fixed subset $\Omega \subset \Delta$ the net of isometric involutions $(u_\sigma \,|\, \sigma \subset \Omega, \operatorname{card}\sigma < \infty)$ strongly converges on E_0 to the isometric involution u_Ω, and therefore the basis $(\epsilon_\delta)_{\delta\in\Delta}$ of E_0 is unconditional with the unconditional constant 1.

b). Fix a permutation $\pi \in S_\Delta$, a vector $x \in E_0$ and $\epsilon > 0$ arbitrarily. Let σ be a finite subset of Δ such that $||(1_{E_0} - p_\sigma)(x)||_{E_0} < \epsilon$. There exists a finite permutation $\pi' \in S_\Delta$ such that $\pi'|\sigma = \pi|\sigma$. Since the Coxeter group B_Δ acts isometrically on E_0, we have

$$||\pi^* p_\sigma(x)||_{E_0} = ||\pi'^* p_\sigma(x)||_{E_0} = ||p_\sigma(x)||_{E_0} \ .$$

Thus, the linear operator π^* is well defined and isometric on the dense subspace $\mathbf{R}^\Delta$ of E_0. Therefore, it can be extended isometrically onto E_0, and since the same is true for $(\pi^{-1})^*$, this extension does belong to the group $\operatorname{Iso} E_0$. This proves *b*). ■

R e m a r k. It is not true, in general, that under the assumptions of this lemma E itself should be an ideal space if all single sign changes are isometries of E. As an example, consider the space c of convergent sequences, which is not an ideal lattice.

We return to the Hilbert decomposition, keeping all the notation and the conventions of Section 5.

6.3. Proposition. *There exists an ideal generalized sequence space X with $\mathcal{B}$ as an index set such that the subspace R_0 of E is isometric to the Banach sum $(\bigoplus_{\beta\in\mathcal{B}} H_\beta)_X$.*

P r o o f. For each $\beta \in \mathcal{B}$ fix a vector $e_\beta \in S(H_\beta)$. Consider the subspace $X = \overline{\operatorname{span}}\,(e_\beta \,|\, \beta \in \mathcal{B}) \subset F \dot{+} R_0$. Since the system of functionals $(e_\beta^* \,|\, \beta \in \mathcal{B})$ is biorthogonal to the system $(e_\beta \,|\, \beta \in \mathcal{B})$ and the reflections $s_{e_\beta, e_\beta^*} \,|\, X$ are isometric, by Lemma 5.9.a), the latter system is an unconditional basis in X with the unconditional constant 1, and so X can be identified with an ideal generalized sequence space on $\mathcal{B}$.

Put $R_1 = (\bigoplus_{\beta\in\mathcal{B}} H_\beta)_X$. We will show that the correspondence

$$\tau : R_0 \ni x \longmapsto (p_\beta(x))_{\beta\in\mathcal{B}} \in R_1$$

is a linear isometry of R_0 onto R_1. Put $e'_\beta = \frac{p_\beta(x)}{||p_\beta(x)||_E} \in S(H_\beta)$ if $p_\beta(x) \neq 0$. Let $\bar{u}_\beta \in \mathrm{O}(H_\beta)$ is such that $\bar{u}_\beta(e'_\beta) = e_\beta$ if $p_\beta(x) \neq 0$ and $\bar{u}_\beta = 1_{H_\beta}$ otherwise. As follows from Theorem 5.7, there exists $u \in \mathrm{Iso}\, R_0$ such that $u \,|\, H_\beta = \bar{u}_\beta$ for all $\beta \in \mathcal{B}$. Since $u(x) \in X$ and u is an isometry, it is clear that $\tau(x) \in R_1$ and $||\tau(x)||_{R_1} = ||x||_{R_0}$.

To show that τ is surjective, fix arbitrary vector $\bar{x} \in R_1$, $\bar{x} = (x_\beta \in H_\beta)_{\beta \in \mathcal{B}}$. Then $x' = \sum_{\beta \in \mathcal{B}} ||x_\beta||_E \, e_\beta \in X \subset R_0$. For each $\beta \in \mathcal{B}$ let $\bar{u}_\beta \in \mathrm{O}(H_\beta)$ be such that $\bar{u}_\beta(x_\beta) = ||x_\beta||_E \, e_\beta$. As above, there exists $u \in \mathrm{Iso}\, R_0$ such that $u \,|\, H_\beta = \bar{u}_\beta$ for all $\beta \in \mathcal{B}$. If $x_0 = u^{-1}(x')$, then we have $\tau(x_0) = \bar{x}$. Thus, τ is an inversible isometry. This completes the proof. □

6.4. Notation. Consider again an ideal generalized sequence space E with an index set Δ. Without lost of generality one may assume that $||\epsilon_\delta||_E = 1$ for all $\delta \in \Delta$. For a subset $\Omega \subset \Delta$ let $E(\Omega)$ be a strip $E(\Omega) = \{x = (x_\delta) \in E \,|\, x_\delta = 0$ for all $\delta \in \Delta \setminus \Omega\}$. Any such strip is biorthogonally complemented in E; indeed, the operator of multiplication by the characteristic function of Ω is a bicontractive projection $p_\Omega : E \to E(\Omega)$ with $1_E - p_\Omega = p_{\Delta \setminus \Omega}$.

Put $E_0(\Omega) = \overline{\mathrm{span}}\,(\epsilon_\delta)_{\delta \in \Omega}$, so that $E = E(\Delta)$, $E_0 = E_0(\Delta)$ and $E_0(\Omega) = E_0 \cap E(\Omega)$. We also preserve in this particular case all the other notation introduced in Section 5. The next proposition shows that the Hilbert and Coxeter decompositions of an ideal generalized sequence space yield an orthogonal decomposition into strips.

A reflection vector ϵ_δ of the single sign change $s_\delta = s_{\epsilon_\delta, \epsilon^*_\delta} \in \mathrm{IR}(E)$ belongs to a certain subspace V_α or H_β. Putting $\Delta_\alpha = \{\delta \in \Delta \,|\, s_\delta \in V_\alpha\}$ and $\Delta_\beta = \{\delta \in \Delta \,|\, s_\delta \in H_\beta\}$, we obtain a disjoint partition of Δ by the subsets $\{\Delta_\alpha, \Delta_\beta\}_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}}$. Put also $\Delta_\mathcal{A} = \bigcup_{\alpha \in \mathcal{A}} \Delta_\alpha$ and $\Delta_\mathcal{B} = \bigcup_{\beta \in \mathcal{B}} \Delta_\beta$, so that $\Delta = \Delta_\mathcal{A} \cup \Delta_\mathcal{B}$.

Proposition 6.5. *In the notation as above one has*

a)

i) $V_\alpha = E_0(\Delta_\alpha)$, $\hat{V}_\alpha = E(\Delta_\alpha)$ *for each* $\alpha \in \mathcal{A}$ *and*

ii) $H_\beta = E(\Delta_\beta) = E_0(\Delta_\beta) = l_2(\Delta_\beta)$ *for each* $\beta \in \mathcal{B}$;

iii) *if* $\mathrm{card}\,(\Delta_\alpha) = \infty$, *then* W_α *is a Coxeter group of type* B_{Δ_α};

b)

i) $F_0 = E_0(\Delta_\mathcal{A})$ *and* $R_0 = E_0(\Delta_\mathcal{B})$, *so that* $E_0 = R_0 \dot{+} F_0$;

ii) $F = E(\Delta_\mathcal{A})$ *and* $\hat{R} = E(\Delta_\mathcal{B})$, *so that* $E = \hat{R} \dot{+} F$;

c)

i) $p_\alpha = p_{\Delta_\alpha} \,|\, (F_0 \dot{+} \hat{R})$ *and* $p_\beta = p_{\Delta_\beta}$ *for all* $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$*;*

ii) $p_{R_0,F} = p_{\Delta_\mathcal{B}} \,|\, (R_0 \dot{+} F)$ *and* $p_{F_0,\hat{R}} = p_{\Delta_\mathcal{A}} \,|\, (F_0 \dot{+} \hat{R})$ *(see Proposition 5.6).*

P r o o f. a). Put $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $V_\gamma = H_\gamma$ for $\gamma \in \mathcal{B}$. By the above definitions, $E_0(\Delta_\gamma) \subset V_\gamma$ for all $\gamma \in \mathcal{C}$. Let $\gamma \in \mathcal{C}$, $\delta \in \Delta_\gamma$ and $\delta' \in \Delta \setminus \Delta_\gamma$. By Lemma 5.4, $s_{\delta'} = s_{\epsilon_{\delta'}, \epsilon^*_{\delta'}} \in \mathrm{IR}(E) \setminus IR_\gamma$ commutes with any reflection $s \in IR_\gamma$, and so $V_\gamma \subset \operatorname{Ker} \epsilon^*_{\delta'}$. Therefore, $\hat{V}_\gamma \subset \operatorname{Ker} \epsilon^*_{\delta'}$, too, and hence $\hat{V}_\gamma \subset \bigcap_{\delta' \in \Delta \setminus \Delta_\gamma} \operatorname{Ker} \epsilon^*_{\delta'} = E(\Delta_\gamma)$. In particular, each reflection vector e of a reflection $s = s_{e,e^*} \in IR(E)$ belongs to one of the strips $E(\Delta_\gamma)$, where $\gamma \in \mathcal{C}$. Namely, $e = (x_\delta) \in E(\Delta_\gamma)$ iff $x_\delta \neq 0$ for at least one $\delta \in \Delta_\gamma$. Furthermore, in the latter case either $e = \pm \epsilon_\delta$ or the reflections s and s_δ do not commute.

Let $\gamma = \alpha \in \mathcal{A}$. From the classification of the infinite Coxeter groups in Section 4 it follows that if W is such a group and $s \in W$, then the set of all reflections in W that do not commute with s contains not more than a finite subset of pairwise commuting reflections. This means that the reflection vector e of any given reflection $s \in IR_\alpha$ has only a finite number of non-zero coordinates, i.e., $e \in \operatorname{span}(\epsilon_\delta \,|\, \delta \in \Delta_\alpha) \subset E_0(\Delta_\alpha)$. Thus, $V_\alpha \subset E_0(\Delta_\alpha)$, and therefore, $V_\alpha = E_0(\Delta_\alpha)$, which is the first statement of $(a.i)$. In particular, the reflection vectors $(\epsilon_\delta)_{\delta \in \Delta_\alpha}$ of sign change reflections $(s_\delta)_{\delta \in \Delta_\alpha} \subset IR_\alpha$ form a complete orthogonal system in V_α.

If $\operatorname{card}(\Delta_\alpha) < \infty$, then clearly $E(\Delta_\alpha) = E_0(\Delta_\alpha) = V_\alpha = \hat{V}_\alpha$. If $\operatorname{card}(\Delta_\alpha) = \infty$, then by Corollary 4.6, the group W_α generated by reflections from IR_α is a Coxeter group of type $A_{\Delta'}$ or $B_{\Delta'}$. But the Coxeter group $A_{\Delta'}$ does not contain a complete set of pairwise commuting reflections, i.e., there is no orhtogonal subsystem of the root system $(\epsilon_\delta - \epsilon_{\delta'} \,|\, \delta, \delta' \in \Delta',\, \delta \neq \delta')$ which would be a Hamel basis of $\mathbf{R}_0^{\Delta'}$. This excludes the first case, and so the group W_α should be a Coxeter group of type $B_{\Delta'}$. It is clear that $\operatorname{card}(\Delta') = \operatorname{card}(\Delta_\alpha)$. This proves $(a.iii)$.

Let, further, $\gamma = \beta \in \mathcal{B}$. Then $E_0(\Delta_\beta)$ is a subspace of the Hilbert space H_β, and the system $(\epsilon_\delta)_{\delta \in \Delta_\beta}$ is an orthonormal basis of $E_0(\Delta_\beta)$. Thus, $E_0(\Delta_\beta) = \mathrm{l}_2(\Delta_\beta)$. Assume that $H_\beta \neq E_0(\Delta_\beta)$. Let $x \in H_\beta$ be a non-zero vector orthogonal to $E_0(\Delta_\beta)$. It is easily seen that $\epsilon^*_\delta(x) = 0$ for all $\delta \in \Delta_\beta$. This is impossible, since $H_\beta \subset E(\Delta_\beta)$ and $x \neq \bar{0}$. Therefore, $H_\beta = E_0(\Delta_\beta) = \mathrm{l}_2(\Delta_\beta)$.

Let $p_\beta : E \to H_\beta$ be the projection as in Theorem 1.b). Suppose that $H_\beta \neq E(\Delta_\beta)$. Then the restriction $p_\beta \,|\, E(\Delta_\beta)$ is a non-identical projection, so that there exists a non-zero vector $x \in \operatorname{Ker} p_\beta \cap E(\Delta_\beta)$. Fixing $\delta \in \Delta_\beta$, consider the plane $L = \operatorname{span}(x, \epsilon_\delta)$. There are two commuting isometric reflections in L, namely $(1_L - 2p_\beta) \,|\, L$ and $s_\delta \,|\, L$. Therefore, $x \in \operatorname{Ker} e^*_\delta$ for all $\delta \in \Delta_\beta$, and so

$x = \bar{0}$, which is a contradiction. Hence $H_\beta = E(\Delta_\beta) = E_0(\Delta_\beta) = \mathrm{l}_2(\Delta_\beta)$. This proves (a), besides the second equality in $(a.i)$, which is proved below.

$b)$. For any $\gamma \in \mathcal{C}$ consider the isometric involution $u_\gamma = 1_E - 2p_{\Delta_\gamma}$ with the spectral subspaces $E(\Delta_\gamma)$ and $E(\Delta \setminus \Delta_\gamma)$. It is easily seen that for any $s = s_{e,e^*} \in IR(E)$ the isometries su_γ and $u_\gamma s$ coincide on the total system of reflection vectors $(\epsilon_\delta)_{\delta\in\Delta}$. From Lemma 4.1 it follows that they coincide on E. Thus, the involution u_γ commutes with each reflection $s_{e,e^*} \in IR(E)$. Therefore, one of its spectral subspaces contains the vector e and another one is contained in the mirror hyperplane $\mathrm{Ker}\, e^*$. Hence, for any $s_{e,e^*} \in IR_\gamma$ one has $\mathrm{Ker}\, e^* \supset E(\Delta\setminus\Delta_\gamma)$.

Let the set $\mathcal{C}$ be devided into two disjoint parts $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$. Put $\Omega' = \bigcup\limits_{\gamma\in\mathcal{C}'} \Delta_\gamma$, $\Omega'' = \bigcup\limits_{\gamma\in\mathcal{C}''} \Delta_\gamma$, so that Ω', Ω'' consist of some parts of the disjoint partition $\Delta = \bigcup\limits_{\gamma\in\mathcal{C}} \Delta_\gamma$. We are going to show, more generally, that $\widehat{E_0(\Omega')} = E(\Omega')$, which easily implies the equalities in $(b.ii)$ and $(a.i)$.

By the considerations above, we have $E(\Omega') \subset \bigcap(\mathrm{Ker}\, e^* \mid s_{e,e^*} \in \mathrm{IR}_{\mathcal{C}''})$, where $\mathrm{IR}_{\mathcal{C}''} = \bigcup\limits_{\gamma\in\mathcal{C}''} \mathrm{IR}_\gamma$. On the other hand, $E(\Omega') = \bigcap\limits_{\delta\in\Omega''} \mathrm{Ker}\, \epsilon^*_\delta \supset \bigcap(\mathrm{Ker}\, e^* \mid s_{e,e^*} \in IR_{\mathcal{C}''})$. Therefore, $E(\Omega') = \bigcap(\mathrm{Ker}\, e^* \mid s_{e,e^*} \in \mathrm{IR}_{\mathcal{C}''}) = \widehat{E_0(\Omega')}$. The last equality is clear from the definition of the envelope, because $s_{e,e^*} \in IR_{\mathcal{C}''}$ iff $E_0(\Omega') \subset \mathrm{Ker}\, e^*$. This proves (b) and the second equality in $(a.i)$.

$c)$. The isometric involutions $u_\beta = 1_E - 2p_{\Delta_\beta}$ and $1_E - 2p_\beta$ coincide on vectors of the system $(\epsilon_\delta)_{\delta\in\Delta}$, so by Lemma 4.1, they coincide on E. This proves the second equality in $(c.i)$. By the same reasoning (see Proposition 5.6.$b).iii$) the first equality in $(c.i)$ holds. The equalities $(c.ii)$ follow from (b), just by the definition of the projections involved. By Proposition 5.6.$(b.i)$, the projection p_α commutes with any sign change reflection s_δ. By the same type of arguments as those used in the proof of (b), it follows that $\mathrm{Ker}\, p_\alpha = \mathrm{Ker}\,(p_{\Delta_\alpha} \mid (F_0 \dot{+} \hat{R}))$. Since the images also coincide, we have the first equality in $(c.i)$. This proves the proposition. ∎

This proposition, together with Theorem 5.7 and the remark that the union of the subspaces H_β, $\beta \in \mathcal{B}$, is invariant with respect to the group $\mathrm{Iso}\, E$, leads to the following

6.6. Corollary. *a)*

$$G_0(E) \subset \bigoplus_{\beta\in\mathcal{B}} G_0(l_2(\Delta_\beta)) \quad \textit{and} \quad \mathrm{Iso}\, E \subset \mathrm{Iso}\, E(\Delta_{\mathcal{A}}) \bigoplus \mathrm{Iso}\, E(\Delta_{\mathcal{B}}) \ ;$$

b) each element of the group $(\mathrm{Iso}\, E) \mid E(\Delta_{\mathcal{B}})$ *is of the form* $(x_\beta) \mapsto (u_\beta(x_\beta))$, *where* $x_\beta \in \mathrm{l}_2(\Delta_\beta)$, $u_\beta : \mathrm{l}_2(\Delta_\beta) \to \mathrm{l}_2(\Delta_{\pi(\beta)})$ *is an isometry of Hilbert spaces for each* $\beta \in \mathcal{B}$, *and* π *is a permutation of the set* $\mathcal{B}$.

6.7. R e m a r k. Let $\alpha \in \mathcal{A}$ be such that $\operatorname{card} \Delta_\alpha = \infty$. Then $\operatorname{Iso} V_\alpha$ contains a Coxeter subgroup of type B_{Δ_α}. It is not true in general that it contains also the symmetric group S_{Δ_α} of shift operators. In fact, this latter group is contained in $\operatorname{Iso} V_\alpha$, but probably in some other representation of V_α as an ideal generalized sequence space. Indeed, consider any symmetric generalized sequence space M on Δ_α, such that the system $(\epsilon_\delta)_{\delta\in\Delta}$ is a Shauder basis in M. Fix a disjoint partition of Δ_α into pairs (δ, δ'). Then by Lemma 6.2.a), the corresponding subsystem $(\epsilon_\delta \pm \epsilon_{\delta'})$ of the root system is an unconditional Shauder basis in M with the unconditional constant 1, and this basis is not symmetric. Thus, using the dual system of functionals, one can represent the strip component $M = V_\alpha$ as an ideal generalized sequence space which is not symmetric and such that the isometry group does not act as permutations and sign changes (the image of a basis vector under an isometric reflection might be a vector with 4 non-zero coordinates!). Recall that a symmetric basis is unique; moreover, a basis which in a sense is symmetric enough, is unique [11]. Thus, here we have an unconditional basis with a relatively small group of symmetries.

Conversely, if $g \in \operatorname{Iso} E$ is such that $g(V_\alpha) = V_{\alpha'}$, where $\alpha \in \mathcal{A}$ is as above, then one can represent V_α resp. $V_{\alpha'}$ as a symmetric generalized sequence space on Δ_α resp. $\Delta_{\alpha'}$, and then g should be an operator of the form $(x_\alpha) \mapsto (\pm x_{\pi(\alpha)})$, where $\pi : \Delta_\alpha \to \Delta_{\alpha'}$ is a bijection (indeed, π must transfer the sign change reflections from IR_α into sign change reflections from $\mathrm{IR}_{\alpha'}$).

In [24, 25] certain conditions on an ideal generalized sequence space are given which guarantee that its isometry group acts by permutations and sign changes. This is always the case in a symmetric sequence spaces different from l_2 [23, Ch. IX, 6] (see also [2, 8] for the complex field).

Next we give several examples related to the results of Sections 5 and 6.

6.8. E x a m p l e s.

1) ([24, 25]). Fix a sequence of real numbers $p_k \geq 1$, $k = 1, \ldots$. *The Orlicz–Nakano space* $E = l(\{p_k\})$ consists of all sequences of real numbers $x = (\xi_k)_{k=1}^\infty$ such that the following norm is finite:

$$||x||_E = \inf \{\lambda > 0 \,|\, \sum_{k=1}^{\infty} |\xi_k/\lambda|^{p_k} \leq 1\} \, .$$

It is an ideal sequence space.

Put $\Delta_q = \{i \in \mathbf{N} \,|\, p_i = q\}$, where $q = q_1, q_2, \ldots$ are pairwise distinct. Then $\mathcal{A} = \{q_i \,|\, i \neq 2\}$ and $\Delta_{\mathcal{A}} = \{i \,|\, p_i \neq 2\}$, $\mathcal{B} = \{2\}$ if $\Delta_2 \neq \emptyset$ and $\mathcal{B} = \emptyset$ otherwise; $E(\Delta_q) = \mathrm{l}_q(\Delta_q)$. The group $\operatorname{Iso} E$ is the direct product of the groups $\mathrm{O}\,(\mathrm{l}_2(\Delta_2))$ and $\operatorname{Iso} \Delta_{\mathcal{A}}$, where $\operatorname{Iso} \Delta_{\mathcal{A}}$ is the group of all permutations of coordinates ξ_i, $i \in \Delta_{\mathcal{A}}$, preserving the partition $\Delta_{\mathcal{A}} = \bigcup \Delta_{q_i}$, and arbitrary sign changes of these coordinates. Indeed, this direct product evidently is a subgroup of Iso E; the converse inclusion follows from the results of [24, 25] in view of the decomposition from Corollary 6.6.

In a similar way one can describe the isometry groups of more general modular sequence spaces or of Banach sums of (symmetric) ideal sequence spaces.

2) Let E be the space of all convergent complex sequences with the supremum norm. Then E is a Banach sum of the real euclidean planes H_i, $i = 1, \ldots$. We have that $E = \hat{R}$, and $E_0 = R_0$ is the subspace of sequences in E convergent to zero. The group Iso E_0 is the semi-direct product of $\mathrm{O}(2)^\omega$ and the infinite symmetric group S_ω, while Iso E is its proper subgroup (indeed, if $g \in$ Iso E, then the corresponding sequence of orthogonal plane transformations from $\mathrm{O}(2)^\omega$ is convergent). This shows that all the inclusions in Theorem 5.7 are strict. Observe that here E is not an ideal sequence space.

3) Consider $E = \mathbf{R}^4$ with the norm

$$||x||_E = ||(\xi_1, \xi_2, \varsigma_1, \varsigma_2)||_E = [((\xi_1^2 + \xi_2^2)^{1/2} + |\varsigma_1|)^2 + \varsigma_2^2]^{1/2}.$$

It is easily seen that here $\hat{R} = R_0 = H = \{x \in E \mid \varsigma_1 = \varsigma_2 = 0\}$ and $F = F_0 = \{x \in E \mid \xi_1 = \xi_2 = 0\}$. Furthermore, $E = F_0 \dot{+} R_0$ is an ideal space, and both strips F_0 and R_0 are euclidean planes. Thus, $G_0(E) \neq G_0(F_0) \oplus G_0(R_0)$, and so Iso $E \neq$ Iso $F_0 \oplus$ Iso R_0 (cf. Corollary 6.6.*a*)).

4) Slightly modifying example 4, consider $E \cong \mathbf{R}^6$ with the norm

$$||x||_E = ||(\xi_1, \xi_2, \eta_1, \eta_2, \varsigma_1, \varsigma_2)||_E = [((\xi_1^2 + \xi_2^2)^{1/2} + |\varsigma_1|)^2 + ((\eta_1^2 + \eta_2^2)^{1/2} + |\varsigma_2|)^2]^{1/2}.$$

Being the direct sum of two euclidean planes H_1 and H_2, which are strips invariant under $G_0(E)$, the subspace R_0 itself is euclidean. Thus, $G_0(R_0) \neq G_0(H_1) \oplus G_0(H_2) = G_0(E)$ (cf. Corollary 6.6.*a*)).

5) Let, further, $\bar{E} = \mathrm{c} \oplus \mathrm{c} \oplus \mathrm{c}$ with the norm $||(x, y, z)||_{\bar{E}} = \sup_{i=1,\ldots} \{(\xi_i^2 + \eta_i^2)^{1/2} + |\varsigma_i|\}$, where $x = (\xi_i)_{i=1}^\infty \in \mathrm{c}$, $y = (\eta_i)_{i=1}^\infty \in \mathrm{c}$, $z = (\varsigma_i)_{i=1}^\infty \in \mathrm{c}$. Consider the hyperplane $E = \{(x, y, z) \in \bar{E} \mid \lim_{i\to\infty} \eta_i = \lim_{i\to\infty} \varsigma_i\}$. Here we have $\hat{R} \approx \mathrm{c} \oplus \mathrm{c}_0$, $F \approx \mathrm{c}_0$. Thus, $\hat{R} \dot{+} F$ is a hyperplane in E, and there is no contractive projection of E onto $\hat{R}$ and onto F, in contrary to the case of ideal sequence spaces (cf. Propositions 5.6 and 6.5.*b*), *c*)).

The following questions are directly related to the subject of this paper.

For a given Banach space E, consider the constant

$$c(E) = \inf_{e \in E,\, e^* \in E^*,\, e^*(e)=1} \{||s_{e,e^*}||_E\}.$$

It is clear that $1 \leq c(E) \leq 3$, and $c(E) = 1$ in the case when there exists an isometric reflection is E. It is easily seen that $c(L_p)$ is a convex function of p

which takes the value 1 only for $p = 2$ and the value 3 only for $p = 1$ and $p = \infty$. For any given finite set of reflections in E one can find an equivalent norm $||\cdot||'$ on E in such a way that the group generated by these reflections will be a subgroup of the isometry group of the new norm. In particular, $c(E, ||\cdot||') = 1$.

Consider, further, the constant

$$\varsigma(E) = \sup_{||\cdot||' \sim ||\cdot||_E} \{c(E, ||\cdot||')\} .$$

By the definition, $\varsigma(E) \in [1, 3]$ is a numerical invariant of isomorphism. Is it non-trivial?

Let $\varsigma(n) = \varsigma(\mathbf{R}^n) = \varsigma(\mathrm{l}_2^n)$. Let M_n be the Minkowski compact of classes of isometric norms in $\mathbf{R}^n$, endowed with the Banach–Mazur distance. Denote by A_n the subset of M_n which consists of the classes of norms having an isometric reflection (or, the same, a hyperplane of symmetry). It is easy to show that $\log \varsigma(n)$ coincides with the radius of the metric factor space M_n/A_n with respect to the distinguish point which corresponds to A_n. It is known that the radius of the M_2 centred at the class of euclidean norms is $\log \sqrt{2}$ (F. Behrend, 1937; see [12, Sect. 7] for this and for some further information). Thus, $\varsigma(2) \leq \sqrt{2}$.

6.9. P r o b l e m. Is it true that

$\varsigma(3) < 3$?

$\varsigma(n) < 3$ for any n ?

$\limsup_{n\to\infty} \varsigma(n) < 3$?

$\varsigma(\mathrm{l}_2) < 3$?

If the answer to any of the above questions is “yes”, which seems to be less plausible, then, of course, the exact value of the corresponding constant ς would be worthwhile to find.

Added in proofs. Interesting generalizations of some results of the present paper* have been recently obtained (after the paper of A. Skorik and M. Zaidenberg) in the preprint by J.B. Guerrero, A.R. Palacios, Isometric reflections on Banach spaces, the University of Granada, Spain, 1996, 14 p. We are grateful to Prof. Palacios for sending us this preprint.

*Or, to be more precise, of the English version of the preprint [27].

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Об изометрических отражениях в банаховых пространствах

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Получена следующая характеризация гильбертовых пространств. Пусть E – банахово пространство, единичная сфера S которого обладает гиперплоскостью симметрии. В этом случае E является гильбертовым пространством тогда и только тогда, когда удовлетворяется какое-либо из следующих двух условий:
а) группа изометрий Iso$E$ пространства E имеет плотную орбиту на S';
б) единичная компонента G_0 группы Iso$E$, наделенной сильной операторной топологией, действует топологически неприводимо на E.

Приводятся некоторые результаты о бесконечномерных группах Коксетера, порожденных изометрическими отражениями, которые позволяют анализировать структуру групп изометрий, содержащих достаточно много отражений.

Про ізометричні відображення у банахових просторах

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Отримано таку характеристику гільбертових просторів. Нехай E – банахів простір, одинична сфера S якого має гиперплощину симетрії. У цьому випадку E є гільбертів простір тоді і тільки тоді, коли вдовольняється будь-яке з наступних двох умов:
а) група ізометрій Iso$E$ простору E має щільну орбіту на S';
б) одинична компонента G_0 групи Iso$E$, що наділена сильною операторною топологією, діє топологічно незвідно на E.

Наводяться деякі результати про нескінченновимірні групи Коксетера, породжені ізометричними відображеннями, які дозволяють аналізувати структуру груп ізометрій, що містять досить багато відображень.