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On quantum systems of particles with singular magnetic interaction in one dimension. M-B statistics

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Quantum one-dimensional systems of particles interacting via singular "collective" (depending on all the position vectors of particles) vector electromagnetic potential is considered in a thermodynamic limit. The reduced density matrices in the limit are computed for the cases short-range interaction and one-dimensional analog of the Chern–Simons interaction (the j-th "collective" vector electromagnetic potential of n particles equals the partial derivative in the position vector of the j-th particle of the Coulomb potential energy of a system of n charged particles).

1. Introduction

We characterize ν-dimensional systems of n-particles with singular magnetic interaction by the "collective" vector electromagnetic potential $a_j(X_n)$, $X_n = (x_1, \ldots, x_n) \in \mathbf{R}^{\nu n}$, which depends on the differences $x_j - x_k$ of the position vectors of particles and has a mild singularity (in the neighborhood of hyperplane $x_j = x_k$ it behaves as $||x_j - x_k||^{-n}$), and the Hamiltonian H_n defined on $C^\infty(\mathbf{R}_0^{\nu n})$, $\mathbf{R}_0^{\nu n} = \mathbf{R}^{\nu n} \setminus \bigcup_{j<k} (x_j = x_k)$,

$$\dot{H}_n = \frac{1}{2} \sum_{j=1}^{n} (p_j - a_j(X_n))^2, \quad X_n = (x_1, \ldots, x_n) \in \mathbf{R}^{\nu n},$$

$$a_j(X_n) \in C^\infty(\mathbf{R}_0^{\nu n}), \quad (p_j - a_j)^2 = \sum_{\alpha=1}^{\nu} (p_j^\alpha - a_j^\alpha)^2, \quad p_j = i^{-1}\partial_j. \tag{1.1}$$

The motivation to study such systems originates from the 2-d Chern–Simons (C–S) system which is believed to describe a phenomena of high temperature superconductivity based on the mechanism of the Bose condensation of clusters of anyons, i.e., particles with exotic statistics [1–3]. C–S system corresponds to the case

$$a_j^\alpha(X_n) = \epsilon^{\alpha\delta}\partial_j^\delta U_C(X_n) = \partial_j^\alpha U_{CS}(X_n), \quad X_n \in \mathbf{R}_0^{2n}, \tag{1.2}$$

where ∂_j^α is the partial derivative with respect to x_j^α, ϵ_δ^α is the antisymmetric tensor, there is a summation over the index δ:

$$U_{C(CS)}(X_n) = \sum_{1\le k<j\le n} \sigma_j\sigma_k\phi_{C(CS)}(x_j - x_k),$$

$$\phi_C(x) = \ln|x|, \quad \phi_{CS}(x) = \arctan\frac{x^2}{x^1}, \quad x = (x^1, x^2), \tag{1.3}$$

σ_j is the charge of the j-th particle. The existence of anyons is explained by the singularity of the C–S potential and equality (2): interaction is gauged out (formally) and the singular phase has discontinuity on union of hyperplanes $x_j = x_k$ that "spoils" symmetricity or anitisymmetricity of a complex wave function.

The C–S particle system is derived in Topological Electrodynamics (the Maxwell term is dropped in the Lagrangian containing the C–S form). There are many interesting conjectures concerning the phase structure of the system [4, 5]. But up to now a mechanism of the Bose condensation was not established. The description of anyons in the zero-temperature 3-d Lattice Scalar Quantum Topological Electrodynamics (QED) in rigorous terms was given by Frohlich and Marchetti in [6]. A change of phase diagram produced by the topological (C–S) term is poorly explored in the zero-temperature Lattice QED. Anyons at non-zero temperature up to now were not discussed in the framework of the Constructive QFT and QSM.

In the vector collective potential a_j satisfies the condition

$$a_j(X_n) = \partial_j U(X_n), \quad x_j \ne x_k, \tag{1.4}$$

there exists the simplest self-adjoint extension H_n of $\dot{H}_n$, which generates a contraction semi-group unitary equivalent to semi-group, whose infinitesimal generator is the minus one-half νn-dimensional Laplacian. It is not difficult to check that for the Dirichlet boundary condition and the Maxwell–Boltzmann (M–B) statistics the grand canonical partition function coincides with the grand partition function of free particles.

The conjecture that the system is equivalent to the free particle system in the thermodynamic limit seems plausible only for the case of short range magnetic interactions (U is expressed through k-particle "magnetic potentials" integrable by $k-1$ variables) when the reduced density matrices are easily computed in the thermodynamic limit. The existence of the matrices for long-range magnetic interactions (k-particle "magnetic potentials" are not integrable) is an open problem (we solve the problem for the simplest "integrable" $1-d$ system).

For the case of the Fermi or Bose statistics the aforementioned self-adjoint extension for the C–S system produces the system of free annoys. Another extension introduces interaction between them.

One-dimensional systems with singular magnetic interactions are also interesting. There are also anyons in the systems but they appear as a result of special self-adjoint extensions of the n-dimensional Laplacian restricted to $C_0^\infty(\mathbf{R}_0^n)$ or $\dot{H}_n$ (a simplest class of them are considered in this paper). The collective vector potential a_j creates interaction between them.

Earlier self-adjoint extensions, corresponding to jumps of partial derivatives of a wave function on the hyperplanes, where the position vectors coincide, were considered in [7, 8].

In this paper, we investigate one-dimensional system of r sorts of particles with the M–B statistics with magnetic interaction for which equality (1.4) holds and

$$U(X_n) = \sum_{1\le k<j\le n} \sigma_j \sigma_k \phi(x_j - x_k)\,. \tag{1.5}$$

At first we compute the reduced density matrices in the thermodynamic limit for the case of short range pair "magnetic potential" $\phi \in C^\infty(\mathbf{R}\backslash 0)\bigcap L^1(\mathbf{R})$ and the class of self-adjoint extensions of $\dot{H}_n$ corresponding to jumps of a wave function on the hyperplanes, where the position vectors of particles coincide. Then we study the system with long range pair "magnetic potential" $\phi = \lambda|x|$. It turns out that if $\sigma_j \in \gamma\mathbf{Z}$ then the reduced density matrices are non-trivial in the thermodynamic limit if the differences of variables sit on the lattice $2\pi\gamma^{-2}\lambda^{-1}\mathbf{Z}$.

It is not difficult to show that this system can be derived from the 2-d electrodynamics with the additional term $A_0\partial^1 A_1$ in the Lagrangian (the Maxwellian term has to be omitted).

2. Main results

Let's consider the Hamiltonian $\dot{H}_n$ with a_j satisfying equalities (1.4), (1.5) and the case $\nu = 1$. From simple equality

$$p_j - \hat{a}_j = \exp\{i\hat{U}_n\} p_j \exp\{-i\hat{U}_n\}$$

it follows that

$$\dot{H}_n = \exp\{i\hat{U}_n\}\dot{H}_n^0 \exp\{-i\hat{U}_n\}\,, \tag{2.1}$$

where $\hat{U}_n$, $\hat{a}_j$ are operators of multiplication by functions $U(X_n)$, $a_j(X_n)$, respectively, and $\dot{H}_n^0$ is the minus one-half n-dimensional Laplacian, restricted to $C^\infty(\mathbf{R}_0^n)$. Now let's define several functions

$$U^\epsilon(X_n) \quad = \sum_{1\le k<j\le n} \epsilon^*(x_j - x_k)\Gamma_{j,k}(\sigma_1, \ldots, \sigma_n)\,,$$

$$\epsilon^*(x) \;=\; \arccos\epsilon(x), \quad \epsilon(x) = \frac{x}{|x|}\,, a.e.\,,$$

$$U^*(X_n) = U(X_n) + U^\epsilon(X_n)\,,$$

where $\Gamma_{j,k}$ are functions on a discrete set. (It's not difficult to see that $\epsilon^*(x)$ can be changed into $\epsilon(x)$, and this change does not alter H_n). By D(A) we'll denote the domain of the operator A and by $\hat{U}_n^{*(\epsilon)}$ the operator of multiplication by the function $U^{*(\epsilon)}(X_n)$. These operators are unitary and the equality

$$\exp\{i\hat{U}_n^*\}C_0^\infty(\mathbf{R}_0^n) = C_0^\infty(\mathbf{R}_0^n) \tag{2.2}$$

holds. As the result the set $\exp\{i\hat{U}_n^*\}D(H_n^0)$ is dense in $L^2(\mathbf{R}^n)$. It is the domain of the self-adjoint operator H_n

$$H_n = \exp\{i\hat{U}^*\}H_n^0\exp\{-i\hat{U}_n^*\}\,. \tag{2.3}$$

Proposition 1. *Operator H_n is a self-adjoint extension of the operator $\dot{H}_n$.*

P r o o f follows immediately from the equality (1.2) and the fact that the operators of partial differentiation commute with the operator $\exp\{+(-)i\hat{U}_n^\epsilon\}$ on $C_0^\infty(\mathbf{R}_0^n)$.

Operator H_n is the infinitesimal generator of the contraction strongly continuos semi-group

$$P_n^t = \exp\{i\hat{U}_n^*\}\exp\{-tH_n^0\}\exp\{-i\hat{U}_n^*\}\,,$$

and by the "core theorem" its core concides with $\exp\{i\hat{U}_n^*\}S(\mathbf{R}_0^n)$ [9].

We'll assume in what follows that

$$\Gamma_{j,k} = \kappa_0\sigma_j\sigma_k\,. \tag{2.4}$$

Now we consider the system in the interval $[-L, L]$ with the Dirichlet boundary condition on its boundary, i.e., with the Hamiltonian $H_{n,L}$:

$$P_{n,L}^t = \exp\{-\beta H_{n,L}\} = \exp\{i\hat{U}_n^*\}P_{0(n,L)}^t\exp\{-i\hat{U}_n^*\}\,, \tag{2.5}$$

where the semi-group $P_{0(n,L)}^t$ is generated by the n-dimensional Laplacian with the Dirichlet boundary condition on the boundary of $[-L, L]$. Let's define the reduced density matrices for the systems of r sorts of particles ($\sigma_j \in \sum(r)$, $\sum(r)$ is the set of r elements) with the M–B statistics [10, 11]. Hense

$$\rho^L(X_m|Y_m) = \Xi_L^{-1}\prod_{k=1}^m z_{\sigma_k}\sum_{n\geq 0}(n!)^{-1}$$

$$\times \sum_{\sigma'_1,\dots,\sigma'_n} \prod_{s=1}^{n} z_{\sigma'_s} \int_{[-L,L]^n} P^{\beta}_{(L)}(X_m, X'_n|Y_m, X'_n)\, dX'_n\,, \tag{2.6}$$

where Ξ_L coincides with the numerator in (1.4) for the case $m=0$, the sums in σ'_j are performed over the set $\sum(r)$, z_σ is the activity of the particle with the "charge" σ, β is the inverse temperature, $P^{\beta}_{(L)}(X_n|Y_n)$ is the kernel of the operator $P^{\beta}_{n,L}$. The reduced density matrices in our case are functions in $\sigma_1,\dots,\sigma_m$, since the Hamiltonian is diagonal in variables that describe the internal degree of freedom. In order to simplify notations we don't indicate this dependence in ρ^Λ.

Lemma. *For the system with the Hamiltonian defined by the equalities* (2.4), (2.5) *the following equality true*:

$$\rho^L(X_m|Y_m) = \exp\{i[U^*(X_m) - U^*(Y_m)]\} \prod_{k=1}^{m} z_{\sigma_k} P^{\beta}_{0(L)}(x_k|y_k) \exp\{G_L(X_m, Y_m)\}\,,$$

$$G_L(X_m, Y_m)$$

$$= \sum_{\sigma} z_\sigma \int_{-L}^{L} \{\exp\{i[\sum_{j=1}^{m} \sigma\sigma_j(\phi^*(x_j - x) - \phi^*(y_j - x))]\} - 1\} P^{\beta}_{0(L)}(x|x)\, dx\,,$$

where $P^{\beta}_{0(L)}(x|y)$ is the integral over the Wiener measure concentrated on paths, starting at zero moment from x and arriving in y at the moment β, of the characteristic function of paths that are strictly inside $[-L, L]$, and

$$\phi^*(x) = \kappa_0 \epsilon^*(x) + \phi(x)\,.$$

Theorem 1. *Let the condition of the lemma be satisfied and $\phi(x) \in C^\infty(\mathbf{R}\backslash 0) \bigcap L^1(\mathbf{R})$, then the thermodynamic limit of the reduced density matrices are given by*

$$\rho(X_m|Y_m) = \lim_{L\to\infty} \rho^L(X_m|Y_m) = \exp\{i[U^*(X_m) - U^*(Y_m)]\} \prod_{k=1}^{m} z_{\sigma_k} P^{\beta}_{0}(x_k|y_k)$$

$$\times \sum_{\pi \in S_{2m}} \exp\{G^{\pi}_0(X_m, Y_m) + G^{\pi}(X_m, Y_m)\}\chi_\pi(X_m, Y_m)\,, \tag{2.7}$$

where S_{2m} is the permutation group of $2m$ elements, χ_π is the characteristic function of the set $v_{\pi(1)} < v_{\pi(2)} < \dots < v_{\pi(2m)}$, $V_{2m} = (X_m, Y_m)$,

$$G^{\pi}_0(X_m, Y_m) = \sum_{\sigma} z_\sigma (2\pi\beta)^{-1/2} \left(\int_{-\infty}^{v_{\pi(1)}} + \int_{v_{\pi(2m)}}^{\infty} \right)$$

$$\times [\exp\{i \sum_{j=1}^{m} \sigma\sigma_j (\phi(v_j - x) - \phi(v_{j+m} - x))\} - 1]\, dx\,,$$

$$G^{\pi}(X_m,Y_m)=\sum_{s=1}^{2m}\sum_{\sigma} z_{\sigma}(2\pi\beta)^{-1/2}$$

$$\times\int_{v_{\pi(s)}}^{v_{\pi(s+1)}}[\exp\{i\sum_{j=1}^{m}\sigma\sigma_j(\phi^*(v_j-x)-\phi^*(v_{j+m}-x))\}-1]\,dx\,.$$

Theorem 2. *Let $\phi(x)=\lambda|x|$, and $\Sigma(r)\subset\gamma\mathbf{Z}$, and the following condition be satisfied:*

$$x_j-y_j\in 2\pi\gamma^{-2}\lambda^{-1}\mathbf{Z}\,, \tag{2.8}$$

then the reduced density matrices in the thermodynamic limit is given by the equality (2.7) provided G_0^{π} is equal to zero. If (2.8) is not satisfied then the matrices in the limit are equal to zero.

3. Proofs

Let's start from the lemma. In all formulas instead of Λ we'll write L.The semi-group $P_{n,L}^{\beta}$ has the kernel

$$P_{(L)}^{\beta}(X_n|Y_n)=\exp\{iU^*(X_n)\}P_{0(L)}^{\beta}(X_n|Y_n)\exp\{-iU^*(Y_n)\}\,,$$

where

$$P_{0(L)}^{\beta}(X_n|Y_n)=\prod_{j=1}^{n}P_{0(L)}^{\beta}(x_j|y_j)\,. \tag{3.1}$$

It is obvious that

$$U^*(X_m,X_n')=U^*(X_m)+U^*(X_n')+W^*(X_m|X_n')\,,$$

where

$$W^*(X_m)|X_n')=\sum_{k=1}^{m}\sum_{j=1}^{n}\sigma_k\sigma_j'\phi^*(x_k-x_j')\,.$$

.

Hence

$$P_{(L)}^{\beta}(X_m,X_n'|Y_m,X_n')=\exp\{i[U^*(X_m)+U^*(Y_m)]\}$$

$$\times\prod_{k=1}^{m}P_{0(L)}^{\beta}(x_k|y_k)\prod_{j=1}^{n}P_{0(L)}^{\beta}(x_j'|x_k')\exp\{i[W^*(x_j'|X_m)-W^*(x_j'|Y_m)]\}\,.$$

Substituting this equality into the equality (1.6), we prove the main formula of the lemma. In order to pass to the thermodynamic limit or to prove Theorem 1 we have to represent the n-dimensional space as a union of not intersecting sets of ordered variables. Each such subset is labelled by the element of the group of permutations of $2m$ elements. Then we split the interval of integration in the expression for $G_L(X_m, Y_m)$ into three intervals. In the first interval $x_j,\ y_j > x$, in the second – $x_j,\ y_j < x$ and the third is the compliment of these intervals to $[-L, L]$. So

$$\exp\{G_L(X_m, Y_m)\} = \sum_{\pi \in S_{2m}} \chi_\pi(X_m, Y_m) \exp\{(G_L^\pi + G^\pi(X_m, Y_m)\}\,.$$

The terms with $\phi^\epsilon(x_j - x)$ cancel exactly the terms with $\phi^\epsilon(y_j - x)$ under the sign of integral in the expression for G_L^π. Since the pair "magnetic potential" $\phi(x)$ is integrable then we pass to the limit $L \to \infty$ in the integral. Since the integral over the third interval (function G^π) does not depend on L we obtain the main formula of Theorem 1, since $P_{0(L)}^\beta(x|x)$ tends to $(2\pi\beta)^{-1/2}$ when L tends to ∞. In order to prove Theorem 2 we have to prove that G_L^π is equal to zero if variables sit on the defined lattice or tends to $-\infty$ if the variables are not on the lattice. This can be shown easily since we can compute the function. Really

$$G_L^\pi(X_m, Y_m) = \sum_\sigma z_\sigma \left[[\exp\{i \sum_{j=1}^m \sigma\sigma_j \lambda(x_j - y_j)\} - 1] \int_{-L}^{x_{v_{\pi(1)}}} P_{0(L)}^\beta(x|x)\, dx \right.$$

$$\left. + \exp\{-i \sum_{j=1}^m \sigma\sigma_j \lambda(x_j - y_j)\} - 1] \int_{v_{\pi(2m)}}^{L} P_{0(L)}^\beta(x|x)\, dx \right].$$

In order to have G_L is equal to zero we have to demand that $x_j - y_j \in 2\pi\lambda^{-1}\gamma^{-2}\mathbf{Z}$. From the computed expression for the function G_L it follows that it tends to $-\infty$ if the differences are not on the lattice and $P_{0(L)}^\beta(x|x)$ tends to $(2\pi\beta)^{-1/2}$ in the limit of infinite L. The theorem is proved.

Discussion. We established that in the thermodynamic limit the behavior of the reduced density matrices for short-range pair magnetic interactions and the long-range C–S type magnetic interaction differs essentially. But there is the common property: on the diagonal they coincide with the free particle reduced density matrices. The question in what respect to the systems differ from the free particle system remains opened. In the next paper, we'll show that the similar results hold for the systems with the Fermi and Bose statistics for two simplest cases: $\kappa_0 = 0, 1$. The second case corresponds to impenetrable free bosons. It is known that there is no condensation in such the system [12] and that is equivalent on the thermodynamic level to the free fermion system (fermionization of the system). It can be stated that impenetrable bosons is an example of simplest anyons. The proof of the absence of the condensation is not trivial. This is a good hint that the thermodynamic equivalence to free particle systems does not automatically yield an equivalence on the level of an algebra of observables and its symmetries. It is known that in one-dimensional Bose gas in an external potential there is a condensation [13]. Is there a condensation in the system of impenetrable bosons with a long-range magnetic interaction? The problem of condensation in systems of 1-d anyons is very interesting and may clarify in some sense the same problem for 2-d anyons. Besides that an investigation of 1-d anyons may clarify rigorous picture of connection of anomalies and bosonization in 2-d systems, including the Schwinger model.

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Квантовые системы частиц с сингулярным магнитным взаимодействием в одном измерении. Статистика М–Б

В.И. Скрипник

Рассматриваются в термодинамическом пределе квантовые одномерные системы частиц, взаимодействие между которыми определяется "коллективным", зависящим от координат всех частиц векторным электромагнитным потенциалом. Вычисляются редукционные матрицы плотности в случаях короткодействующего взаимодействия и одномерный аналог взаимодействия Черна–Саймонса (j-й "коллективный" векторный потенциал n-частиц равен частной производной по координате j-й частицы кулоновской энергии системы n заряженных частиц).

Про квантові системи частинок з сингулярною магнітною взаємодією в одному вимірі. Статистика М–Б

В.І. Скрипник

Розглядаються в термодинамічній границі квантові одновимірні системи частинок, взаємодія між якими визначається "колективним", що залежить від координат усіх частинок, векторним електромагнітним потенціалом. Обчислюються редуковані матриці густини для випадків короткодіючої взаємодії та одновимірного аналогу взаємодії Черна–Саймонса (j-й "колективний" векторний потенціал n-частинок дорівнює частковій похідній за координатою j-ї частинки кулонівської енергії системи n заряджених частинок).