Matematicheskaya fizika, analiz, geometriya
2000, v. 4, No. 3, p. 348–359

Dynamical entropy for Bogoliubov actions of $Z/nZ \oplus Z/nZ \oplus \ldots$ on CAR-algebra

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Received July 25, 1996

The notion of dynamical entropy for actions of torsion Abelian groups $Z/nZ \oplus Z/nZ \oplus \ldots, n \geq 2$, by automorphisms of C^*-algebras is considered. The properties of this entropy are studied. These results are applied to Bogoliubov actions of those groups on the CAR-algebra. It is shown that the entropy of Bogoliubov actions corresponding to the singular spectrum is equal to zero.

1. Introduction

In paper [1] Connes, Narnhofer and Thirring extended the definition of entropy for automorphisms of finite von Neumann algebras to the case of automorpisms of C^*-algebras invariant with respect to a given state. It is a natural generalization of the Kolmogorov–Sinai entropy. In last year this entropy has been studied extensively by many authors from different points of view. Störmer and Voiculescu in [2] computed the entropy for Bogoliubov automorphisms of the CAR-algebra with respect to invariant quasifree states and prove that the singular part for $\mathbf{Z}$ case doesn't affect the entropy. Bezuglyi and Golodets [3] generalized this result for action of free Abelian groups on the CAR-algebra.

In this paper we study the dynamical entropy for Bogoliubov actions of torsion Abelian group on CAR-algebra. We extend some results of [1, 2] to groups $Z/nZ \oplus Z/nZ \oplus \ldots$. The structure of this paper is such: in Section 2 we introduce the dynamical entropy of an action of $Z/nZ \oplus Z/nZ \oplus \ldots, n \geq 2$, on C^*-algebra, list the properties of this entropy and prove the fact (among others) that the entropy of the tensor product of actions of a group with respect to states on various C^*-algebras is not less than a sum of the entropies of each action of the group with respect to respectively state. In Section 3 a connection between entropy of the group and entropy of the subgroup is studied. In particular, the formula is found for the entropy of a finite index subgroup. In Section 4 the entropy of the Bogoliubov action for any invariant state corresponding to singular spectrum is shown to be equal to zero.

In the next work we are going to calculate the entropy for the Bogolubov action of $Z/nZ \oplus Z/nZ \oplus \ldots$, $n \geq 2$, with completely continuous spectrum.

2. Dynamical entropy of $Z/nZ \oplus Z/nZ \oplus \ldots$-actions

In the sequel completely positive maps will play a crucial role. For reader's convenience we remind briefly some terminology and definitions (see [1, 2] more details). The term "unital" stands for containing or preserving the unit element. A completely positive unital map ϕ between two unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a positive unital map such that the map ϕ between $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ the $n \times n$ matrices with elements from $\mathcal{A}$ (respectively $\mathcal{B}$), $(\phi(a))_{ij} = \phi(a_{ij})$ is positive. If $\mathcal{A}$ or $\mathcal{B}$ are Abelian, any positive map is completely positive. In the vector space of linear maps the completely positive unital maps constitute a closed convex set. If $\mathcal{B} \subset \mathcal{A}$ a positive unital map with $\phi(b_1 a b_2) = b_1 \phi(a) b_2, b_i \in \mathcal{B}$, a $\in \mathcal{A}$, is called a unital conditional expectation. It is automatically completely positive.

Let $\mathcal{A}$ be a unital C^*-algebra, $C_1, \ldots, C_k$ finite dimensional C^*-algebras, and $\gamma_j : C_j \to \mathcal{A}$ unitial completely positive maps, $j = 1, \ldots, k$. Let ϕ be a state on $\mathcal{A}$ and P a unital completely positive map from $\mathcal{A}$ into a finite dimensional Abelian C^*-algebra $\mathcal{B}$ such that there is a state μ on $\mathcal{B}$ for which $\mu \circ P = \phi$. If $p_1, \ldots, p_r$ are the minimal projections in $\mathcal{B}$, then there are states $\phi_i, i = 1, \ldots, r$ on $\mathcal{A}$ such that

$$P(x) = \sum_{i=1}^{r} \phi_i(x) p_i, \quad x \in \mathcal{A},$$

and

$$\phi = \sum_{i=1}^{r} \mu(p_i) \phi_i.$$

We set up

$$\epsilon_\mu(P) = \sum_{i=1}^{r} S(\phi|\phi_i),$$

where $S(\phi|\phi_i)$ is the relative entropy of the ϕ and ϕ_i (see [1]). The entropy defect $s_\mu(P)$ is given by

$$s_\mu(P) = S(\mu) - \epsilon_\mu(P).$$

Let B_j, $j = 1, \ldots, k$, be C^*-subalgebras of $\mathcal{B}$ and $E_j : \mathcal{B} \to B_j$ a μ-invariant conditional expectation. Then $(\mathcal{B}, E_j, P, \mu)$ is called an Abelian model for $(\mathcal{A}, \phi, \gamma_1, ..., \gamma_k)$. The entropy of such an Abelian model is defined to be

$$S(\mu| \bigvee_{j=1}^{k} B_j) - \sum_{j=1}^{k} s_\mu(\rho_j),$$

where $\rho_j = E_j \circ P \circ \gamma_j : C_j \to B_j$. The supremum of entropies for all such Abelian models is denoted by $H_\phi(\gamma_1, ..., \gamma_k)$. Properties of this function can be found in [1, Proposition III.6]. If α is a ϕ-preserving automorphism of $\mathcal{A}$ and $\gamma : C \to \mathcal{A}$ is a unital completely positive map of a finite dimensional C^*-algebra C, then we denote by

$$h_{\phi,\alpha}(\gamma) = \lim_{k\to\infty} \frac{1}{k} H_\phi(\gamma, \alpha\circ\gamma, ..., \alpha^{k-1}\circ\gamma).$$

The entropy of α with respect to ϕ is defined by the formula

$$h_\phi(\alpha) = \sup_\gamma h_{\phi,\alpha}(\gamma).$$

To simplify the exposition, we restrict our observations concerning $Z/nZ \oplus Z/nZ \oplus ...$-actions, $n \in \mathbf{N}$, the case $n = 2$ only. It is clear that the general case may be considered in a similar way.

Let $\mathcal{A}$ be as above, $\Gamma = \bigoplus_{i=1}^{\infty} Z_i, \Gamma[n, m] = \bigoplus_{i=n}^{m} Z_i, \Gamma[n] = \bigoplus_{i=n}^{\infty} Z_i$, where $Z_i = Z/2Z$. Let ϕ be a state on $\mathcal{A}$ and $\alpha : \Gamma \to Aut(\mathcal{A}, \phi)$ a ϕ-preserving action of Γ on $\mathcal{A}$ by $*$-automorphisms, i.e., α is an injective homomorphism from Γ into $Aut(\mathcal{A})$ such that $\alpha(\xi)$ is a ϕ-preserving $*$-automorphism of $\mathcal{A}$ for any $\xi \in \Gamma$. Let π be a representation of $\mathcal{A}$ corresponding to ϕ via the GNS-construction, $M = \pi(\mathcal{A})''$.

Define the dynamical entropy of the action α of Γ on $\mathcal{A}$. According to the above definition of entropy, if $\gamma : C \to \mathcal{A}$ is a unital completely positive linear map of a finite dimensional C^*-algebra C, then the function $H_\phi(\alpha(\xi)\circ\gamma, \xi \in \Gamma[1, n])$ can be defined as a supremum of the entropies for Abelian models of $(\mathcal{A}, \phi, \alpha(\xi)\circ\gamma, \xi \in \Gamma[1, n])$.

Set up $H_n(\gamma) = H(\gamma, \alpha, \Gamma[1, n]) = H_\phi(\alpha(\xi)\circ\gamma, \xi \in \Gamma[1, n])$. It follows from the subadditivity of $H_n(\gamma)$ (see [1, Proposition III.6.(d)]) that $H_{n+1}(\gamma) \leq 2H_n(\gamma)$.

Conversely, there exists a limit

$$\lim_{n\to\infty} \frac{1}{2^n} H_n(\gamma) = h_{\phi,\alpha}(\gamma, \Gamma).$$

Definition 1. *The dynamical entropy of Γ-action α with respect to ϕ on a C^*-algebra $\mathcal{A}$ is*

$$h_\phi(\alpha, \Gamma) = \sup_\gamma h_{\phi,\alpha}(\gamma, \Gamma)\,.$$

We formulate now without proof the statements that generalize those from [2, Section 3].

Proposition 2. *Let ϕ be a pure state on a unital C^*-algebra $\mathcal{A}$, and α a ϕ-preserving Γ-action on $\mathcal{A}$. Then $h_\phi(\alpha,\Gamma)=0$.*

Proposition 3. *Let ϕ be a state on a unital C^*-algebra $\mathcal{A}$, and α a ϕ-preserving Γ-action on $\mathcal{A}$. Suppose $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ such that there is an expectation $E:\mathcal{A}\to\mathcal{B}$ with $\phi\circ E=\phi$, and $\alpha(\xi)E=E\alpha(\xi),\xi\in\Gamma$. Then $\alpha|\mathcal{B}$ is an action of Γ on $\mathcal{B}$ and $h_\phi(\alpha|\mathcal{B},\Gamma)\le h_\phi(\alpha,\Gamma)$.*

Proposition 4. *Let $\mathcal{A}$ be a C^*-algebra, ϕ a state, and α a ϕ-preserving action of Γ on $\mathcal{A}$. Let $\{\mathcal{A}_j\}_{j=1}^{\infty}$ be an increasing sequence of C^*-subalgebras such that the expectations $E_j:\mathcal{A}\to\mathcal{A}_j$ satisfy the following conditions:*
(i) $\alpha(\xi)E_j=E_j\alpha(\xi),\quad \xi\in\Gamma,\ j=1,2,\dots;$
(ii) $E_{j+1}E_j=E_jE_{j+1}=E_j,\ j=1,2,\dots;$
(iii) $E_j\to id_{\mathcal{A}}$ in the pointwise-norm topology.
Suppose that the norm closure of $\bigcup_j\mathcal{A}_j$ is $\mathcal{A}$. Then $\alpha|\mathcal{A}_j$ is an action on $\mathcal{A}_j$ for $j\in\mathbf{N}$ and

$$h_\phi(\alpha,\Gamma)\le\liminf_j h_\phi(\alpha|\mathcal{A}_j,\Gamma).$$

Moreover, if $\phi\circ E_j=\phi, j\in\mathbf{N}$, then

$$h_\phi(\alpha,\Gamma)=\lim_j h_\phi(\alpha|\mathcal{A}_j,\Gamma).$$

Proposition 5. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two C^*-algebras with states ϕ_1 and ϕ_2, respectively. Let α_i be of Γ-action on $(\mathcal{A}_i,\phi_i)$ such that $\phi_i\circ\alpha_i=\phi_i, i=1,2$. Then*

$$h_{\phi_1\otimes\phi_2}(\alpha_1\otimes\alpha_2,\Gamma)\ge h_{\phi_1}(\alpha_1,\Gamma)+h_{\phi_2}(\alpha_2,\Gamma)\,.$$

P r o o f. Let $(\mathcal{A}_1,\,\phi_1,\alpha_1(\xi)\circ\gamma_1,\xi\in\Gamma[1,n])$ be given with an Abelian model $(\mathcal{B}_1,\,E_{1,j},P_1,\mu_1)$ and subalgebras $B_{1,j}$ with $E_{1,j}$ the μ_1-preserving expectation of $\mathcal{B}_1$ onto $B_{1,j}$. Assume also that we have a similar setup for $(\mathcal{A}_2,\phi_2,\alpha_2(\xi)\circ\gamma_2,\xi\in\Gamma[1,n])$. Since the relative entropy and the entropy of states are additive on tensor products, we deduce the additivity of $\epsilon_\mu, S(\mu), s_\mu$. We may assume $\mathcal{B}_1=\bigvee_i B_{1,i}$, $\mathcal{B}_2=\bigvee_i B_{2,i}$. Since $\mathcal{B}_1\otimes\mathcal{B}_2$ contains $B_{1,i}\otimes 1$ and $1\otimes B_{2,j}$ for any $i,\,j=1,\dots,2^n$, one has $\mathcal{B}_1\otimes\mathcal{B}_2=\bigvee_i B_{1,i}\otimes B_{2,i}$. Thus

$$S(\mu_1\otimes\mu_2\mid\bigvee_i B_{1,i}\otimes B_{2,i})=S(\mu_1)+S(\mu_2).$$

It follows easily that the entropy of the tensor product Abelian model for $(\mathcal{A}_1 \otimes \mathcal{A}_2, \phi_1 \otimes \phi_2, \alpha_1(\xi) \circ \gamma_1 \otimes \alpha_2(\xi) \circ \gamma_2, \xi \in \Gamma[1,n]\ )$ is the sum of entropies of two Abelian models for $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Take supremum over all tensor product Abelian models as above to get

$$H_{\phi_1}(\alpha_1(\xi) \circ \gamma_1, \xi \in \Gamma[1,n]\ ) + H_{\phi_2}(\alpha_2(\xi) \circ \gamma_2, \xi \in \Gamma[1,n]\ )\,. \tag{1}$$

However, to get

$$H_{\phi_1 \otimes \phi_2}((\alpha_1 \otimes \alpha_2)(\xi) \circ (\gamma_1 \otimes \gamma_2), \xi \in \Gamma[1,n]\ )\,, \tag{2}$$

one has to take the sup over a larger family of Abelian models. Thus the expression in (2) is greater than that in (1). It remainds to take the sup through all possible γ's, and the conclusion of the lemma follows. ■

3. Properties of the entropy of Γ-action on a C^*-algebra

Proposition 6. *Let $\mathcal{A}$ be a nuclear C^*-algebra, $\mathcal{A}_n$ be finite dimensional C^*- algebras, α and ϕ be as above. Let τ_n be a sequence of completely positive unital maps $\tau_n : \mathcal{A}_n \to \mathcal{A}$ such that for suitable completely positive unital maps $\sigma_n : \mathcal{A} \to \mathcal{A}_n$ one has $\tau_n \circ \sigma_n :\to id_A$ in the pointwise norm topology. Then*

$$\lim_{n\to\infty} h_{\phi,\alpha}(\tau_n, \Gamma) = h_\phi(\alpha, \Gamma)\,.$$

P r o o f. The analogues of this proposition for the case $\Gamma = Z$ were proved in [1, Theorem V.2]. Our statement can be proved by a similar method, remembering definition 1 and above observations. ■

Proposition 7. *Let $\mathcal{A}$ be a nuclear C^*-algebra, α and ϕ be as above. Let $M = \pi_\phi(\mathcal{A})''$. Then*

$$\sup_\gamma h_{\phi,\alpha}(\gamma, \Gamma) = \sup_N h_{\phi,\alpha}(N)\,,$$

where N runs through all the finite dimensional subalgebras of M.

Proposition 8. *Let M, ϕ, α be as above. Let N_k be an ascending sequence of finite dimensional von Neumann algebras with $\bigcup_k N_k$ weakly dense in M. Then*

$$h_\phi(\alpha, \Gamma) = \lim_{k\to\infty} h_{\phi,\alpha}(N_k)\,.$$

Proposition 9. *For all automorphisms σ of $\mathcal{A}$ (respectively M) we have*

$$h_\phi(\alpha, \Gamma) = h_{\phi\circ\sigma}(\sigma \circ \alpha \circ \sigma^{-1}, \Gamma)\,,$$

where $\alpha \in Aut(\mathcal{A})$.

Propositions 7–9 are proved in [1, VII.3–VII.5] for the case $\Gamma = Z$. But the same method can be used to prove it for our group Γ.

Proposition 10.

$$h_\phi(\alpha, \Gamma[k]) = 2^{k-1} h_\phi(\alpha, \Gamma) .$$

P r o o f. We prove the proposition for the case $k = 2$. It is clear that in the case of an arbitrary k it can be proved analogously. From Definition 1 and [1, Proposition III.6] we have

$$h_\phi(\alpha, \Gamma) \geq \lim_{n\to\infty} \frac{1}{2^n} H(\gamma, \alpha, \Gamma[1, n]) \geq \lim_{n\to\infty} \frac{1}{2^n} H(\gamma, \alpha, \Gamma[2, n]) .$$

Since $\forall\epsilon > 0, \exists\gamma$ such that $h_\phi(\alpha, \Gamma[2]) - \epsilon \leq h_{\phi,\alpha}(\gamma, \Gamma[2])$, then

$$h_\phi(\alpha, \Gamma) \geq \frac{1}{2} \lim_{n\to\infty} \frac{1}{2^{n-1}} H(\gamma, \alpha, \Gamma[2, n]) = \frac{1}{2} h_{\phi,\alpha}(\gamma, \Gamma[2]) \geq \frac{1}{2}(h_\phi(\alpha, \Gamma[2]) - \epsilon) .$$

Therefore

$$h_\phi(\alpha, \Gamma) \geq \frac{1}{2}(h_\phi(\alpha, \Gamma[2] - \epsilon) .$$

On the other hand, it follows from GNS-construction that α can be extended to the entire M. This extension we also denote by α.

Let N be a finite dimensional subalgebra of M, and $\delta > 0$. One can observe from the proof of theorem VII.4 [1] that there exist finite dimensional subalgebra $B \subset M$ and completely positive maps

$\gamma_0 : N \to B$ and $\gamma_1 : \alpha(\delta_1)N \to B$, such that $\|\gamma_0(a) - a\|_\phi \leq \delta$ and $\|\gamma_1(a) - \alpha(\delta_1)(a)\|_\phi \leq \delta$, where $\delta_1 = (1, 0, 0...)$, $\delta_2 = (0, 1, 0...)$, ..., $\delta_i \in \Gamma$, $a \in N_1$, N_1 a unit ball of N. Since $\phi \circ \alpha(\xi) = \phi$, $\|\alpha(\xi)(\gamma_0(a) - a)\|_\phi \leq \delta$ and $\|\alpha(\xi)(\gamma_1(a) - \alpha(\delta_1)(a))\|_\phi \leq \delta$, where $\xi \in \Gamma[2, n]$.

By [1, Theorem VI.3], for any $\epsilon_1 > 0$

$$\begin{aligned} H_\phi(\alpha(\xi)N, \xi \in \Gamma[1, n]) &\leq H(\alpha(\xi) \circ \gamma_0, \alpha(\xi) \circ \gamma_1, \xi \in \Gamma[2, n]) + \epsilon_1 2^{n-1} \\ &\leq H(\alpha(\xi)B, \xi \in \Gamma[2, n]) + \epsilon_1 2^{n-1} . \end{aligned}$$

That is why for an appropriate N

$$h_\phi(\alpha, \Gamma) - \epsilon \leq \lim_{n\to\infty} \frac{1}{2^n} H_\phi(\alpha(\xi)N, \xi \in \Gamma[1, n])$$

$$\leq \frac{1}{2} \lim_{n\to\infty} \frac{1}{2^{n-1}} H(\alpha(\xi)B, \xi \in \Gamma[2, n]) + \frac{\epsilon_1}{2} \leq \frac{1}{2}(h_\phi(\alpha, \Gamma[2]) + \epsilon_1) . \quad \blacksquare$$

Proposition 11. $h_{\lambda\phi_1+(1-\lambda)\phi_2}(\alpha,\Gamma)=\lambda h_{\phi_1}(\alpha,\Gamma)+(1-\lambda)h_{\phi_2}(\alpha,\Gamma)$, *for all* $0\le\lambda\le 1$.

P r o o f. Using Proposition 10 and [1, Proposition III.6(e)], we have

$$|h_{\lambda\phi_1+(1-\lambda)\phi_2}(\alpha,\Gamma)-\lambda h_{\phi_1}(\alpha,\Gamma)-(1-\lambda)h_{\phi_2}(\alpha,\Gamma)|$$
$$=\frac{1}{2^{k-1}}|h_{\lambda\phi_1+(1-\lambda)\phi_2}(\alpha,\Gamma[k])-\lambda h_{\phi_1}(\alpha,\Gamma[k])-(1-\lambda)h_{\phi_2}(\alpha,\Gamma[k])|$$
$$\le\frac{1}{2^{k-1}}[-\lambda\log\lambda-(1-\lambda)\log(1-\lambda)].$$

With k going to infinity one gets

$$|h_{\lambda\phi_1+(1-\lambda)\phi_2}(\alpha,\Gamma)-\lambda h_{\phi_1}(\alpha,\Gamma)-(1-\lambda)h_{\phi_2}(\alpha,\Gamma)|\le 0\,.$$ ■

Proposition 12. *Let* $D\in\Gamma$ *be a subgroup of the group* Γ, ϕ *and* α *be as above. Then*

$$h_\phi(\alpha,\Gamma)\le h_\phi(\alpha,D)\,.$$

P r o o f. Let $D_n=D\bigcap\Gamma[1,n]$. Then $D_n\in\Gamma[1,n]$ is a subgroup of the group $\Gamma[1,n]$. Let $2^{k_n}=|\Gamma[1,n]:D_n|$ be an index of D_n in $\Gamma[1,n]$, and $\Gamma[1,n]=\bigcup_{i=1}^{2^{k_n}}g_iD_n, g_i\in\Gamma$ a decomposition into cosets. If $\gamma:C\to\mathcal{A}$ be a unital completely positive linear unitar map of finite dimensional C^*-algebra C, then

$$H_\phi(\alpha(\xi)\circ\gamma,\xi\in\Gamma[1,n])=H_\phi(\alpha(\xi)\circ\gamma,\xi\in\bigcup_{i=1}^{2^{k_n}}g_iD_n)$$
$$\le 2^{k_n}H_\phi(\alpha(\xi)\circ\gamma,\xi\in D_n)\,.$$

The inequality is due to [1, Proposition III.6]. Therefore

$$\frac{1}{2^n}H_\phi(\alpha(\xi)\circ\gamma,\xi\in\Gamma[1,n])\le\frac{1}{2^{n-k_n}}H_\phi(\alpha(\xi)\circ\gamma,\xi\in D_n)\,.$$

That is why

$$h_{\phi,\alpha}(\gamma,\Gamma)\le h_{\phi,\alpha}(\gamma,D),$$

and hence

$$h_\phi(\gamma,\Gamma)\le h_\phi(\gamma,D)\,.$$ ■

Proposition 13. *Let $D \subset \Gamma$ be a subgroup of Γ such that $\Gamma = D \times D'$, where D' is also a subgroup of Γ, and ϕ , α be as above. If $h_\phi(\alpha, D) < \infty$ and $cardD' = \infty$ then $h_\phi(\alpha, \Gamma) = 0$.*

P r o o f. Let D'_k be any subgroup of finite index in D' such that $|D' : D'_k| = 2^k$. Then $|\Gamma : \Gamma_k| = 2^k$, where $\Gamma_k = D \times D'_k$. In view of Proposition 10

$$h_\phi(\alpha, \Gamma_k) = 2^{k-1} h_\phi(\alpha, \Gamma)\,.$$

As far as $D \subset \Gamma_k$ then, by a virtue of Proposition 12, we have

$$h_\phi(\alpha, \Gamma_k) \le h_\phi(\alpha, D)\,.$$

Therefore

$$2^{k-1} h_\phi(\alpha, \Gamma) \le h_\phi(\alpha, D)$$

that is $h_\phi(\alpha, \Gamma) \le \frac{1}{2^{k-1}} h_\phi(\alpha, D)$. Since k it take arbitrary, $h_\phi(\alpha, \Gamma) = 0$. ∎

4. Entropy of the Bogoliubov action of Γ on CAR-algebra

Firstly we remind some definitions concerning the CAR-algebra. Let H be a Hilbert space. The CAR-algebra $\mathcal{A}(H)$ is a C^*-algebra with the property that there is a linear map $f \mapsto a(f)$ of H into $\mathcal{A}(H)$ whose range generates $\mathcal{A}(H)$ as a C^*-algebra and satisfyies the canonical anticommutation relations

$$a(f)a(g)^* + a(g)^* a(f) = (f, g)\mathbf{1}\,,$$

$$a(f)a(g) + a(g)a(f) = 0, \quad f, g \in H\,,$$

where $(.,.)$ is the inner product on H, and $\mathbf{1}$ is the unit of $\mathcal{A}(H)$. Let $0 \le A \le 1$ be an operator on H. The quasifree state ω_A on $\mathcal{A}(H)$ is defined by its values on products of the form $a(f_n)^* ... \, a(f_1)^* a(g_1) \, ... \, a(g_m)$, $n, m \in \mathbf{N}$, given by

$$\omega_A(a(f_n)^* ... \, a(f_1)^* a(g_1) \, ... \, a(g_m)) = \delta_{nm} \det((Ag_i, f_j)).$$

If U is a unitary operator on H, then U defines an automorphism α_U of $\mathcal{A}(H)$ called the Bogoliubov automorphism : $\alpha_U(a(f)) = a(Uf)$, $f \in H$. If $UA = AU$, then $\omega_A \circ \alpha_U = \omega_A$. It is well known that $\mathcal{A}(H) \simeq \bigotimes_{k=1}^{\infty} M_2(\mathbf{C})_k$, and if A has a pure point spectrum, then $\omega_A = \bigotimes_{k=1}^{\infty} \omega_{\lambda_k}$, where $\{\lambda_k\}$ is the set of eigenvalues for A and

$$\omega_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (1-\lambda)a + \lambda d\,.$$

If U is a unitary representation of Γ on H such that $U(\xi)A = AU(\xi)$, $\xi \in \Gamma$, then U defines an action α of Γ on $\mathcal{A}(H)$ by

$$\alpha_{U(\xi)}(a(f)) = a(U(\xi)(f)),\ \xi \in \Gamma\,.$$

α_U is called the Bogoliubov action.

Let $\Gamma = Z/2Z \oplus Z/2Z \oplus \dots$, X be a dual group ($X = \hat{\Gamma}$) and μ be the Haar measure on X. X can be realized as space of infinite sequences $x = x_1x_2\dots$, where $x_i = 0$ or 1; and μ is a product measure on X that is $\mu = \prod_{i=1}^{\infty}\mu_i$, where μ_i is a measure on $X_i = \{0, 1\}$ given by $\mu_i(0) = \mu_i(1) = \frac{1}{2}$. Every element $\xi \in \Gamma$ is a sequence $\xi = (\xi_1, \xi_2, \dots)$, with every ξ_i equals to zero or one, and only finitely many components ξ_i being non zero. To every $\xi \in \Gamma$ one can assosiate a character χ_ξ of the group X that

$$\chi_\xi(x) = \prod_k (-1)^{x_k},\ k \in \{t|\xi_t = 1\}\,.$$

If we have a unitar representation of Γ in the Hilbert space H, then by a result of Mackey (see [5]), it can be disintegrated into irreducible one-dimensional representations, and the space H is represented as a direct integral of Hilbert space

$$H = \int_X \oplus H_x d\nu(x)\,,$$

where ν is the Borel measure on X. If $f \in H$ then f corresponds to a measurable vector function $\overline{f(x)}$ on X ($f \sim \overline{f(x)}$), and $f(x) \in H_x$. Moreover, if $f \sim \overline{f(x)}$ and $g \sim \overline{g(x)}$, then

$$(f, g) = \int (f(x), g(x))d\nu(x)\,,$$

where $(f(x), g(x))$ is the inner product of vectors $f(x)$ and $g(x)$ in H_x, and $(f(x), g(x))$ is a measurable function on X. If $\xi \to U(\xi)$ is a unitary representation of Γ in H then $U(\xi)f \sim \overline{\chi_\xi(x)f(x)}$, where $\chi_\xi(x)$ is a character of X which corresponds to $\xi \in \Gamma$.

The measure ν is a direct sum $\nu = \mu_a + \mu_s$, where measure μ_a is absolutely continuous with respect to the Haar measure μ, and μ_s is a singular measure. In accordance with this the representation $\xi \to U(\xi)$ of Γ is a direct sum $U(\xi) = U_a(\xi) \oplus U_s(\xi)$, where U_a is the absolutely continuous part of U, and U_s is the singular part. We will study the case when the unitary representation U of Γ has nontrivial singular part U_s. Our aim is prove that the entropy corresponding to the singular spectrum of the action is equal to zero.

We first prove

Proposition 14. *Let U be a unitary representation of Γ on H with the spectral measure singular with respect to the Haar measure μ on X. Assume that*

P is a finite rank orthogonal projection onto a subspace of H and let $\epsilon > 0$ be given. Then there is $k_0 \in \mathbf{N}$ such that for each $k \geq k_0$ there exists a finite rank projection Q_k with the properties:

(i) $\|(1 - Q_k)U(\xi)P\| < \epsilon,\ \xi \in \Gamma[1, k]$;

(ii) $dim Q_k \ \leq 2^k \epsilon$.

P r o o f. Since the spectral measure of U is singular and P has finite rank, there exists a subset X' of X ($X' \subset X$) such that for given $\delta > 0$ we have:

(1) $X' = \bigcup_{\xi \in \Xi} X(\xi)$, where $X(\xi) = \{x \in X, x_1 = \xi_1, ..., x_t = \xi_t\}$, $\xi = (\xi_1, ..., \xi_t) \in \Gamma[1, t]$, $\Xi \subset \Gamma[1, t]$;

(2) $\mu(X') = \sum_{\xi \in \Xi} \mu(X(\xi)) = \frac{m}{2^t} < \delta$, $0 < \ m \ < 2^t$, $\ Card \Xi = m$;

(3) if $E(X')$ is a spectral projection of U, then $\|(I - E(X'))P\| < \delta$.

Really, since the spectral measure of representation U of the group Γ is singular, conditions (1)–(3) are the direct consequences of the definition of a singular measure.

Let $E(X(\xi))$ be a spectral projection of U corresponding to the cylinder set $X(\xi), \xi \in \Gamma[1, t]$. Let us consider the subspace $H_t = \bigoplus_{\xi \in \Xi} E(X(\xi))PH \subset H$. We have $dim H_t \leq m \ dim P$, where $m = card\ \Xi$.

Let $c(\xi, \zeta) = \chi_\xi(x)|X(\zeta)$, for $\xi,\ \zeta \in \Gamma[1, t]$. It is obvious, that $c(\xi, \zeta) = 1$ or -1. Since $(U(\xi)f)(x) = \chi_\xi(x)f(x)$, the definition of numbers $c(\xi, \zeta)$ and (3) imply that for $f \in PH$ with $\|f\| = 1$ the distance

$$d(U(\xi)f, H_t)^2 \leq \| \sum_{\zeta \in \Xi} c(\xi, \zeta)E(X(\zeta))f - U(\xi)f\|^2$$

$$\leq 2m \sum_{\zeta \in \Xi} \|c(\xi, \zeta)E(X(\zeta))f - U(\xi)E(\zeta)f\|^2 + 2\|U(\xi)E(X')f - U(\xi)f\|^2$$

$$= 2\|U(\xi)E(X')f - U(\xi)f\|^2 \leq 2\delta^2 .$$

Next, in viev of (2), $dim H_t \leq m \ dim P \leq 2^t \delta dim P$.

Take now $\delta > 0$ that $\sqrt{2\delta}\ dim P < \epsilon$ and $k \geq t$ to get (i) and (ii). ■

Now we are going to prove the main theorem.

Theorem 15. *Let $U = U(\xi), \xi \in \Gamma$, be a unitary representation of Γ on H whose spectral measure is singular with respect to the Haar measure. If ϕ is a state on $\mathcal{A}(H)$ such that $\alpha_{U(\xi)}$ is ϕ-preserving, then*

$$h_\phi(\alpha_{U(\xi)}) = 0\,.$$

P r o o f. Let P be an orthogonal projection in $B(H)$ of finite rank, and $j : P(H) \to H$ the inclusion map. Then there are unital completely positive maps $\alpha_j : \mathcal{A}(P(H)) \to \mathcal{A}(H), \alpha_p : \mathcal{A}(H) \to \mathcal{A}(P(H))$ [4] such that

$$\alpha_j(a(f)) = a(jf), \quad \alpha_p(a(f)) = a(Pf)\,.$$

If $P_n \nearrow 1$ is a sequence of projections then with $j_n : P_n(H) \to H$ being the inclusions, $\alpha_{j_n} \circ \alpha_{p_n} \to 1_{\mathcal{A}(H)}$ in pointwise-norm topology. By Proposition 6,

$$h_\phi(\alpha_U, \Gamma) = \lim_n h_{\phi,\alpha_U}(\alpha_{j_n}, \Gamma)\,.$$

Thus it suffices to show that $h_{\phi,\alpha_U}(\alpha_j, \Gamma) = 0$. Since $dimP < \infty$, for a given $\delta > 0$, there is $\eta > 0$ such that if $W_1, W_2 : P(H) \to H$ are isometries with $\|W_1 - W_2\| < \eta$, then $\|\alpha_{W_1} - \alpha_{W_2}\| < \delta$, where $\alpha_W(a(f)) = a(Wf)$ [4]. Take Q_k as in Proposition 14. Denote by $\mathrm{pol}(Q_kU(\xi)|_{P(H)}), \xi \in \Gamma[1,k]$ the partial isometry W_2 appearing in the polar decomposition

$$Q_kU(\xi)|_{P(H)} = W_2|Q_kU(\xi)|_{P(H)}|\,.$$

Let $W_1 = U(\xi)|_{P(H)}$. Since

$$\|U(\xi)P - Q_kU(\xi)P\| < \epsilon \ \text{ for } \ \xi \in \Gamma[1,\mathrm{k}],$$

we can easily infer

$$\|U(\xi)|_{P(H)} - \mathrm{pol}(Q_kU(\xi)|_{P(H)})\| \le 3\epsilon.$$

Choosing $\epsilon < \eta/3$ and $k \ge k_0$, we obtain that

$$\|\alpha_{U(\xi)|_{P(H)}} - \alpha_{\mathrm{pol}(Q_kU(\xi)|_{P(H)})}\| < \delta, \ \text{for} \ \ \alpha \in \Gamma[1,\mathrm{k}].$$

By [1, Proposition IV.3] for any $\tau > 0$ and $\epsilon > 0$, there is $k_0 \in \mathbf{N}$ such that if $k \ge k_0$ and Q_k as in Proposition 14, then

$$H_\phi(\alpha_j, \alpha_{U(\xi)}\alpha_j,\ \xi \in \Gamma[1,k]) \le 2^k\tau + H_\phi(\alpha_{\mathrm{pol}(Q_kj)}, \alpha_{\mathrm{pol}(Q_kU(\xi)j)},\ \xi \in \Gamma[1,k])\,. \quad (3)$$

If we define $v : Q_k(H) \to H$ to be the inclusion map, then $\alpha_{\mathrm{pol}(Q_kj)} = \alpha(v) \circ \alpha_{\mathrm{pol}(Q_kj)}$. Whence, by [1, Proposition III.6 (a),(c)],

$$H_\phi(\alpha_{\mathrm{pol}(Q_kj)}, \alpha_{\mathrm{pol}(Q_kU(\xi)j)},\ \xi \in \Gamma[1,k]) \le H_\phi(\alpha(v), ..., \alpha(v)) = H_\phi(\alpha(v))\,. \quad (4)$$

On the other hand, by definition of H_ϕ we have

$$H_\phi(\alpha(v)) = S(\phi \circ \alpha(v))\,,$$

where $\phi \circ \alpha(v)$ is a state on $\mathcal{A}(Q_k(H))$, a C^*-algebra of dimension less than $2^{\epsilon 2^k}$. Thus

$$H_\phi(\alpha(v)) \le \log 2^{\epsilon 2^k} = \epsilon 2^k \log 2\,.$$

Hence, by (3) and (4),

$$\frac{1}{2^k} H_\phi(\alpha_j, \alpha_{U(\xi)}\alpha_j,\ \xi \in \Gamma[1,k]) \le \tau + \epsilon \ \log 2\,.$$

Since τ and ϵ are arbitrary, $h_{\phi,\alpha_U}(\alpha_j, \Gamma) = 0$. ■

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Динамическая энтропия боголюбовских действий $Z/nZ \oplus Z/nZ \oplus \ldots$ на CAR-алгебре

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Рассматривается динамическая энтропия для действий абелевых групп $Z/nZ \oplus Z/nZ \oplus \ldots$, $n \geq 2$, автоморфизмами C^*-алгебр. Изучены свойства такой энтропии. Полученные результаты применены к боголюбовским действиям этих групп на CAR-алгебре. Показано, что энтропия боголюбовского действия, соответствующая сингулярному спектру, равна нулю.

Динамічна ентропія боголюбівських дій $Z/nZ \oplus Z/nZ \oplus \ldots$ на CAR-алгебрі

В.М. Олексенко

Розглядається динамічна ентропія для дій абелевих груп $Z/nZ \oplus Z/nZ \oplus \ldots$, $n \geq 2$, автоморфізмами C^*-алгебр. Вивчено властивості такої ентропії. Одержані результати застосовані до боголюбівських дій цих груп на CAR-алгебрі. Показано, що ентропія боголюбівської дії, яка відповідає сингулярному спектру, дорівнює нулю.