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Unique polynomials of entire and meromorphic functions

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In this paper, by making use of the theory on Riemann surface and value distribution theory we obtained some unique polynomials $P(z)$ such that, for any nonconstant meromorphic functions f and g, if $P(f) \equiv P(g)$ then $f \equiv g$.

1. Introduction

Let $P(z)$ be a nonconstant polynomial in the complex plane. If the condition $P(f) = P(g)$ implies $f \equiv g$ for any two nonconstant meromorphic (entire) functions f and g, then $P(z)$ is called a unique polynomial of meromorphic (entire) functions, and we say P is UPM (UPE) in brief (cf. Li and Yang [7]). This relates closely to the concept of Unique Range Set introduced by Gross and Yang [4]. A set S is called a unique range set of meromorphic (entire) functions if $E_f(S) = E_g(S)$ for any meromorphic (entire) functions f and g implies $f \equiv g$, where $E_f(S) = \cup\{z|f(z)-a=0,\ \ a \in S\}$, here a zero of $f(z)-a$ of multiplicity m appears m times in $E_f(S)$. Now we construct a polynomial $P(z)$ such that it has no multiple roots and the set of the roots coincides with S. Then the condition $E_f(S) = E_g(S)$ means that $P(f)$ and $P(g)$ have the same zeros with the same multiplicities. This is weaker than $P(f) = P(g)$. Thus any unique range set of meromorphic (entire) functions will lead to a corresponding unique polynomial. The converse is not true in general.

E x a m p l e 1. Let $P(z) = z^4+2z^3-9z^2-2z+8 = (z-1)(z+1)(z-2)(z+4)$, $S = \{1, -1, 2, -4\}$. Then $P(z)$ is a UPE by the following Theorem A. However,

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for two different entire functions $f(z) = \frac{3}{2}\sqrt{5}e^z + \frac{7}{2}$ and $g(z) = \frac{3}{2}\sqrt{5}e^{-z} + \frac{7}{2}$ we can easily check that $E_f(S) = E_g(S)$. Thus S is not a unique range set of entire functions.

Concerning UPE and UPM, Li and Yang [7] proved the following results.

Theorem A. *Any polynomial of degree less than 4 is not a UPE and any polynomial of degree less than 5 is not a UPM. If $P(z) = z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$, then P is a UPE if and only if $a_3^3/8 - a_2a_3/2 + a_1 \neq 0$.*

In this paper, we shall consider some general polynomial of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \quad (n \geq 4) \tag{1}$$

and prove the following results.

Theorem 1. *Let $P(z)$ be defined as (1). If there exists an integer t with $1 \leq t < n-2$ and $(n, t) = 1$ such that $a_{n-1} = \cdots = a_{t+1} = 0$ but $a_t \neq 0$, then $P(z)$ is a UPE.*

R e m a r k 1. The following results show that the conditions $1 \leq t$ and $(n, t) = 1$ are needed.

E x a m p l e 2. Let $t = 0$ and $P(z) = z^n + a_0$. Then for any function f and n-th root of unity u we have $P(f) = P(uf)$.

E x a m p l e 3. Let $P(z) = z^6 + z^2 + 1$. Then $t = 2 < n - 2 = 4$ but $(t, n) \neq 1$. Obviously, $P(f) = P(-f)$ for any f.

Concerning UPM, we have the following result.

Theorem 2. *Let $P(z) = z^n + a_mz^m + a_0$ be a polynomial such that $(n, m) = 1$ and $a_m \neq 0$. If $n \geq 5$ and $1 \leq m < n - 1$, then $P(z)$ is a UPM.*

R e m a r k 2. From Theorem A we see that the condition $n \geq 5$ is necessary. Combining the two examples above and the following example, we see that the conditions $1 \leq m < n - 1$ and $(n, m) = 1$ are necessary.

E x a m p l e 4. Let $P(z) = z^n - z^{n-1} + a_0$, where $m = n - 1$. Then for any non-constant meromorphic function $h(z)$ we have

$$P\left(\frac{\sum_{j=1}^{n-1} h^j}{\sum_{j=0}^{n-1} h^j}\right) = P\left(\frac{\sum_{j=0}^{n-2} h^j}{\sum_{j=0}^{n-1} h^j}\right).$$

R e m a r k 3. If we take $m = n - 1$ in Theorem 2, then $P(z)$ is a UPE.

2. Some lemmas

For the proof of our results, we make use of the theory of Riemann surface. The following lemma is the famous Picard's theorem on uniformization of algebraic curves (see Selberg [10]), cf. Ozawa [9]).

Lemma 1 (Picard's Theorem). *Let $R(u,v)$ be an irreducible polynomial in $C[u,v]$. If there are non-constant meromorphic functions $f(z)$ and $g(z)$ such that*

$$R(f(z), g(z)) \equiv 0\,,$$

then the Riemann surface defined by $R(u,v)=0$ is of genus ≤ 1.

From this result we can prove the following lemma which is similar to Ford [1, Theorems 7 and 8] (cf. [2]).

Lemma 2. *Let $P(z)$ be defined as (1). If $f(z)$ and $g(z)$ are non-constant entire functions satisfying*

$$P(f(z)) \equiv P(g(z))\,, \tag{2}$$

then there exist an entire function $h(z)$ and rational functions

$$R_1(z) = \sum_{j=s}^{k} b_j z^j, \quad R_2(z) = \sum_{j=s}^{k} c_j z^j \quad (-\infty < s \le k < +\infty) \tag{3}$$

such that

$$f(z) = R_1(h(z)), \quad g(z) = R_2(h(z))\,. \tag{4}$$

P r o o f. Factorize $P(u) - P(v)$ into irreducible factors in $C[u,v]$. By (2), one of these irreducible factors, $R(u,v)$, say, will satisfy

$$R(f(z), g(z)) \equiv 0\,. \tag{5}$$

Without loss of generality, we suppose that $deg_v R(u,v) \ge 1$. Note that f and g are entire, by Lemma 1, $R(u,v)=0$ defines a $deg_v R(u,v)$-sheeted Riemann surface H of genus 0, since Riemann surfaces of genus one can only be uniformized by elliptic functions, not by entire functions (see Ford [1, Section 91]).

Thus there exists a conformal mapping $\psi(y) = h$ of the points y of the Riemann surface H onto the point h of the Riemann sphere. Without loss of generality, we may assume that $h=\infty$ corresponds to $u=\infty$. Except at a finite number of branch points of H we may use u as local uniformizing parameter, so that h induces a holomorphic function $\phi(u)$ of u near all points of H except the branch points. Therefore the mapping

$$z \to (f(z), g(z)) = y \to h = \psi(y) = \phi(f(z)) = h(z)$$

is holomorphic near all z except perhaps those for which $(f(z), g(z)) = (u_1, v_1) = y_1$ is a branch point of H. These values of z form a discrete set E. If $z \to z_1 \in E$, then $h(z) \to h(y_1)$, thus z_1 is a removable singularity of $h(z)$, $h(z)$ is entire. Since the genus of H is zero, then H is conformally equivalent to the Riemann sphere so that any holomorphic function on H can be written as a holomorphic function of h defined on the Riemann sphere, i.e., a rational function. Thus there exist two rational functions $R_1(z)$ and $R_2(z)$ such that

$$u = R_1(h(z)), \quad v = R_2(h(z)) \,. \tag{6}$$

Therefore

$$f(z) = R_1(h(z)), \quad g(z) = R_2(h(z)) \,. \tag{7}$$

From the relation (2) we may suppose that both f and g are transcendental entire functions. Then R_1, R_2 have at most one pole z_0 and z_0 is a Picard value of $h(z)$. By a linear transformation we may let $z_0 = 0$. From (2) we see that R_1 and R_2 are of the form (3).

The proof is complete.

Lemma 3. *Let $P(z)$ be defined as (1). If there are entire functions $f(z)$ and $g(z)$ such that (2) holds, then there exists a complex number b such that*

$$m\left(r, \frac{f}{bf - g}\right) = \circ\{T(r, g)\} \,.$$

P r o o f. For any non-constant meromorphic function $h(z)$, we know that

$$N\left(r, \frac{1}{h - a}\right) \sim T(r, h)$$

except for a set $E(h) \subset \mathbb{C}$ which is of zero capacity (see [8, p. 276]). According to Lemma 2, (3) and (4) hold. If $R_1(z) \equiv R_2(z)$, then we take $b = 0$ and obtain the desired result. If $R_1(z) \not\equiv R_2(z)$, we define

$$s^- = \begin{cases} 0, & s \geq 0, \\ |s|, & s < 0. \end{cases}$$

By (3) there exist two relatively prime polynomials p_1 and p_2 and rational function $R(z)$ such that

$$R_1(z) = R(z)p_1(z), \quad R_2(z) = R(z)p_2(z) \,,$$

and

$$m = \deg \mathrm{p}_1 = \deg \mathrm{p}_2 \leq k + s^- \,.$$

Now we choose a complex number b such that $bp_1 - p_2$ has m zeros $\{z_1, \ldots, z_m\}$ (counting multiplicities) and $z_j \notin E(h)$ for $j = 1, \ldots, m$. Then we have

$$N\left(r, \frac{1}{h - z_j}\right) \sim T(r, h), \quad (j = 1, \ldots, m).$$

Combining these and (4), we obtain

$$\begin{aligned} N\left(r, \frac{f}{bf - g}\right) &= N\left(r, \frac{p_1(h)}{bp_1(h) - p_2(h)}\right) = \sum_{j=1}^{m} N\left(r, \frac{1}{h - z_j}\right) \\ &\sim mT(r, h) = T\left(r, \frac{f}{bf - g}\right) + O(1). \end{aligned}$$

Therefore, by using Valiron's theorem (see Gol'dberg and Ostrovskii [3, p. 47, Theorem, 6.1], the desired result holds. This completes the proof.

The following lemma can be easily deduced from Nevanlinna [8, p. 279] (cf. Hayman [5] or Yang [12]).

Lemma 4. *Any non-constant meromorphic function has no more than* $\left[\frac{2k}{k-1}\right]$ *k-ramified values, here a value b is called a k-ramified value of a meromorphic function $f(z)$ if $f(z) - b$ has no zeros of order less than k, and the square bracket denotes the integer function.*

3. Proof of Theorem 1

Suppose on the contrary that there exist two entire functions $f(z)$ and $g(z)$ with

$$f(z) \not\equiv g(z) \tag{8}$$

such that

$$P(f(z)) \equiv P(g(z)). \tag{9}$$

This and (1) give

$$f^n + a_t f^t + \cdots + a_1 f + a_0 = g^n + a_t g^t + \cdots + a_1 g + a_0.$$

Thus by (8), the above equation can head to

$$\sum_{j=0}^{n-1} f^j g^{n-1-j} + a_t \sum_{j=0}^{t-1} f^j g^{t-1-j} + \cdots + a_2(f + g) + a_1 \equiv 0.$$

Let

$$\beta = \frac{f}{g}. \tag{10}$$

Then we have

$$g^{n-1}\sum_{j=0}^{n-1}\beta^j + a_t g^{t-1}\sum_{j=0}^{t-1}\beta^j + \cdots + a_2 g(\beta+1) + a_1 = 0\,. \tag{11}$$

If β is a constant ($\neq 1$), then the above equation implies that g is a constant, a contradiction. Thus β can not be a constant. Therefore

$$\sum_{j=0}^{n-1}\beta^j \not\equiv 0\,.$$

Next we suppose that

$$m(r,\beta) = \circ\{T(r,g)\}\,, \tag{12}$$

otherwise, we replace g by $bf - g$ in the following discussions, where b satisfies Lemma 3. Note that g is entire, it follows from (11) that

$$(n-1)T(r,g) = (n-1)m(r,g)$$

$$\leq m\left(r, \frac{1}{\sum_{j=0}^{n-1}\beta^j}\right) + m\left(r, \sum_{k=1}^{t} a_k g^{k-1}\sum_{j=0}^{k-1}\beta^j\right) + \bigcirc(1)\,. \tag{13}$$

Let

$$J = \{\theta : \theta \in [0,2\pi), |g(re^{i\theta})| \geq 1\}, \quad J^c = [0,2\pi) - J\,. \tag{14}$$

Then by (10), (12), and Hellerstein–Rubel [6],

$$m\left(r, \sum_{k=1}^{t} a_k g^{k-1}\sum_{j=0}^{k-1}\beta^j\right) \leq (t-1)m(r,g) + \circ\{T(r,g)\}\,. \tag{15}$$

Let $\{z_1,\ldots,z_{n-1},1\}$ be all the n-th roots of unity. By the second fundamental theorem, we deduce from Lemma 2, (9), and (10) that

$$\begin{aligned}
m\left(r, \frac{1}{\sum_{j=0}^{n-1}\beta^j}\right) &= (n-1)T(r,\beta) - N\left(r, \frac{1}{\sum_{j=0}^{n-1}\beta^j}\right) + \bigcirc(1) \\
&= (n-1)T(r,\beta) - \sum_{j=1}^{n-1} N\left(r, \frac{1}{\beta - z_j}\right) + \bigcirc(1) \\
&\leq (n-1)T(r,\beta) - (n-1-2)T(r,\beta) + \circ(T(r,\beta)) \\
&\leq 2T(r,\beta) + \circ(T(r,f) + T(r,g)) \\
&= 2N(r,\beta) + \circ(T(r,g)) \\
&\leq 2N(r,\frac{1}{g}) + \circ(T(r,g)) \\
&\leq 2T(r,g) + \circ(T(r,g))
\end{aligned}$$

possibly outside a set of r of finite linear measure. Substituting this and (15) into (13), we obtain

$$(n-t-2)T(r,g) \leq \circ(T(r,g))$$

possibly outside a set of r of finite linear measure. Since $t < n-2$, the above inequality implies that g must be a constant, a contradiction. The proof is complete, and Theorem 1 is thus proved.

4. Proof of Theorem 2

Suppose on the contrary that $P(z)$ is not a UPM, then there exist two distinct non-constant meromorphic functions f and g such that

$$f^n + a_m f^m + a_0 = g^n + a_m g^m + a_0\,. \tag{16}$$

Let

$$f = \beta g\,. \tag{17}$$

Then $\beta \not\equiv 1$. Substituting (17) into (16), we obtain

$$g^{n-m} = -a_m \frac{\beta^m - 1}{\beta^n - 1} = -a_m \frac{\beta^{m-1} + \cdots + \beta + 1}{\beta^{n-1} + \cdots + \beta + 1}\,. \tag{18}$$

Since $(n,m) = 1$, the denominator and the numerator above are relatively prime. Now we consider two cases.

Case 1. $n \geq 5$ and $1 \leq m \leq 2$. Then $n-m \geq 3$ and $n-1 \geq 4$. From the denominator of (18) we see that β has at least $n-1$ (≥ 4) 3-ramified values. It follows from Lemma 4 that β must be a constant, so does g by (18), a contradiction.

Case 2. $n \geq 5$ and $3 \leq m < n-1$. Then $n-m \geq 2$, $m-1 \geq 2$ and $n-1 \geq 4$. From the denominator and the numerator of (18) we see that β has at least $n-1+m-1$ (≥ 6) 2-ramified values. Again from Lemma 4 we get a contradiction.

This completes the proof.

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Полиномы единственности целых и мероморфных функций

Чжунь–Чун Ян, Ксин–Ху Хуа

Используя теорию римановых поверхностей и теорию обобщенных функций, получены полиномы единственности $P(z)$ такие, что если $P(f) \equiv P(g)$ для любой пары мероморфных функций f и g, то $f \equiv g$.

Поліноми єдиності цілих та мероморфних функцій

Чжунь–Чун Ян, Ксін–Ху Хуа

Використовуючи теорію риманових поверхонь та теорію узагальнених функцій, одержано поліноми єдиності $P(z)$ такі, що коли $P(f) \equiv P(g)$ для будь-якої пари мероморфних функцій f і g, то $f \equiv g$.