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Homogenization of semilinear parabolic equations with asymptotically degenerating coefficients

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An initial boundary value problem for semilinear parabolic equation

$$\frac{\partial u^\varepsilon}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), \quad x \in \Omega, \ t \in (0,T);$$

with the coefficients $a_{ij}^\varepsilon(x)$ depending on a small parameter ε is considered. We suppose that $a_{ij}^\varepsilon(x)$ are of the order of $\varepsilon^{3+\gamma}$ $(0 \le \gamma < 1)$ on a set of spherical annuluses G_ε^α of a thickness $d_\varepsilon = d\varepsilon^{2+\gamma}$. The annuluses are periodically with a period ε distributed in Ω. On the set $\Omega \backslash U_\alpha G_\varepsilon^\alpha$ these coefficients are constants. We study the asymptotical behaviour of the solutions $u^\varepsilon(x,t)$ of the problem as $\varepsilon \to 0$. It is shown that the asymptotic behaviour of the solutions is described by a system of a parabolic p.d.e. coupled with an o.d.e.

1. Introduction

The aim of the paper is to study the asymptotic behaviour of the solutions of the semilinear initial boundary value problem

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), & x \in \Omega, \ t \in (0,T); \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0, & x \in \partial\Omega, \ t \in (0,T); \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), & x \in \Omega, \end{cases} \tag{1.1}$$

as $\varepsilon \to 0$. We suppose that $a_{ij}^{\varepsilon}(x)$ depend on a parameter ε and for any ε these coefficients satisfy the following condition:

$$\alpha^{(\varepsilon)}(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}^{\varepsilon}(x)\xi_i\xi_k \le \beta^{(\varepsilon)}(x)|\xi|^2, \tag{1.2}$$

where

$$0 \le \alpha^{(\varepsilon)}(x) \le \beta^{(\varepsilon)}(x) < \infty, \quad x \in \Omega.$$

The existence and uniqueness of the generalized solution of problem (0.1) in the appropriate classes (see Theorem 2.1 below) under hypothesis (0.2) and natural assumptions on $u_0^{\varepsilon}(x)$, $h^{\varepsilon}(x)$, and $f(u)$ follows from standard parabolic theory.

If there exist some constants α and β such that

$$0 < \alpha \le \alpha^{(\varepsilon)}(x) \le \beta^{(\varepsilon)}(x) \le \beta < \infty, \quad x \in \Omega,$$

then the homogenized equation has a form of the initial equation. The problem of passing to the limit in the problems of such type was studied by a number of authors: it is difficult to give here a complete bibliography, but one can find the extensive bibliography in the monographs [1–3].

If on the contrary condition (0.2) is not valid, i.e., there exist some subsets $\mathcal{G}_\varepsilon$ such that $\sup \alpha^{(\varepsilon)}(x) \to 0$ as $\varepsilon \to 0$ or $\inf \beta^{(\varepsilon)}(x) \to \infty$ as $\varepsilon \to 0$, then the homogenized equations have more complex form, essentially depending on the structure of the set $\mathcal{G}_\varepsilon$.

In this paper we suppose that $a_{ij}^{\varepsilon}(x)$ are of the order $\varepsilon^{3+\gamma}$ $(0 \le \gamma < 1)$ on the union $\mathcal{G}_\varepsilon$ of spherical annuluses G_ε^α having the thickness $d_\varepsilon = d\varepsilon^{2+\gamma}$. These annuluses are periodically, with a period ε, distributed along the directions of the axes in Ω. On the set $\Omega \setminus \bigcup_\alpha G_\varepsilon^\alpha$ these coefficients are equal to the Kronecker symbol δ_{ij}.

We study the asymptotic behaviour of the solutions $u^\varepsilon(x,t)$ of problem (0.1), as $\varepsilon \to 0$. We show that the homogenization of this problem leads to a system of a semilinear parabolic equation coupled with an ordinary differential equation with respect to the variable t:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum\limits_{i,j=1}^{n} b_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j} + b_1(u-v) + f(u) = h_1(x), & x \in \Omega, \ t \in (0,T); \\ \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \ t \in (0,T); \quad u(x,0) = u_0(x), & x \in \Omega; \\ \dfrac{\partial v}{\partial t} + b_2(v-u) + f(v) = h_2(x), & x \in \Omega, \ t \in (0,T); \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where the coefficients $b_{ij}(i,j=1,2,...,n)$ and $b_k(k=1,2)$ are calculated from the solutions of cellular problems and the parameters of the structure.

A problem that is very close to above mentioned problem is the problem of homogenization of the non–linear parabolic equations in the domains with "traps", considered by L. Boutet de Monvel, I. Chueshov, and E. Khruslov [4] and by A. Bourgeat and L. Pankratov [5]. Problem (0.1) with $f(u)\equiv 0$ was considered by E. Khruslov (see, for example, [6]).

2. Problem statement and uniform estimates

Let Ω be a bounded domain from $\mathbf{R}^n$, $n\geq 3$. Let us introduce the notation

$$G_\varepsilon^\alpha=\{x\in\Omega: r_\varepsilon-d_\varepsilon<|x-x^\alpha|<r_\varepsilon\},\quad B_\alpha(r_\varepsilon-d_\varepsilon)=\{x\in\Omega: |x-x^\alpha|<r_\varepsilon-d_\varepsilon\},$$

$$\mathcal{G}_\varepsilon=\bigcup_{\alpha\in N_\varepsilon}G_\varepsilon^\alpha;\quad \mathcal{B}_\varepsilon=\bigcup_{\alpha\in N_\varepsilon}B_\alpha(r_\varepsilon-d_\varepsilon);\quad \Omega_\varepsilon=\Omega\setminus\left(\mathcal{G}_\varepsilon\bigcup\mathcal{B}_\varepsilon\right),$$

where $x^\alpha=\alpha\varepsilon$ $(\alpha\in\mathbf{Z}^n)$ and N_ε is a set of multi–indices such that $G_\varepsilon^\alpha\subset\Omega$, $r_\varepsilon=r\varepsilon$, $d_\varepsilon=d\varepsilon^{2+\gamma}$ $(0\leq\gamma<1)$. In the domain $\Omega_T=\Omega\times(0,T)$ we consider the boundary value problem:

$$\frac{\partial u^\varepsilon}{\partial t}-\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right)+f(u^\varepsilon)=h^\varepsilon(x),\quad x\in\Omega,\ t\in(0,T); \tag{2.1}$$

$$\frac{\partial u^\varepsilon}{\partial n}=0,\quad x\in\partial\Omega,\ t\in(0,T); \tag{2.2}$$

$$u^\varepsilon(x,0)=u_0^\varepsilon(x),\quad x\in\Omega\,, \tag{2.3}$$

where n is a external normal vector to the domain Ω, $f(u)$ is a smooth function in $\mathbf{R}$, the functions $h^\varepsilon(x)$, $u_0^\varepsilon(x):\Omega\to\mathbf{R}$ are given. We study the asymptotical behaviour of the solutions $u^\varepsilon(x,t)$ of problem (2.1)–(2.3), as $\varepsilon\to 0$.

If r is chosen such that $r<1/4$ the annuluses G_ε^α are non-intersecting, each containing in "its" cell $K_\alpha(\varepsilon)$:

$$K_\alpha(\varepsilon)=\left(x^\alpha-\frac{\varepsilon}{2},\ x^\alpha+\frac{\varepsilon}{2}\right)^n.$$

We assume for sake of simplicity that $K_\alpha(\varepsilon)\subset\Omega$ for each $\alpha\in N_\varepsilon$.

The coefficients $a_{ij}^\varepsilon(x)$ of equation (2.1) are defined as follows:

$$a_{ij}^\varepsilon(x)=\begin{cases}\delta_{ij}, & x\in\Omega\setminus\mathcal{G}_\varepsilon;\\ a_\varepsilon\delta_{ij}\equiv a\delta_{ij}\varepsilon^{3+\gamma}, a>0, & x\in\mathcal{G}_\varepsilon.\end{cases} \tag{2.4}$$

We assume that the function $f(u) \in C^2(\mathbf{R})$ has the following properties :

$$\sup\{|f'(u)| : u \in \mathbf{R}\} < \infty \tag{2.5}$$

and there exist constants B_1, B_2, B_3 such that

$$uf(u) \geq B_1 u^2 - B_2\,, \tag{2.6}$$

$$\mathcal{F}(u) \equiv \int_0^u f(\xi)d\xi \geq B_1 u^2 - B_3\,. \tag{2.7}$$

We denote by $V_2^{1,0}(\Omega_T)$ the Banach space of functions that are continuous with respect to the variable t in $L^2(\Omega)$ with a norm

$$\langle\!\langle u \rangle\!\rangle_{\Omega_T} = \max_{0\leq t\leq T} \|u(x,t)\|_{2,\Omega} + \|\nabla u\|_{2,\Omega_T}\,,$$

where $\|\cdot\|_{2,\Omega}$ is a norm in the space $L^2(\Omega)$ and

$$\|\nabla u\|_{2,\Omega_T} = \sqrt{\int_0^T \int_\Omega |\nabla u|^2 dx dt}.$$

We denote also by $V_2^{1,1/2}(\Omega_T)$ a subset of functions $u(x,t)$ from the space $V_2^{1,0}(\Omega_T)$ such that

$$\int_0^{T-h} \int_\Omega h^{-1}[u(x,t+h) - u(x,t)]^2 dx dt \to 0,$$

as $h \to 0$.

We say that $u^\varepsilon(x,t)$ is a generalized solution from $V_2^{1,0}(\Omega_T)$ (or from $V_2^{1,1/2}(\Omega_T)$) of problem (2.1)–(2.3) if this solution satisfies the equation

$$-\int_{\Omega_T} u^\varepsilon(x,t)\frac{\partial\phi}{\partial t}(x,t)dxdt + \int_{\Omega_T} \sum_{i,j=1}^n a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}(x,t)\frac{\partial \phi}{\partial x_i}(x,t)dxdt$$

$$+\int_{\Omega_T} (f(u^\varepsilon) - h^\varepsilon(x))\phi(x,t)dxdt = \int_\Omega u_0^\varepsilon(x)\phi(x,0)dx, \tag{2.8}$$

for any $\phi(x,t) \in W_2^{1,1}(\Omega_T)$, $\phi(x,T) = 0$, where $W_2^{1,1}(\Omega_T)$ is a Hilbert space with a scalar product

$$(u,v)_{2,\Omega_T}^{1,1} = \int_{\Omega_T} \left(uv + \sum_{k=1}^n u_{x_k} v_{x_k} + u_t v_t \right) dxdt\,.$$

The Theorem is valid (see [7]) :

Теорема 1 *i). Let $h^\varepsilon(x), u_0^\varepsilon(x) \in L^2(\Omega)$. Then problem (2.1)–(2.3) has a unique generalized solution in the class $V_2^{1,1/2}(\Omega_T)$ and moreover this solution satisfies the estimate*

$$\max_{0\le t\le T} \|u^\varepsilon(x,t)\|_{2,\Omega} + \|\nabla u^\varepsilon\|_{2,\Omega_\varepsilon\times(0,T)} + \varepsilon^{3/2+\gamma/2}\|\nabla u^\varepsilon\|_{2,\mathcal{G}_\varepsilon\times(0,T)} + \|\nabla u^\varepsilon\|_{2,\mathcal{B}_\varepsilon\times(0,T)}$$

$$\le A_1[\|u_0^\varepsilon\|_{2,\Omega} + \|h^\varepsilon\|_{2,\Omega}] + A_2, \tag{2.9}$$

where A_1, A_2 are the constants independent of ε.

ii). Let $h^\varepsilon(x), u_0^\varepsilon(x) \in W_2^1(\Omega)$. Then problem (2.1)–(2.3) has a unique generalized solution in the class $W_2^{1,1}(\Omega_T)$ and we have the equality

$$\Lambda_\varepsilon(u^\varepsilon) + \int_0^t \|u_t^\varepsilon(x,\tau)\|_{2,\Omega}^2 d\tau = \Lambda_\varepsilon(u_0^\varepsilon), \tag{2.10}$$

where

$$\Lambda_\varepsilon(u^\varepsilon) = \frac{1}{2}\left\{\|\nabla u^\varepsilon\|_{2,\Omega_\varepsilon}^2 + a_\varepsilon\|\nabla u^\varepsilon\|_{2,\mathcal{G}_\varepsilon}^2 + \|\nabla u^\varepsilon\|_{2,\mathcal{B}_\varepsilon}^2\right\} + \int_\Omega \{\mathcal{F}(u^\varepsilon) - h^\varepsilon u^\varepsilon\}\, dx.$$

Lemma 2.1. *Let $u^\varepsilon(x,t)$ be the solution of problem (2.1)–(2.3), then*

$$\int_0^{T-\Delta}\int_\Omega |u^\varepsilon(x,t+\Delta t) - u^\varepsilon(x,t)|^2 dx dt \le A_3\cdot\Delta t\cdot[\|u_0^\varepsilon\|_{2,\Omega}^2 + \|h^\varepsilon\|_{2,\Omega}^2 + 1], \tag{2.11}$$

where A_3 is a constant independent of ε.

The proof of this Lemma is of standard character and relies on the methods presented in [7].

3. Formulation of main result

We suppose for sake of simplicity that $0 \in \Omega$. Let K be a cube in $\mathbf{R}^n$:

$$K = \{x \in \mathbf{R}^n;\ |x_i| < \frac{1}{2r};\ i = 1,2,\ldots,n\};$$

and B is the unit ball in $\mathbf{R}^n$:

$$B = \{x \in K; \sum_{i=1}^n x_i^2 < 1\}.$$

Define now in $P = K \setminus B$ the functions $v_i(x)$, $i = 1, 2, ..., n$, that are the solutions of the following auxiliary problem:

$$\begin{cases} \Delta v_i = 0, & x \in P = K \setminus \bar{B}; \\ \dfrac{\partial v_i}{\partial n} = (x_i, n), & x \in \partial B; \\ v_i(x), Dv_i(x) \quad \text{are } K - \text{periodic}, \end{cases} \tag{3.1}$$

where n is the external normal vector to B. It is known that this problem has a unique solution $v_i(x)$ (up to a constant).

Let $\{x^\alpha = \alpha\varepsilon, \alpha \in \mathbf{Z}^n\}$ be a lattice in $\mathbf{R}^n$. We associate with this lattice a covering of the domain Ω by the cubes centered at x^α and edges of length ε. Let Q_ε be a linear interpolation operator that is defined by its value in each node of the lattice as follows. For each node of the sublattice $\{x^\alpha = \alpha\varepsilon,\ \alpha \in N_\varepsilon\}$ we set

$$(Q_\varepsilon u)(x^\alpha) = \frac{1}{\mathrm{Vol}(B_\alpha(r_\varepsilon - d_\varepsilon))} \int\limits_{B_\alpha(r_\varepsilon - d_\varepsilon)} u(x)dx, \quad \alpha \in N_\varepsilon\,,$$

where $\mathrm{Vol}(B_\alpha(r_\varepsilon - d_\varepsilon))$ is the volume of the ball $B_\alpha(r_\varepsilon - d_\varepsilon)$ of radius $(r_\varepsilon - d_\varepsilon)$ centered at x^α. For each node $\{x^\alpha = \alpha\varepsilon,\ \alpha \notin N_\varepsilon\}$ we set $(Q_\varepsilon u)(x^\alpha)$ being equal to a mean between the values of $(Q_\varepsilon u)$ in the nearest nodes of the lattice. Taking now the restriction of the operator Q_ε obtained on the domain Ω and keeping for it the same notation, we obtain the linear bounded operator from $L^2(\Omega)$ to $W_2^1(\Omega)$.

The main result of the paper is the following

Theorem 3.1. *Let $u^\varepsilon(x,t)$ be the solution of problem (2.1)–(2.3). We assume that*

i) for any $\varepsilon \in (0, \varepsilon_0)$

$$\|u_0^\varepsilon\|_{2,\Omega}^{(1)} + \|\nabla Q_\varepsilon u_0^\varepsilon\|_{2,\Omega} + \|h^\varepsilon\|_{2,\Omega}^{(1)} + \|\nabla Q_\varepsilon h^\varepsilon\|_{2,\Omega} \le C,$$

where C denotes any constant independent of ε;

ii) there exist the functions u_0, v_0, h_1, h_2 from $L^2(\Omega)$ such that

$$\|u_0^\varepsilon - u_0\|_{2,\Omega_\varepsilon} \to 0, \quad \|h^\varepsilon - h_1\|_{2,\Omega_\varepsilon} \to 0\,;$$

and

$$\|u_0^\varepsilon - v_0\|_{2,\mathcal{B}_\varepsilon} \to 0, \quad \|h^\varepsilon - h_2\|_{2,\mathcal{B}_\varepsilon} \to 0\,,$$

as $\varepsilon \to 0$.

Then we have

$$\lim_{\varepsilon\to 0}\{\|u^\varepsilon(x,t)-u(x,t)\|^2_{2,\Omega_\varepsilon\times(0,T)}+\|u^\varepsilon(x,t)-v(x,t)\|^2_{2,\mathcal{B}_\varepsilon\times(0,T)}\}=0\,,$$

where the pair of functions $\{u(x,t),v(x,t)\}$ *is the solution of problem (0.3). The coefficients* b_{ij} *and* b_k *in (0.3) are calculated from cellular problem (3.1) solutions by*

$$b_{ij}=\delta_{ij}\left[1-\frac{r^n}{1-\mu}\int\limits_P(\nabla v_i,\nabla v_j)dx\right]\,, \tag{3.2a}$$

$$b_1=\frac{b_2\mu}{1-\mu},\quad \mu=\frac{\pi^{n/2}r^n}{\Gamma\left(\frac{n}{2}+1\right)},\quad b_2=\frac{an}{rd}\,. \tag{3.2b}$$

In (3.2) above δ_{ij} is the Kronecker symbol and Γ is the Gamma function.

The proof of Theorem 3.1 consists of two parts. In the first part we obtain the uniform estimate

$$\int\limits_0^T\|\nabla Q_\varepsilon u^\varepsilon(t)\|^2_{2,\Omega}dt<C\,.$$

Then in the second part we pass to the limit in equation (2.1) using a special test function.

The proof of the existence and uniqueness of the generalized solution $U(x,t)=(u(x,t),v(x,t))$ of problem (0.3) under conditions (2.5)–(2.7) for $U_0=(u_0,v_0)\in\mathbf{L}_0=L^2(\Omega)\times L^2(\Omega)$ in the class $C(0,T;\mathbf{L}_0)$ is of a standard character and relies on the methods presented in [9] (see also [4]).

4. Properties of operator Q_ε

In this section we obtain some properties of the solutions of problem (2.1)–(2.3).

Let P_ε be a linear extension operator from Ω_ε to Ω (as defined for instance in [10]) having the properties:

i) $P_\varepsilon : W_2^l(\Omega_\varepsilon)\to W_2^l(\Omega)$ for $l=0,1$ such that

$$\|P_\varepsilon u\|^{(l)}_{2,\Omega}\le C\|u\|^{(1)}_{2,\Omega_\varepsilon},\quad l=0,1,$$

where C is a constant independent of ε;

ii) $P_\varepsilon u=u$ on Ω_ε for any $u\in L^2(\Omega_\varepsilon)$.

The main result of this section will be Lemma 4.1 giving a bound for

$$\int\limits_0^T \left(\|Q_\varepsilon u^\varepsilon\|_{2,\Omega}^{(1)}\right)^2 dt\,.$$

This Lemma and estimates for $P_\varepsilon u^\varepsilon$ following from (2.9), (2.10) will make possible to extract from $\{P_\varepsilon u^\varepsilon\}$ and $\{Q_\varepsilon u^\varepsilon\}$ subsequences strongly convergent in $L^2(\Omega_T)$. We will assume in the following that the conditions (i), (ii) of Theorem 3.1 are fulfilled.

According to (2.9), (2.10) the solution $u^\varepsilon(x,t)$ of problem (2.1)–(2.3) satisfies the estimate

$$\|u^\varepsilon(x,t)\|_{2,\Omega_\varepsilon}^2 + \|\nabla u^\varepsilon(x,t)\|_{2,\Omega_\varepsilon}^2 \le C_1\,, \tag{4.1}$$

for a constant C_1 independent of ε. Therefore we have for the extension:

$$\|P_\varepsilon u^\varepsilon(x,t)\|_{2,\Omega}^2 + \|\nabla P_\varepsilon u^\varepsilon(x,t)\|_{2,\Omega}^2 \le C_2\,. \tag{4.2}$$

In a very similar way using (2.9), (2.10) we have for all $t \in (0,T)$

$$\|Q_\varepsilon u^\varepsilon(x,t)\|_{2,\Omega}^2 \le C_3\,. \tag{4.3}$$

Now in remaining part of this section we will prove similar estimate for $\nabla Q_\varepsilon u^\varepsilon(x,t)$.

Let us introduce the following notation:

$$G_\varepsilon = \{x \in \Omega : r_\varepsilon - d_\varepsilon < |x| < r_\varepsilon\}, \quad B(r_\varepsilon - d_\varepsilon) = \{x \in \Omega : |x| < r_\varepsilon - d_\varepsilon\}\,,$$

$$B(2r_\varepsilon) = \{x \in \Omega : |x| < 2r_\varepsilon\}, \quad \Pi_\varepsilon = \{x \in \Omega : r_\varepsilon < |x| < 2r_\varepsilon\}\,;$$

$$u_\alpha^\varepsilon(x,t) = u^\varepsilon(x^\alpha + x, t), \quad u_{0,\alpha}^\varepsilon(x) = u_0^\varepsilon(x^\alpha + x), \quad x^\alpha = \alpha\varepsilon,\ \alpha \in N_\varepsilon,\ x \in B(2r_\varepsilon);$$

$$u_\alpha^{\varepsilon,in}(t) = \frac{1}{\mathrm{Vol}(B(r_\varepsilon - d_\varepsilon))} \int\limits_{B(r_\varepsilon - d_\varepsilon)} u_\alpha^\varepsilon(x,t)dx\,,$$

$$u_{0,\alpha}^{\varepsilon,in} = \frac{1}{\mathrm{Vol}(B(r_\varepsilon - d_\varepsilon))} \int\limits_{B(r_\varepsilon - d_\varepsilon)} u_{0,\alpha}^\varepsilon(x)dx\,,$$

$$u_\alpha^{\varepsilon,ex}(t) = \frac{1}{\mathrm{Vol}(\Pi_\varepsilon)} \int\limits_{\Pi_\varepsilon} u_\alpha^\varepsilon(x,t)dx\,, \quad u_{0,\alpha}^{\varepsilon,ex} = \frac{1}{\mathrm{Vol}(\Pi_\varepsilon)} \int\limits_{\Pi_\varepsilon} u_{0,\alpha}^\varepsilon(x)dx\,,$$

where $u^\varepsilon(x,t)$ is the solution of problem (2.1)–(2.3).

We also use the notation

$$w^\varepsilon \equiv w_{\alpha\beta}^\varepsilon(x,t) = u_\alpha^\varepsilon(x,t) - u_\beta^\varepsilon(x,t), \quad w_0^\varepsilon \equiv w_{0,\alpha\beta}^\varepsilon(x) = u_{0,\alpha}^\varepsilon(x) - u_{0,\beta}^\varepsilon(x)\,,$$

$$x \in B(2r_\varepsilon), \ \ \alpha, \beta \in N_\varepsilon;$$

and we will denote indistinctly by $w^{\#}$ or $w^{\#}_{\alpha\beta}$ for

$$w^{\#}=w^{\#}_{\alpha\beta}(t)=u^{\varepsilon,\#}_{\alpha}(t)-u^{\varepsilon,\#}_{\beta}(t),\quad w^{\#}_0=w^{\#}_{0,\alpha\beta}=u^{\varepsilon,\#}_{0,\alpha}-u^{\varepsilon,\#}_{0,\beta},\quad \alpha,\beta\in N_\varepsilon,$$

where # is either "*in*" or "*ex*".

The main result of the section is

Lemma 4.1. *Under assumptions of Theorem 3.1 we have a priori estimate*

$$\int\limits_0^T \|\nabla Q_\varepsilon u^\varepsilon(x,t)\|^2_{2,\Omega}dt\le C\,, \tag{4.4}$$

for a constant C independent of ε.

In order to prove this lemma it is sufficient to obtain appropriate estimates for $w_{\alpha\beta}(t)$, for this we will use the following preliminary lemmas.

Lemma 4.2. *Let $w(x)\in W^1_2(G_\varepsilon\bigcup\Pi_\varepsilon)$. Then we have*

$$\int\limits_{G_\varepsilon} w^2(x)dx\le C\{\varepsilon^{n+1+\gamma}|w^{ex}|^2+\varepsilon^{3+\gamma}\|\nabla w\|^2_{2,\Pi_\varepsilon}+\varepsilon^{4+2\gamma}\|\nabla w\|^2_{2,G_\varepsilon}\}\,. \tag{4.5}$$

Let us introduce the function $v^\varepsilon(x)$ defined on $B(2r_\varepsilon)$ as follows:

$$v^\varepsilon(x)=\begin{cases}\lambda_1^\varepsilon\rho^2, & x\in\Pi_\varepsilon;\\[2mm] \mathcal{L}_1^\varepsilon\rho^{2-n}+\mathcal{L}_2^\varepsilon, & x\in G_\varepsilon;\\[2mm] 1+\lambda_2^\varepsilon\rho^2, & x\in B(r_\varepsilon-d_\varepsilon).\end{cases} \tag{4.6}$$

We choose the coefficients in (4.6) from the following conditions:

$$\begin{cases}(v^\varepsilon)^+=(v^\varepsilon)^-, & \rho=r_\varepsilon;\\[2mm] \left(\dfrac{\partial v^\varepsilon}{\partial\rho}\right)^+=a_\varepsilon\left(\dfrac{\partial v^\varepsilon}{\partial\rho}\right)^-, & \rho=r_\varepsilon;\end{cases} \tag{4.7}$$

$$\begin{cases}(v^\varepsilon)^+=(v^\varepsilon)^-, & \rho=r_\varepsilon-d_\varepsilon;\\[2mm] a_\varepsilon\left(\dfrac{\partial v^\varepsilon}{\partial\rho}\right)^+=\left(\dfrac{\partial v^\varepsilon}{\partial\rho}\right)^-, & \rho=r_\varepsilon-d_\varepsilon,\end{cases} \tag{4.8}$$

where we denote by "+" a value of the function $v^\varepsilon(x)$ and its derivative on the external surface of $\partial B(r_\varepsilon)$ (or $\partial B(r_\varepsilon - d_\varepsilon)$) and by "–" we denote a value of the function $v^\varepsilon(x)$ and its derivative on the internal surface of $\partial B(r_\varepsilon)$ (or $\partial B(r_\varepsilon - d_\varepsilon)$). It also follows from (4.6) that $\Delta v^\varepsilon(x) = 0$ for $x \in G_\varepsilon$. The function $v^\varepsilon(x)$ is a model of behaviour of problem (2.1)–(2.3) solution in a neighbourhood of the set G_ε. Calculating the coefficients in the explicit form it is easy to verify that

$$\lim_{\varepsilon\to 0} \lambda_1^\varepsilon = \lim_{\varepsilon\to 0} \lambda_2^\varepsilon = -\frac{a}{2rd}. \tag{4.9}$$

Let us also introduce the function $\mathcal{V}^\varepsilon(x)$ defined on $B(2r_\varepsilon)$ as follows :

$$\mathcal{V}^\varepsilon(x) = v^\varepsilon(x) - 4\lambda_1^\varepsilon r_\varepsilon^2. \tag{4.10}$$

Lemma 4.3. *Let $\mathcal{V}^\varepsilon(x)$ is defined by (4.10). If we denote*

$$\mathcal{R}_{\alpha\beta}^\varepsilon(t) = \frac{d}{dt} \int\limits_{B(2r_\varepsilon)} w^\varepsilon(x,t)\mathcal{V}^\varepsilon(x)dx + (I_{\alpha\beta}^\varepsilon(t) - 2n\lambda_2^\varepsilon) \int\limits_{B(2r_\varepsilon)} w^\varepsilon(x,t)\mathcal{V}^\varepsilon(x)dx\,, \tag{4.11}$$

where

$$I_{\alpha\beta}^\varepsilon(t) = \frac{1}{M_\varepsilon}\int\limits_0^1 d\xi \int\limits_{B(r_\varepsilon - d_\varepsilon)} f'(u_\beta^\varepsilon(x,t) + \xi(u_\alpha^\varepsilon(x,t) - u_\beta^\varepsilon(x,t)))\mathcal{V}^\varepsilon dx\,,$$

$$M_\varepsilon = \int\limits_{B(r_\varepsilon - d_\varepsilon)} \mathcal{V}^\varepsilon(x)dx\,,$$

then quantity $\mathcal{R}_{\alpha\beta}^\varepsilon(t)$ is bounded:

$$|\mathcal{R}_{\alpha\beta}^\varepsilon(t)| \le B_1\varepsilon^n \left(|w^{ex}(t)| + |h^{\varepsilon,in}| + |h^{\varepsilon,ex}|\right)$$

$$+B_2\varepsilon^{(n/2+1)}\Big\{\|w^\varepsilon\|_{2,B(2r_\varepsilon)} + \|\nabla w^\varepsilon\|_{2,\Pi_\varepsilon} + \varepsilon^{3/2+3\gamma/2}\|\nabla w^\varepsilon\|_{2,G_\varepsilon}$$

$$+\|\nabla w^\varepsilon\|_{2,B(r_\varepsilon - d_\varepsilon)} + \|h_{\alpha\beta}^\varepsilon\|_{2,B(2r_\varepsilon)}^{(1)}\Big\}. \tag{4.12}$$

Here $B_i (i = 1,2)$ are constants independent of ε, $h_{\alpha\beta}^\varepsilon(x)$, $h^{\varepsilon,ex}$, and $h^{\varepsilon,in}$ are defined in the same way as $w_{\alpha\beta}^\varepsilon(x,t)$, $w^{\varepsilon,ex}(t)$, and $w^{\varepsilon,in}(t)$.

P r o o f o f L e m m a 4.3. Let us consider the equation

$$\frac{d}{dt}\int\limits_{B(2r_\varepsilon)} w^\varepsilon(x,t)\mathcal{V}^\varepsilon(x)dx - \int\limits_{B(2r_\varepsilon)} \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial w^\varepsilon}{\partial x_j}\right)\mathcal{V}^\varepsilon(x)dx$$

$$+\int\limits_{B(2r_\varepsilon)}(f(u^\varepsilon_\alpha(x,t))-f(u^\varepsilon_\beta(x,t)))\mathcal{V}^\varepsilon(x)dx=\int\limits_{B(2r_\varepsilon)}h^\varepsilon(x)\mathcal{V}^\varepsilon(x)dx, \qquad (4.13)$$

which follows from (2.1). Denoting $\mathcal{R}^\varepsilon_{\alpha\beta}(t)$ the quantity

$$\mathcal{R}^\varepsilon_{\alpha\beta}(t)=\left\{\int\limits_{B(2r_\varepsilon)}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}(x)\frac{\partial w^\varepsilon}{\partial x_j}\right)\mathcal{V}^\varepsilon(x)dx-2n\lambda^\varepsilon_2\int\limits_{B(2r_\varepsilon)}w^\varepsilon(x,t)\mathcal{V}^\varepsilon(x)dx\right\}$$

$$+\left\{I^\varepsilon_{\alpha\beta}(t)\int\limits_{B(2r_\varepsilon)}w^\varepsilon(x,t)\mathcal{V}^\varepsilon(x)dx-\int\limits_{B(2r_\varepsilon)}(f(u^\varepsilon_\alpha(x,t))-f(u^\varepsilon_\beta(x,t)))\mathcal{V}_\varepsilon(x)dxd\tau\right\}$$

$$+\int\limits_{B(2r_\varepsilon)}h^\varepsilon(x)\mathcal{V}_\varepsilon(x)dx\equiv\mathcal{E}^{\alpha\beta}_1(t,\varepsilon)+\mathcal{E}^{\alpha\beta}_2(t,\varepsilon)+\mathcal{E}^{\alpha\beta}_3(t,\varepsilon)\,, \qquad (4.14)$$

we immediately obtain equation (4.11).

Let us prove that $\mathcal{R}^\varepsilon_{\alpha\beta}(t)$ verifies estimate (4.12). Consider the first term in (4.14). It is clear that $\mathcal{V}^\varepsilon(x)=0$, for $x\in\partial B(2r_\varepsilon)$, and $\Delta\mathcal{V}^\varepsilon(x)=0$, for $x\in G_\varepsilon$. Using the properties of the solution of problem (2.1)–(2.3), (4.7), and (4.8), we get

$$\mathcal{E}^{\alpha\beta}_1(t,\varepsilon)=\int\limits_{\Pi_\varepsilon}w^\varepsilon\Delta\mathcal{V}^\varepsilon dx-2n\lambda^\varepsilon_2\int\limits_{\Pi_\varepsilon}w^\varepsilon\mathcal{V}^\varepsilon dx-2n\lambda^\varepsilon_2\int\limits_{G_\varepsilon}w^\varepsilon\mathcal{V}^\varepsilon dx$$

$$+\int\limits_{\partial B(2r_\varepsilon)}w^\varepsilon\frac{\partial\mathcal{V}^\varepsilon}{\partial\nu}d\sigma+\int\limits_{B(r_\varepsilon-d_\varepsilon)}w^\varepsilon\cdot(\Delta\mathcal{V}^\varepsilon-2n\lambda^\varepsilon_2\mathcal{V}^\varepsilon)dx\,. \qquad (4.15)$$

Using Lemma 4.2, we obtain from (4.15)

$$|\mathcal{E}^{\alpha\beta}_1(t,\varepsilon)|\le C_1\varepsilon^n|w^{ex}(t)|+C_2\varepsilon^{(n/2+2)}\left[\|w^\varepsilon\|_{2,\Pi_\varepsilon}+\|w^\varepsilon\|_{2,B(r_\varepsilon-d_\varepsilon)}\right]$$

$$+C_3\varepsilon^{(n/2+1)}\left[\|\nabla w^\varepsilon\|_{2,\Pi_\varepsilon}+\varepsilon^{3/2+3\gamma/2}\|\nabla w^\varepsilon\|_{2,G_\varepsilon}\right]\,. \qquad (4.16)$$

Consider the second term in (4.14). We have

$$\mathcal{E}^{\alpha\beta}_2(t,\varepsilon)=-\int\limits_{\Pi_\varepsilon}(f(u^\varepsilon_\alpha)-f(u^\varepsilon_\beta))\mathcal{V}^\varepsilon(x)dx-\int\limits_{G_\varepsilon}(f(u^\varepsilon_\alpha)-f(u^\varepsilon_\beta))\mathcal{V}^\varepsilon(x)dx$$

$$+\left\{-\int\limits_{B(r_\varepsilon-d_\varepsilon)}(f(u^\varepsilon_\alpha)-f(u^\varepsilon_\beta))\mathcal{V}^\varepsilon(x)dx+\int\limits_0^t w^{in}(t)I^\varepsilon_{\alpha\beta}(t)\int\limits_{B(r_\varepsilon-d_\varepsilon)}\mathcal{V}^\varepsilon(x)dx\right\}$$

$$+I^\varepsilon_{\alpha\beta}(t)\left\{-w^{in}(t)\int\limits_{B(r_\varepsilon-d_\varepsilon)}\mathcal{V}^\varepsilon(x)dx+\int\limits_{B(r_\varepsilon-d_\varepsilon)}w^\varepsilon(x,t)\mathcal{V}^\varepsilon(x)dx\right\}. \qquad (4.17)$$

It follows from (2.5) that

$$|I^\varepsilon_{\alpha\beta}(t)|\leq L\,. \qquad (4.18)$$

Therefore, using Lemma 4.2, we have from (4.17)

$$|\mathcal{E}^{\alpha\beta}_2(t,\varepsilon)|\leq C_4\varepsilon^{n+1+\gamma}|w^{ex}|+C_5\varepsilon^{(n/2+2)}\|w^\varepsilon\|_{2,\Pi_\varepsilon}$$

$$+C_6\varepsilon^{(n/2+1)}\left[\varepsilon^{1+\gamma}\|\nabla w^\varepsilon\|_{2,\Pi_\varepsilon}+\varepsilon^{3/2+3\gamma/2}\|\nabla w^\varepsilon\|_{2,G_\varepsilon}+\|\nabla w^\varepsilon\|_{2,B(r_\varepsilon-d_\varepsilon)}\right]. \qquad (4.19)$$

For the third term in (4.14) we find

$$|\mathcal{E}^{\alpha\beta}_3(t,\varepsilon)|\leq C_7\varepsilon^n(\varepsilon^{1+\gamma}|h^{\varepsilon,ex}|+|h^{\varepsilon,in}|)+C_8\varepsilon^{(n/2+2)}\|h^\varepsilon_{\alpha\beta}\|_{2,\Pi_\varepsilon}$$

$$+C_9\varepsilon^{(n/2+1)}\left[\varepsilon^{1+\gamma}\|\nabla h^\varepsilon_{\alpha\beta}\|_{2,\Pi_\varepsilon}+\varepsilon^{3/2+3\gamma/2}\|\nabla h^\varepsilon_{\alpha\beta}\|_{2,G_\varepsilon}+\|\nabla h^\varepsilon_{\alpha\beta}\|_{2,B(r_\varepsilon-d_\varepsilon)}\right]. \qquad (4.20)$$

Now Lemma 4.3 immediately follows from (4.16), (4.19), (4.20).

P r o o f o f L e m m a 4.1. Consider equation (4.11). Denoting by $y^\varepsilon(t)$ the value

$$y^\varepsilon(t)=\int\limits_{B(2r_\varepsilon)}w^\varepsilon(x,t)\mathcal{V}^\varepsilon(x)dx\,,$$

we obtain for any $\delta>0$

$$\frac{d}{dt}(y^\varepsilon(t))^2\leq 2\vartheta_\varepsilon(y^\varepsilon(t))^2+\frac{1}{2\delta}(\mathcal{R}^\varepsilon_{\alpha\beta})^2\,, \qquad (4.21)$$

where

$$\vartheta_\varepsilon=L+2n|\lambda^\varepsilon_2|+\delta.$$

According to the Gronwall lemma we find

$$(y^\varepsilon(t))^2\leq\frac{e^{2T\vartheta_\varepsilon}}{2\delta}\int\limits_0^T[\mathcal{R}^\varepsilon_{\alpha\beta}(t)]^2dt+e^{2T\vartheta_\varepsilon}\left(y^\varepsilon(0)\right)^2\,, \qquad (4.22)$$

where

$$y^\varepsilon(0)=\int\limits_{B(2r_\varepsilon)}w^\varepsilon(x,0)\mathcal{V}^\varepsilon(x)dx\,, \qquad (4.23)$$

and

$$|y^\varepsilon(0)|\leq C_1\varepsilon^n(|w^{\varepsilon,ex}_0|+|w^{\varepsilon,in}_0|)+C_2\varepsilon^{(n/2+1)}\|w^\varepsilon_0\|^{(1)}_{2,B(2r_\varepsilon)}\,. \qquad (4.24)$$

From (4.12) we get

$$|\mathcal{R}^\varepsilon_{\alpha\beta}(t)|^2 \le C_3\varepsilon^{2n}\left\{|w^{ex}(t)|^2 + |h^{\varepsilon,in}|^2 + |h^{\varepsilon,ex}|^2\right\} + C_4\varepsilon^{n+2}(\Upsilon^\varepsilon_\alpha + \Upsilon^\varepsilon_\beta)\,,$$

where

$$\Upsilon^\varepsilon_j = \|u^\varepsilon\|^2_{2,B_j(2r_\varepsilon)} + \|\nabla u^\varepsilon\|^2_{2,\Pi^j_\varepsilon} + \|\nabla u^\varepsilon\|^2_{2,B_j(r_\varepsilon - d_\varepsilon)}$$
$$+\varepsilon^{3+\gamma}\|\nabla u^\varepsilon\|^2_{2,G^j_\varepsilon} + \left(\|h^\varepsilon\|^{(1)}_{2,B_j(2r_\varepsilon)}\right)^2 .$$

Here

$$B_j(2r_\varepsilon) = j\varepsilon + B(2r_\varepsilon), \quad \Pi^j_\varepsilon = j\varepsilon + \Pi_\varepsilon, \quad G^j_\varepsilon = j\varepsilon + G_\varepsilon,$$
$$B_j(r_\varepsilon - d_\varepsilon) = j\varepsilon + B(r_\varepsilon - d_\varepsilon),$$

and $j \in N_\varepsilon$. Therefore

$$\sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)} \varepsilon^{-n}[\mathcal{R}^\varepsilon_{\alpha\beta}(t)]^2 \le C_5\varepsilon^2$$

$$\times\left\{\|u^\varepsilon(x,t)\|^2_{2,\Omega} + \|\nabla u^\varepsilon\|^2_{2,\Omega_\varepsilon} + \varepsilon^{3+\gamma}\|\nabla u^\varepsilon\|^2_{2,\mathcal{G}_\varepsilon} + \|\nabla u^\varepsilon\|^2_{2,\mathcal{B}_\varepsilon} + \left(\|h^\varepsilon\|^{(1)}_{2,\Omega}\right)^2\right\}$$

$$+C_6\sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\varepsilon^n\left\{|u^{\varepsilon,ex}_\alpha(t) - u^{\varepsilon,ex}_\beta(t)|^2 + |h^{\varepsilon,ex}_\alpha - h^{\varepsilon,ex}_\beta|^2 + |h^{\varepsilon,in}_\alpha - u^{\varepsilon,in}_\beta|^2\right\}, \quad (4.25)$$

where $\sigma(\alpha) = \bar\sigma(\alpha)\bigcap N_\varepsilon$ and $\bar\sigma(\alpha)$ is the set in $\mathbf{Z}^n$ of the nearest neighbors of α.

To estimate the second term in (4.25) and $|y^\varepsilon(0)|$ we use the inequalities:

$$\sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\varepsilon^n|u^{\varepsilon,ex}_\alpha(t) - u^{\varepsilon,ex}_\beta(t)|^2 \le C_7\varepsilon^2\|\nabla u^\varepsilon\|^2_{2,\Omega_\varepsilon}\,, \quad (4.26)$$

$$\sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\varepsilon^n|h^{\varepsilon,ex}_\alpha - h^{\varepsilon,ex}_\beta|^2 \le C_8\varepsilon^2\|\nabla h^\varepsilon\|^2_{2,\Omega_\varepsilon}\,, \quad (4.27)$$

$$\sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\varepsilon^n|u^{\varepsilon,ex}_{0,\alpha} - u^{\varepsilon,ex}_{0,\beta}|^2 \le C_9\varepsilon^2\|\nabla u^\varepsilon_0\|^2_{2,\Omega_\varepsilon}\,, \quad (4.28)$$

$$C_{10}\varepsilon^2\|\nabla Q_\varepsilon h^\varepsilon\|^2_{2,\Omega} \le \sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\varepsilon^n|h^{\varepsilon,in}_\alpha - u^{\varepsilon,in}_\beta|^2 \le C_{11}\varepsilon^2\|\nabla Q_\varepsilon h^\varepsilon\|^2_{2,\Omega}\,, \quad (4.29)$$

$$C_{12}\varepsilon^2\|\nabla Q_\varepsilon u^\varepsilon_0\|^2_{2,\Omega} \le \sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\varepsilon^n|u^{\varepsilon,in}_{0,\alpha} - u^{\varepsilon,in}_{0,\beta}|^2 \le C_{13}\varepsilon^2\|\nabla Q_\varepsilon u^\varepsilon_0\|^2_{2,\Omega}\,. \quad (4.30)$$

Therefore, using (4.26)–(4.30), one can obtain from (4.25)

$$\sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\varepsilon^{-n}[\mathcal{R}^\varepsilon_{\alpha\beta}(t)]^2 \le C_{14}\varepsilon^2$$

$$\times\Big\{\|u^\varepsilon(x,t)\|^2_{2,\Omega}+\|\nabla u^\varepsilon\|^2_{2,\Omega_\varepsilon}+\varepsilon^{3+\gamma}\|\nabla u^\varepsilon\|^2_{2,\mathcal{G}_\varepsilon}$$

$$+\|\nabla u^\varepsilon\|^2_{2,\mathcal{B}_\varepsilon}+\left(\|h^\varepsilon\|^{(1)}_{2,\Omega}\right)^2+\|\nabla Q_\varepsilon h^\varepsilon\|^2_{2,\Omega}\Big\}. \tag{4.31}$$

Using (4.29) with $u^\varepsilon(x,t)$ instead of $h^\varepsilon(x)$, we get

$$\varepsilon^2\int\limits_\Omega|\nabla Q_\varepsilon u^\varepsilon|^2dx\le C_{15}\sum_{\alpha\in N_\varepsilon}\sum_{\beta\in\sigma(\alpha)}\frac{\varepsilon^n}{M^2_\varepsilon}\left\{(y^\varepsilon(t))^2+(w^{in}M_\varepsilon-y^\varepsilon(t))^2\right\}. \tag{4.32}$$

Now the statement of Lemma 4.1 follows from (4.22), (4.24), (4.26)–(4.32), and (2.9), (2.10).

5. Convergence proof

Let $U_\varepsilon(x,t)=P_\varepsilon u^\varepsilon(x,t)$ and $V_\varepsilon(x,t)=Q_\varepsilon u^\varepsilon(x,t)$, where $u^\varepsilon(x,t)$ is the solution of problem (2.1)–(2.3). Then it follows from (4.2)–(4.4) and Lemma 2.1 that the family $\{U_\varepsilon(x,t);V_\varepsilon(x,t)\}$ is a precompact set in the space $L^2(\Omega_T)\times L^2(\Omega_T)$, as $\varepsilon\to0$.

In this section we prove that any cluster value $(u(x,t);v(x,t))\in L^2(\Omega_T)\times L^2(\Omega_T)$ of the family $\{(U_\varepsilon(x,t);V_\varepsilon(x,t)):\varepsilon\to0\}$ is a weak solution of problem (0.3).

At first we rewrite problem (2.1)–(2.3) in a weak form. Let $u^\varepsilon(x,t)$ be the solution of (2.1)–(2.3). This solution satisfies the equalities (see [7])

$$\begin{cases}(u^\varepsilon)^+=(u^\varepsilon)^-, & x\in\partial B_\alpha(r_\varepsilon);\\[4pt] \left(\dfrac{\partial u^\varepsilon}{\partial n}\right)^+=a_\varepsilon\left(\dfrac{\partial u^\varepsilon}{\partial n}\right)^-, & x\in\partial B_\alpha(r_\varepsilon);\end{cases}$$

$$\begin{cases}(u^\varepsilon)^+=(u^\varepsilon)^-, & x\in\partial B_\alpha(r_\varepsilon-d_\varepsilon);\\[4pt] a_\varepsilon\left(\dfrac{\partial u^\varepsilon}{\partial n}\right)^+=\left(\dfrac{\partial u^\varepsilon}{\partial n}\right)^-, & x\in\partial B_\alpha(r_\varepsilon-d_\varepsilon),\end{cases}$$

where $\alpha=1,...,N_\varepsilon$, and we denote by "+" a value of the function $u^\varepsilon(x,t)$ and its normal derivative on the external surface of $\partial B_\alpha(r_\varepsilon)$ (or $\partial B_\alpha(r_\varepsilon-d_\varepsilon)$), and by "–" we denote a value of the function $u^\varepsilon(x,t)$ and its normal derivative on the internal surface of $\partial B_\alpha(r_\varepsilon)$ (or $\partial B_\alpha(r_\varepsilon-d_\varepsilon)$).

Consider (2.8). Let us choose a test function $\phi^\varepsilon(x,t)\in W^{1,1}_2(\Omega_T)$ as follows:

$$\frac{\partial\phi^\varepsilon}{\partial n}(x,t)=0,\ x\in\partial\Omega,\ t\in(0,T);\quad \phi^\varepsilon(x,T)=0,\ x\in\Omega;$$

locally $\phi^\varepsilon(x,t)$ is the function such that $\phi^\varepsilon(x,t) \in W_2^{2,1}(\Omega_\varepsilon \times (0,T))$; $\phi^\varepsilon(x,t) \in W_2^{2,1}(\mathcal{G}_\varepsilon \times (0,T))$; $\phi^\varepsilon(x,t) \in W_2^{2,1}(\mathcal{B}_\varepsilon \times (0,T))$, where $W_2^{2,1}(\Omega_T)$ is the Banach space with the norm

$$\|u\|^{(2)}_{2,\Omega_T} = \sum_{j=0}^{2} \langle\!\langle u \rangle\!\rangle^{(j)}_{2,\Omega_T}, \quad \langle\!\langle u \rangle\!\rangle^{(j)}_{2,\Omega_T} = \sum_{2r+s=j} \|D_t^r D_x^s u\|_{2,\Omega_T};$$

and $\phi^\varepsilon(x,t)$ satisfies the same equalities on $\partial B_\alpha(r_\varepsilon)$ and $\partial B_\alpha(r_\varepsilon - d_\varepsilon)$ as the solution of problem (2.1)–(2.3). Now, integrating by parts in (2.8), we obtain the following equality:

$$J_\varepsilon(u^\varepsilon;\phi^\varepsilon;\Omega_\varepsilon) + J_\varepsilon(u^\varepsilon;\phi^\varepsilon;\mathcal{B}_\varepsilon) + J_\varepsilon(u^\varepsilon;\phi^\varepsilon;\mathcal{G}_\varepsilon) = 0\,, \tag{5.1}$$

where

$$\begin{aligned} J_\varepsilon(u^\varepsilon;\phi^\varepsilon;\Omega_\varepsilon) = &-\int_{\Omega_\varepsilon} u_0^\varepsilon(x)\phi^\varepsilon(x,0)dx - \int_0^T\!\!\int_{\Omega_\varepsilon} u^\varepsilon(x,t)\frac{\partial\phi^\varepsilon}{\partial t}(x,t)dxdt \\ &-\int_0^T\!\!\int_{\Omega_\varepsilon} u^\varepsilon(x,t)\Delta\phi^\varepsilon(x,t)dxdt + \int_0^T\!\!\int_{\Omega_\varepsilon} f(u^\varepsilon(x,t))\phi^\varepsilon(x,t)dxdt \\ &-\int_0^T\!\!\int_{\Omega_\varepsilon} h^\varepsilon(x)\phi^\varepsilon(x,t)dxdt\,, \end{aligned} \tag{5.2}$$

$$\begin{aligned} J_\varepsilon(u^\varepsilon;\phi^\varepsilon;\mathcal{G}_\varepsilon) = &-\int_{\mathcal{G}_\varepsilon} u_0^\varepsilon(x)\phi^\varepsilon(x,0)dx - \int_0^T\!\!\int_{\mathcal{G}_\varepsilon} u^\varepsilon(x,t)\frac{\partial\phi^\varepsilon}{\partial t}(x,t)dxdt \\ &-a_\varepsilon\int_0^T\!\!\int_{\mathcal{G}_\varepsilon} u^\varepsilon(x,t)\Delta\phi^\varepsilon(x,t)dxdt + \int_0^T\!\!\int_{\mathcal{G}_\varepsilon} f(u^\varepsilon(x,t))\phi^\varepsilon(x,t)dxdt \\ &-\int_0^T\!\!\int_{\mathcal{G}_\varepsilon} h^\varepsilon(x)\phi^\varepsilon(x,t)dxdt\,, \end{aligned} \tag{5.3}$$

$$\begin{aligned} J_\varepsilon(u^\varepsilon;\phi^\varepsilon;\mathcal{B}_\varepsilon) = &-\int_{\mathcal{B}_\varepsilon} u_0^\varepsilon(x)\phi^\varepsilon(x,0)dx - \int_0^T\!\!\int_{\mathcal{B}_\varepsilon} u^\varepsilon(x,t)\frac{\partial\phi^\varepsilon}{\partial t}(x,t)dxdt \\ &-\int_0^T\!\!\int_{\mathcal{B}_\varepsilon} u^\varepsilon(x,t)\Delta\phi^\varepsilon(x,t)dxdt + \int_0^T\!\!\int_{\mathcal{B}_\varepsilon} f(u^\varepsilon(x,t))\phi^\varepsilon(x,t)dxdt \end{aligned}$$

$$-\int_0^T \int_{\mathcal{B}_\varepsilon} h^\varepsilon(x)\phi^\varepsilon(x,t)dxdt. \tag{5.4}$$

Let us cover the domain Ω by the cubes $K_\alpha(\varepsilon+\kappa)$ with the centers $x^\alpha \in \Omega$ and edges of length $\varepsilon+\kappa$, $\kappa = o(\varepsilon)$, $\kappa > 0$, oriented along the coordinate axes. We assume that each ball $B_\alpha(r_\varepsilon)$ is located in $K_\alpha(\varepsilon+\kappa)$ and has the same center. Let us associate to this covering a partition of unity $\{\psi_\alpha(x)\} : 0 \le \psi_\alpha(x) \le 1$; $\psi_\alpha(x) = 0$ for $x \notin K_\alpha(\varepsilon+\kappa)$; $\psi_\alpha(x) = 1$ for $x \in K_\alpha(\varepsilon+\kappa) \setminus \bigcup_{\beta\neq\alpha} K_\beta(\varepsilon+\kappa)$; $\sum_\alpha \psi_\alpha(x) = 1$ for $x \in \Omega$; $|\nabla\psi_\alpha(x)| \le A\kappa^{-1}$.

Let us introduce the functions

$$v_{i\varepsilon}(x) = r_\varepsilon v_i(\frac{x}{r_\varepsilon}), \quad w_{i\varepsilon}(x) = x_i - x_i^\alpha - v_{i\varepsilon}(x),$$

where $v_i(x)$ is the solution of problem (3.1). We denote by $\tilde{w}_{i\varepsilon}(x)$ a continuation of $w_{i\varepsilon}(x)$ on $\Omega \setminus \Omega_\varepsilon$. Let $\omega(x) = \bar{\omega}(|x|)$ be a function from $C_0^\infty(\mathbf{R}^n)$ such that $0 \le \omega(x) \le 1$; $\bar{\omega}(\rho) = 1$ for $\rho \le r$; $\bar{\omega}(\rho) = 0$ for $\rho \ge 2r$.

Now we choose $\phi^\varepsilon(x,t) = b(t)z_\varepsilon(x)$, where $b(t) \in C^1(0,T)$, $b(T) = 0$, and $z_\varepsilon(x)$ is a function constructed as follows:

$$\begin{aligned} z_\varepsilon(x) = \sum_\alpha \Bigg\{ \zeta(x^\alpha) + \varphi_\varepsilon(x) \Bigg[\sum_{i=1}^n \frac{\partial\zeta}{\partial x_i}(x^\alpha)\tilde{w}_{i\varepsilon}(x) \\ + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2\zeta}{\partial x_i \partial x_j}(x^\alpha)\tilde{w}_{i\varepsilon}(x)\tilde{w}_{j\varepsilon}(x) \Bigg] \Bigg\} \psi_\alpha(x) \\ + \sum_\alpha (\eta(x^\alpha) - \zeta(x^\alpha))v^\varepsilon(x-x^\alpha)\omega\left(\frac{x-x^\alpha}{\varepsilon}\right). \end{aligned} \tag{5.5}$$

Here, in (5.5), $\zeta(x)$ and $\eta(x)$ are chosen smooth enough in Ω, the function $v^\varepsilon(x)$ is defined by (4.6), $\varphi_\varepsilon(x) \in C^2(\Omega)$ is a function such that

$$\varphi_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_\varepsilon; \\ 0, & x \in \mathcal{B}_\varepsilon, \end{cases}$$

and $|D^k\varphi_\varepsilon| < Ad_\varepsilon^{-k}$ $(k = 1,2)$.

Now let us consider the properties of the function $z_\varepsilon(x)$.

Lemma 5.1. *The function $z_\varepsilon(x)$ is such that*

$$\int_\Omega |\Delta z_\varepsilon(x)|^2 dx \le C, \tag{5.6}$$

where C is a constant independent of ε.

P r o o f o f L e m m a 5.1. Let us represent the integral in the left hand side of (5.6) in the form

$$\int\limits_{\Omega} |\Delta z_\varepsilon(x)|^2 dx = \sum_{\alpha\beta} \int\limits_{K_\alpha(\varepsilon+\kappa)\bigcap K_\beta(\varepsilon+\kappa)\bigcap\Omega} |\Delta z_\varepsilon(x)|^2 dx$$

$$+\sum_{\alpha} \int\limits_{K_\alpha(\varepsilon-\kappa)\bigcap\Omega} |\Delta z_\varepsilon(x)|^2 dx. \tag{5.7}$$

First we obtain the estimate for the first term in (5.7). It follows from the definition of the function $\omega(x)$, that the function $z_\varepsilon(x)$ on $\bigcup_{\alpha,\beta} K_\alpha(\varepsilon+\kappa)\bigcap K_\beta(\varepsilon+\kappa)$ has a form

$$z_\varepsilon(x) = \sum_\alpha \Bigg\{ \zeta^\alpha + \sum_{i=1}^n \zeta^\alpha_{,i}(x_i - x_i^\alpha - v_{i\varepsilon}(x))$$

$$+\frac{1}{2}\sum_{i,j=1}^n \zeta^\alpha_{,ij}(x_i - x_i^\alpha - v_{i\varepsilon}(x))(x_j - x_j^\alpha - v_{j\varepsilon}(x))\Bigg\}\psi_\alpha(x)\,, \tag{5.8}$$

where

$$\zeta^\alpha = \zeta(x^\alpha), \quad \zeta^\alpha_{,i} = \frac{\partial\zeta}{\partial x_i}(x^\alpha), \quad \zeta^\alpha_{,ij} = \frac{\partial^2\zeta}{\partial x_i\partial x_j}(x^\alpha).$$

Taking $\kappa = \varepsilon^{1+\theta}$ for any $0 < \theta < 2/3$, one obtain that for any set $K_\alpha(\varepsilon+\kappa)\bigcap K_\beta(\varepsilon+\kappa)$

$$\int\limits_{K_\alpha(\varepsilon+\kappa)\bigcap K_\beta(\varepsilon+\kappa)} |\Delta z_\varepsilon(x)|^2 dx = o(\varepsilon^n). \tag{5.9}$$

Therefore the first term in the right hand side of (5.7) tends to zero as $\varepsilon \to 0$.

Now let us estimate the second term in (5.7). Consider the set $K_\alpha(\varepsilon-\kappa)\setminus B_\alpha(2r_\varepsilon)$. On this set we have for sufficiently small ε that $\bar\omega(\rho)\equiv 0$. Therefore

$$z_\varepsilon(x) = \zeta^\alpha + \sum_{i=1}^n \zeta^\alpha_{,i}\cdot(x_i - x_i^\alpha - v_{i\varepsilon}(x)) + \frac{1}{2}\sum_{i,j=1}^n \zeta^\alpha_{,ij}\cdot(x_i - x_i^\alpha - v_{i\varepsilon}(x))(x_j - x_j^\alpha - v_{j\varepsilon}(x)),$$

and

$$\Delta z_\varepsilon(x) = \sum_{i,j=1}^n \zeta^\alpha_{,ij}\left\{\sum_{k=1}^n \frac{\partial}{\partial x_k}(x_i - x_i^\alpha - v_{i\varepsilon}(x))\frac{\partial}{\partial x_k}(x_j - x_j^\alpha - v_{j\varepsilon}(x))\right\}.$$

From this equality we obtain

$$\int\limits_{K_\alpha(\varepsilon-\kappa)\setminus B_\alpha(2r_\varepsilon)} |\Delta z_\varepsilon(x)|^2 dx = O(\varepsilon^n). \tag{5.10}$$

Consider the annulus $\Pi_\varepsilon^\alpha = B_\alpha(2r_\varepsilon) \setminus \bar{B}_\alpha(r_\varepsilon)$. On this subset $z_\varepsilon(x)$ has the form

$$z_\varepsilon(x) = \left\{ \zeta^\alpha + \sum_{i=1}^n \zeta_{,i}^\alpha w_{i\varepsilon}(x) + \frac{1}{2} \sum_{i,j=1}^n \zeta_{,ij}^\alpha w_{i\varepsilon}(x) w_{j\varepsilon}(x) \right\}$$
$$+ \left\{ (\eta^\alpha - \zeta^\alpha) v^\varepsilon(x - x^\alpha) \omega \left(\frac{x - x^\alpha}{\varepsilon} \right) \right\} = p_1^\varepsilon(x) + p_2^\varepsilon(x) \,.$$

First we obtain appropriate estimate for $p_1^\varepsilon(x)$. We get

$$p_1^\varepsilon(x) = \zeta^\alpha + \sum_{i=1}^n \zeta_{,i}^\alpha (x_i - x_i^\alpha - v_{i\varepsilon}(x)) + \frac{1}{2} \sum_{i,j=1}^n \zeta_{,ij}^\alpha \cdot (x_i - x_i^\alpha - v_{i\varepsilon}(x))(x_j - x_j^\alpha - v_{j\varepsilon}(x)),$$

and

$$\Delta p_1^\varepsilon(x) = \sum_{i,j=1}^n \zeta_{,ij}^\alpha \left\{ \sum_{k=1}^n \frac{\partial}{\partial x_k}(x_i - x_i^\alpha - v_{i\varepsilon}(x)) \frac{\partial}{\partial x_k}(x_j - x_j^\alpha - v_{j\varepsilon}(x)) \right\}.$$

Therefore

$$\int_{\Pi_\varepsilon^\alpha} |\Delta p_1^\varepsilon(x)|^2 dx = O(\varepsilon^n). \tag{5.11}$$

Consider $p_2^\varepsilon(x)$. We find:

$$\Delta p_2^\varepsilon = (\eta^\alpha - \zeta^\alpha)[\Delta v^\varepsilon \omega + v^\varepsilon \Delta \omega].$$

It follows from the definition of the function v^ε (4.6)–(4.10) and the definition of the function $\omega(x)$ that

$$\int_{\Pi_\varepsilon^\alpha} |\Delta p_2^\varepsilon(x)|^2 dx = O(\varepsilon^n). \tag{5.12}$$

On the set G_ε^α according to the definitions of $\varphi_\varepsilon(x)$ and v^ε for $0 \le \gamma < 1$ we get

$$\int_{G_\varepsilon^\alpha} |\Delta z_\varepsilon(x)|^2 dx = o(\varepsilon^n). \tag{5.13}$$

Let now consider the ball $B_\alpha(r_\varepsilon - d_\varepsilon)$. We have

$$\Delta z^\varepsilon = (\eta^\alpha - \zeta^\alpha) \cdot \Delta v^\varepsilon = 2n\lambda_2^\varepsilon(\eta^\alpha - \zeta^\alpha).$$

Therefore

$$\int_{B_\alpha(r_\varepsilon - d_\varepsilon)} |\Delta z_\varepsilon(x)|^2 dx = O(\varepsilon^n). \tag{5.14}$$

The statement of the Lemma follows now from (5.9)–(5.14).

Let us introduce the following functions:

$$\tilde{z}_\varepsilon(x) = \begin{cases} z_\varepsilon(x), & x \in \Omega_\varepsilon; \\ 0, & x \in \Omega \setminus \Omega_\varepsilon; \end{cases}$$

and

$$\tilde{Z}_\varepsilon(x) = \begin{cases} \Delta z_\varepsilon(x), & x \in \Omega_\varepsilon; \\ 0, & x \in \Omega \setminus \Omega_\varepsilon. \end{cases}$$

Lemma 5.2. *The functions $\tilde{z}_\varepsilon(x)$ and $\tilde{Z}_\varepsilon(x)$ have the convergence properties:*

$$\tilde{z}_\varepsilon(x) \to (1-\mu)\zeta(x) \quad \text{weakly in} \quad L^2(\Omega), \tag{5.15}$$

and

$$\tilde{Z}_\varepsilon(x) \to (1-\mu)\sum_{i,j=1}^{n} b_{ij}\frac{\partial^2\zeta}{\partial x_i \partial x_j}(x) + l_n\mu(\eta(x)-\zeta(x)) \quad \text{weakly in} \quad L^2(\Omega), \ (5.16),$$

as $\varepsilon \to 0$. Here above

$$b_{ij} = \delta_{ij}\left[1 - \frac{r^n}{1-\mu}\int\limits_P (\nabla v_i, \nabla v_j)dx\right], \quad \mu = \frac{\pi^{n/2} r^n}{\Gamma\left(\frac{n}{2}+1\right)}, \quad l_n = \frac{an}{rd},$$

and $\zeta(x)$, $\eta(x)$ are defined in (5.5).

P r o o f o f L e m m a 5.2. The result (5.15) comes directly from (5.5) and the definition of the function $v^\varepsilon(x)$.

Now we shall obtain the result (5.16). It follows from Lemma 5.1 that

$$\int\limits_\Omega |\tilde{Z}_\varepsilon(x)|^2 dx \le C_1,$$

for a constant C_1 independent of ε. Therefore the family $\{\tilde{Z}_\varepsilon(x)\}$ is bounded in $L^2(\Omega)$, and we have weak convergence for any smooth function $\theta(x) \in C_0^\infty(\Omega)$ of a subsequence $\{\tilde{Z}_\varepsilon\theta\}$.

Let us consider the integral

$$\int\limits_\Omega \tilde{Z}_\varepsilon(x)\theta(x)dx = \int\limits_{\Omega_\varepsilon} \Delta z_\varepsilon(x)\theta(x)dx$$

$$= \sum_{\alpha} \Bigg\{ \int\limits_{K_\alpha(\varepsilon)\setminus B_\alpha(r_\varepsilon)} \theta(x) \sum_{i,j=1}^{n} \zeta_{,ij}^{\alpha} \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(x_i - x_i^\alpha - v_{i\varepsilon}(x)) \frac{\partial}{\partial x_k}(x_j - x_j^\alpha - v_{j\varepsilon}(x)) dx$$

$$+ \int\limits_{\Pi_\varepsilon^\alpha} (\eta^\alpha - \zeta^\alpha) \Delta \left[v^\varepsilon(x - x^\alpha) \omega \left(\frac{x - x^\alpha}{\varepsilon} \right) \right] \theta(x) dx \Bigg\} + o(1), \tag{5.17}$$

as $\varepsilon \to 0$. Consider the first term in (5.17). We have

$$\int\limits_{K_\alpha(\varepsilon)\setminus B_\alpha(r_\varepsilon)} \theta(x) \sum_{i,j=1}^{n} \zeta_{,ij}^{\alpha} \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(x_i - x_i^\alpha - v_{i\varepsilon}(x)) \frac{\partial}{\partial x_k}(x_j - x_j^\alpha - v_{j\varepsilon}(x)) dx$$

$$= \varepsilon^n (1 - \mu) \theta(x^\alpha) \sum_{i,j=1}^{n} b_{ij} \zeta_{,ij}^{\alpha} + o(\varepsilon^n) \, . \tag{5.18}$$

For the second term in (5.17) we find:

$$\int\limits_{\Pi_\varepsilon^\alpha} (\eta^\alpha - \zeta^\alpha) \Delta \left[v^\varepsilon(x - x^\alpha) \omega \left(\frac{x - x^\alpha}{\varepsilon} \right) \right] \theta(x) dx$$

$$= -\theta(x^\alpha)(\eta^\alpha - \zeta^\alpha) \int\limits_{\partial B(r_\varepsilon)} \frac{\partial v^\varepsilon}{\partial \rho} \cdot \omega \left(\frac{x}{\varepsilon} \right) d\sigma + o(\varepsilon^n)$$

$$= l_n \mu \varepsilon^n \theta(x^\alpha)(\eta^\alpha - \zeta^\alpha) + o(\varepsilon^n) \, . \tag{5.19}$$

Now the assertion of the Lemma follows from (5.18), (5.19).

If $(u; v)$ is the cluster value in $L^2(\Omega_T) \times L^2(\Omega_T)$ of the family $\{U_\varepsilon(x,t), V_\varepsilon(x,t)\}$ as $\varepsilon \to 0$, then there exist a subsequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, such that

$$\|u(x,t) - U_{\varepsilon_k}(x,t)\|_{2,\Omega_T} + \|v(x,t) - V_{\varepsilon_k}(x,t)\|_{2,\Omega_T} \to 0 \, , \tag{5.20}$$

as $k \to \infty$. Therefore Lemmas 5.1 and 5.2 imply that

$$\lim_{k \to \infty} J_{\varepsilon_k}(u^{\varepsilon_k}; z_{\varepsilon_k}(x) b(t); \Omega_{\varepsilon_k}) = (1 - \mu) J_1(u; \eta, \zeta) \, , \tag{5.21}$$

where

$$J_1(u; \eta, \zeta) = -\int\limits_{\Omega} u_0(x) b(0) \zeta(x) dx - \int\limits_0^T \int\limits_{\Omega} u(x,t) \zeta(x) b'(t) dx dt$$

$$- \int\limits_0^T \int\limits_{\Omega} u(x,t) \left\{ \sum_{ij=1}^{n} b_{ij} \frac{\partial^2 \zeta}{\partial x_i \partial x_j}(x) + \frac{l_n \mu}{(1-\mu)} (\eta(x) - \zeta(x)) \right\} b(t) dx dt$$

$$+\int_0^T\int_\Omega (f(u(x,t)) - h_1(x))\zeta(x)b(t)dt\,,$$

and

$$\lim_{k\to\infty} J_{\varepsilon_k}(u^{\varepsilon_k}; z_{\varepsilon_k}(x)b(t); \mathcal{G}_{\varepsilon_k}) = 0\,. \tag{5.22}$$

Now we study the asymptotical behaviour of the quantity $J_{\varepsilon_k}(u^{\varepsilon_k}; z_{\varepsilon_k}(x)b(t); \mathcal{B}_{\varepsilon_k})$.

Lemma 5.3. *Let*

$$q_\varepsilon = \sum_{\alpha\in N_\varepsilon} \int_{B_\alpha(r_\varepsilon - d_\varepsilon)} f(u^\varepsilon(x,t))z_\varepsilon(x)dx.$$

Then we have

$$\left| q_\varepsilon - \sum_{\alpha\in N_\varepsilon} f(u_\alpha^{\varepsilon,in}(t))\eta(x^\alpha)\mathrm{Vol}(B_\alpha(r_\varepsilon - d_\varepsilon))\right| \le C\varepsilon. \tag{5.23}$$

Proof of Lemma 5.3. The Poincaré inequality, the structure of $z_\varepsilon(x)$ in $B_\alpha(r_\varepsilon - d_\varepsilon)$, and (2.7) give

$$\left| \int_{B_\alpha(r_\varepsilon - d_\varepsilon)} f(u^\varepsilon)z_\varepsilon(x)dx - f(u_\alpha^{\varepsilon,in}(t)) \int_{B_\alpha(r_\varepsilon - d_\varepsilon)} z_\varepsilon(x)dx \right|$$

$$\le C_1\varepsilon^{(n/2+1)}\|\nabla u^\varepsilon\|_{2,B_\alpha(r_\varepsilon - d_\varepsilon)}\,.$$

In the ball $B_\alpha(r_\varepsilon - d_\varepsilon)$ the function $z_\varepsilon(x)$ has the form

$$z_\varepsilon(x) = \zeta^\alpha + (\eta^\alpha - \zeta^\alpha)\cdot v^\varepsilon(x - x^\alpha) = \eta^\alpha + O(\varepsilon^2)\,.$$

Therefore

$$q_\varepsilon = \sum_{\alpha\in N_\varepsilon} \int_{B_\alpha(r_\varepsilon - d_\varepsilon)} f(u^\varepsilon(x,t))z_\varepsilon(x)dx$$

$$= \sum_{\alpha\in N_\varepsilon} f(u_\alpha^{\varepsilon,in}(t))\eta^\alpha\mathrm{Vol}(B_\alpha(r_\varepsilon - d_\varepsilon)) + O(\varepsilon)\,.$$

Now Lemma 5.3 immediately follows from this equality.

In a very similar way we conclude that

$$\left| \sum_{\alpha\in N_\varepsilon} \left\{ \int_{B_\alpha(r_\varepsilon - d_\varepsilon)} u^\varepsilon(x,t)z_\varepsilon(x)dx - u_\alpha^{\varepsilon,in}(t)\eta^\alpha\mathrm{Vol}(B_\alpha(r_\varepsilon - d_\varepsilon)) \right\} \right| \le C_2\varepsilon\,, \tag{5.24}$$

for all $t \in [0,T]$, and

$$\left| \sum_{\alpha \in N_\varepsilon} \left\{ \int\limits_{B_\alpha(r_\varepsilon - d_\varepsilon)} h^\varepsilon(x) z_\varepsilon(x) dx - h^\varepsilon(x^\alpha)\eta^\alpha \mathrm{Vol}(B_\alpha(r_\varepsilon - d_\varepsilon)) \right\} \right| \le C_3 \varepsilon \,. \quad (5.25)$$

Using the definition of $v^\varepsilon(x)$, we have

$$\Delta z_\varepsilon = 2n\lambda_2^\varepsilon(\eta^\alpha - \zeta^\alpha), \quad x \in B_\alpha(r_\varepsilon - d_\varepsilon)\,.$$

Therefore

$$\sum_{\alpha \in N_\varepsilon} \int\limits_{B_\alpha(r_\varepsilon - d_\varepsilon)} u^\varepsilon(x,t)\Delta z_\varepsilon(x) dx = l_n \mu \varepsilon^n \sum_{\alpha \in N_\varepsilon} u_\alpha^{\varepsilon,in}(t)(\zeta^\alpha - \eta^\alpha) + O(\varepsilon)\,. \quad (5.26)$$

It follows from (5.23)–(5.26) that

$$\lim_{k\to\infty} J_{\varepsilon_k}(u^{\varepsilon_k}(x,t); b(t) z_{\varepsilon_k}(x); \mathcal{B}_\varepsilon) = \mu J_2(v;\eta,\zeta)\,, \quad (5.27)$$

where

$$J_2(v;\eta,\zeta) = -\int\limits_\Omega v_0(x) b(0)\eta(x) dx - \int\limits_0^T \int\limits_\Omega v(x,t)\eta(x) b'(t) dx dt$$

$$-l_n \int\limits_0^T \int\limits_\Omega v(x,t)(\zeta(x) - \eta(x)) b(t) dx dt + \int\limits_0^T \int\limits_\Omega (f(v(x,t)) - h_2(x))\eta(x) b(t) dx dt\,.$$

Thus, it follows from (5.1), (5.21), (5.22), and (5.27) that

$$(1-\mu)J_1(u;\eta,\zeta) + \mu J_2(v;\eta,\zeta) = 0 \quad (5.28)$$

for any cluster value $(u(x,t); v(x,t)) \in L^2(\Omega_T) \times L^2(\Omega_T)$ of the family $\{(U_\varepsilon(x,t); V_\varepsilon(x,t)) : \varepsilon \to 0\}$. Here $J_1(u;\eta,\zeta)$ and $J_2(v;\eta,\zeta)$ are defined by the equalities, where $b(t) \in C^1(0,T)$, $b(T) = 0$, and $\eta(x)$, $\zeta(x)$ are any smooth functions on Ω. Therefore according to (5.28) $(u(x,t), v(x,t))$ is a weak solution of problem (0.3).

From the consideration presented above it also follows the existence theorem for solutions from $C(0,T; L^2(\Omega_T) \times L^2(\Omega_T))$ of problem (0.3) under certain conditions concerning to the functions u_0, v_0, h_1 and h_2 (see the assumption (ii) of Theorem 3.1).

In order to complete the proof of Theorem 3.1 we only need to prove the uniqueness theorem for system (0.3). It can be done by the same way as in the paper [4]. Theorem 3.1 is proved.

R e m a r k 5.1 Above we considered the case $n \ge 3$. For the case $n = 2$ the considerations should be repeated word by word with slight modification in estimates.

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Усреднение нелинейных параболических уравнений с асимптотически вырождающимися коэффициентами

Л. Панкратов

Рассматривается начально–краевая задача для нелинейного параболического уравнения вида

$$\frac{\partial u^\varepsilon}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), \quad x \in \Omega, \ t \in (0,T),$$

коэффициенты $a_{ij}^\varepsilon(x)$ которого зависят от малого параметра ε, так что $a_{ij}^\varepsilon(x)$ имеют порядок $\varepsilon^{3+\gamma}$ $(0 \leq \gamma < 1)$ на множестве сферических колец G_ε^α толщины $d_\varepsilon = d\varepsilon^{2+\gamma}$. Кольца периодически, с периодом ε, распределены в области Ω. На множестве $\Omega \backslash U_\alpha G_\varepsilon^\alpha$ эти коэффициенты равны постоянной величине. Изучается асимптотическое поведение решений $u^\varepsilon(x,t)$ этой задачи при $\varepsilon \to 0$. Показано, что асимптотическое поведение решений описывается системой нелинейных уравнений, которая состоит из параболического уравнения в частных производных и связанного с ним обыкновенного дифференциального уравнения.

Усереднення нелінійних параболічних рівнянь з коефіцієнтами, що асимптотично вироджуються

Л. Панкратов

Розглядається початково-крайова задача для нелінійного параболічного рівняння

$$\frac{\partial u^\varepsilon}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), \quad x \in \Omega, \ t \in (0,T),$$

коефіцієнти $a_{ij}^\varepsilon(x)$ якого залежать від малого параметру ε, так що $a_{ij}^\varepsilon(x)$ мають порядок $\varepsilon^{3+\gamma}$ $(0 \leq \gamma < 1)$ на множині сферичних кілець G_ε^α товщини $d_\varepsilon = d\varepsilon^{2+\gamma}$. Кільця періодично, з періодом ε, розподілені в області Ω. На множині $\Omega \backslash U_\alpha G_\varepsilon^\alpha$ ці коефіцієнти дорівнюють сталій величині. Вивчається асимптотична поведінка розв'язків $u^\varepsilon(x,t)$ цієї задачі при $\varepsilon \to 0$. Доведено, що асимптотична поведінка розв'язків описується системою нелінійних рівнянь, що складається з нелінійного параболічного рівняння у частинних похідних та звичайного диференціального рівняння.