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An example of isometric immersion of a domain of 3-dimensional Lobachevsky space into E^6 with a section as the Veronese surface

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Some example of isometric immersion of a domain of the Lobachevsky space L^3 into E^6 is constructed in such a way that every intersection of the obtained submanifold with coordinate hyperplane $x^6 = const$ be the Veronese surface. The submanifold is not orientable and admits a 2-parametric family of motions along itself. It is also proved general statements on existence of immersions of some domain of L^3 into E^k, $k > 5$, in the form of special submanifolds.

In the article [1] toroidal submanifolds were introdused, in particular isometric immersions of the Lobachevsky space L^n into E^{2n-1} as toroidal submanifolds were considered in [1, 2]. The intersection such submanifold with a coordinate space is toroid. In the following work we consider isometric immersions of a domain of L^3 into E^6 with the property that every intersection of a submanifold with the coordinate hyperplane $x_6 = const$ be the Veronese surface. More precisely, if $x_1, ..., x_6$ be cartesian coordinates in E^6, then a position vector r of a submanifold $F^3 \subset E^6$ has the following form:

$$x_1 = f_1 \frac{uv}{\sqrt{3}},\ x_2 = f_1 \frac{uw}{\sqrt{3}},\ x_3 = f_1 \frac{vw}{\sqrt{3}},$$

$$x_4 = f_1 \frac{u^2 - v^2}{2\sqrt{3}},\ x_5 = f_1 \frac{u^2 + v^2 - 2w^2}{6},\ x_6 = f_2,$$

where $f_i = f_i(t)$ and $u^2 + v^2 + w^2 = 3$.

The condition $x_6 = const$ gives us $t = const$, and therefore the intersection F^2 of the submanifold F^3 with hyperplane $x_6 = const$ is some Veronese surface. We call F^2 standard Veronese surface which lies in the sphere $S^4 \subset E^5$ with unit radius.

Theorem 1. *There exists 1-parametric family of immersions of domains on L^3 into E^6 with sections by hyperplanes in the form of the Veronese surfaces.*

Remark that the submanifold F^3 is not orientable, it admits a 2-parametric family of translations by itself. Let $e_1, ..., e_{n+2}$ is orthonormal basis in E^{n+2}. Theorem 1 follows from the Theorem 2.

Theorem 2. *Let 2-dimensional metric ds_0^2 with constant curvature K_0 has isometric immersion into the unit sphere $S^n \subset E^{n+1}$ with the position vector $\rho = \rho(\alpha, \beta)$. Then the Lobachevsky space L^3 has isometric immersion into E^{n+2} of the following form:*

$$r(\alpha, \beta, t) = f_1(t)\rho(\alpha, \beta) + f_2(t)e_{n+2}, \tag{1}$$

iff the curvature $K_0 < 1$.

We remark that the standard Veronese surface has curvature $K_0 = \frac{1}{3}$ and lies in S^4. Later the surface F^2 we call generating. Its metric ds_0^2 can be writing in the following form:

$$ds_0^2 = R^2[(d\alpha)^2 + G^2(\alpha)(d\beta)^2]\,, \tag{2}$$

where $R^2 = \frac{1}{K_0}, G(\alpha) = \cos(\alpha)$, if $K_0 > 0$; $R^2 = 1$, $G(\alpha) = 1$, if $K_0 = 0$; and $R^2 = -\frac{1}{K_0}, G(\alpha) = e^{\alpha}$, if $K_0 < 0$. For Veronese surface $R^2 = 3$.

We put $\alpha = u^1, \beta = u^2, t = u^3$ and let us find the coefficients of the first quadratic form $ds^2 = g_{ij}du^i du^j$ of F^3. We have

$$r_\alpha = f_1\rho_\alpha, \quad r_\beta = f_1\rho_\beta, \quad r_t = \rho f_1' + e_6 f_2'\,.$$

As F^2 lies in the unit sphere S^4 with the center at the origin of coordinate system, so

$$\rho^2 = 1, \quad (\rho\rho_\alpha) = (\rho\rho_\beta) = 0\,,$$
$$\rho_\alpha^2 = R^2, \quad \rho_\beta^2 = R^2G^2\,.$$

Hence

$$g_{11} = R^2f_1^2,\ g_{22} = R^2f_1^2G^2,\ g_{33} = f_1'^2 + f_2'^2\,,$$
$$g_{12} = g_{13} = g_{23} = 0\,.$$

We remark here that the point with $f_1 = 0$ is singular on the submanifold F^3. We put that the parameter t be the length of arc of the curve $f_1(t), f_2(t)$ on the plane f_1, f_2. Then $g_{33} = 1$. We can use the expressions of Riemannian tensor components of ds^2 from [3]:

$$R_{ijij} = -H_i H_j \left[\frac{\partial}{\partial u_i} \left(\frac{\partial H_j}{H_i \partial u_i} \right) + \frac{\partial}{\partial u_j} \left(\frac{\partial H_i}{H_j \partial u_j} \right) + \frac{1}{H_k^2} \frac{\partial H_i}{\partial u_k} \frac{\partial H_j}{\partial u_k} \right], \tag{3}$$

where $H_i^2 = g_{ii}$ and i, j, k are distinct;

$$R_{ijkj} = -H_j \left[\frac{\partial^2 H_j}{\partial u_i \partial u_k} - \frac{\partial H_j}{\partial u_i} \frac{\partial H_k}{H_k \partial u_k} - \frac{\partial H_j}{\partial u_k} \frac{\partial H_i}{H_i \partial u_i} \right], \tag{4}$$

where $i \neq k \neq j$. There are six essential components of the Riemannian tensor which be the components of the following symmetric curvature matrix:

$$\begin{pmatrix} R_{2323} & R_{1323} & R_{1232} \\ * & R_{1313} & R_{2131} \\ * & * & R_{1212} \end{pmatrix}. \tag{5}$$

By supposition F^3 is the manifold with the curvature equal to -1, so the Gauss equations give us

$$R_{ijkl} = -(g_{ik} g_{jl} - g_{il} g_{jk}). \tag{6}$$

Let us put $i = k = 2$, $j = l = 3$. From (3) and (6) we obtain

$$g_{22} g_{33} = H_2 H_3 \left[\frac{\partial}{\partial u_2} \left(\frac{\partial H_3}{H_2 \partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{\partial H_2}{H_3 \partial u_3} \right) + \frac{1}{H_1^2} \frac{\partial H_2}{\partial u_1} \frac{\partial H_3}{\partial u_1} \right].$$

Substitution of the expression (2) of g_{ij} gives us

$$f_1'' = f_1. \tag{7}$$

The equation (6) with component R_{1313} gives us the equation (7) again. Let us put $i = k = 1$, $j = l = 2$. From (3) and (6) we have

$$g_{11} g_{22} = H_1 H_2 \left[\frac{\partial}{\partial u_1} \left(\frac{\partial H_2}{H_1 \partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{\partial H_1}{H_2 \partial u_2} \right) + \frac{1}{H_3^2} \frac{\partial H_1}{\partial u_3} \frac{\partial H_2}{\partial u_3} \right].$$

Taking into attention the equation $G_{\alpha\alpha} = -K_0 R^2 G$, after simple calculation we obtain

$$f_1'^2 = K_0 + f_1^2. \tag{8}$$

Without difficulties we can verify that the equations (6), which correspond to matrix elements of (5) outside of the principal diagonal, do not give us more news. So, the submanifold $F^3 \subset E^3$ will have the sectional curvature equal to -1 iff the equations (7), (8) are fulfiled. From equation (7) we have

$$f_1 = Ae^t + Be^{-t}\,, \tag{9}$$

where A and B are constants. From (8) we obtain one condition on A and B

$$AB = -\frac{K_0}{4}\,. \tag{10}$$

From the equation $f_1'^2 + f_2'^2 = 1$ and (8) we obtain at first: $K_0 < 1$ and then

$$f_2 = \int\limits_0^t \sqrt{1 - f_1'^2(\tau)}\, d\tau\,.$$

Hence the Theorem 2 is proved.

Now let us investigate the question on regularity of submanifold. From the condition $f'^2 < 1$ we have

$$A^2 e^{2t} - 2AB + B^2 e^{-2t} < 1\,.$$

From here we obtain that the function $\lambda = e^{2t}$ satisfies the inequalities $\lambda_1 \leq \lambda \leq \lambda_2$, where

$$\lambda_1 = \frac{2 - K_0 - 2\sqrt{1 - K_0}}{4A^2},\ \lambda_2 = \frac{2 - K_0 + 2\sqrt{1 - K_0}}{4A^2}\,.$$

The numbers λ_1, λ_2 correspond to singular points of F^3. The set of these points gives us two borders. The point for which $f_1 = 0$ is singular also. It exists only for $K_0 > 0$. Let $K_0 > 0$ and λ_0 be the meaning of λ corresponding to this point. We obtain

$$\lambda_0 = \frac{K_0}{4A^2}\,.$$

Without difficulty we have: $\lambda_1 < \lambda_0 < \lambda_2$. So, in the case $K_0 > 0$ submanifold F^3 has the form of a bobbin with one "conic" singular point beetwen borders. If $K_0 \leq 0$, then F^3 has the form of ordinary bobbin.

Let us indicate the length of profil line. If t_i corresponds to λ_i , then l is the length of arc of profil line beetwen two borders is $l = t_2 - t_1$. We obtain

$$l = \frac{1}{2} \ln \frac{2 - K_0 + 2\sqrt{1 - K_0}}{2 - K_0 - 2\sqrt{1 - K_0}}\,.$$

This expression does not depend on A.

It is well-known fact that for every isometric immersion of a domain of L^n into E^{2n-1} at every point of immersed domain there are n principil directions and 2^{n-1} asymptotic one. Let us consider the question on its existence when the the generating surface F^2 is the standard Veronese surface. For this we find the second quadratic forms of F^3. Let n_1, n_2 be unit vector fields of normals of the Veronese surface F^2 tangent to the sphere S^4. Because

$$(r_\alpha n_i) = f_1(\rho_\alpha n_i) = 0,\ (r_\beta n_i) = f_1(\rho_\beta n_i) = 0\,,$$
$$(r_t n_t) = f'(\rho n_i) = 0\,, \tag{12}$$

so the vectors n_i, $i = 1, 2$, be normals to F^3 too. Without difficulties it is possible to verify that the vector field $n_3 = \rho f_2' - e_6 f_1'$ is the third normal unit vector field. We have expressions for the second derivatives of the position vector r:

$$\begin{aligned} r_{\alpha\alpha} &= f_1\rho_{\alpha\alpha},\ r_{\alpha t} = f_1'\rho_\alpha\,,\\ r_{\alpha\beta} &= f_1\rho_{\alpha\beta},\ r_{\beta t} = f_1'\rho_\beta\,,\\ r_{\beta\beta} &= f_1\rho_{\beta\beta},\ r_{tt} = f_1''\rho + e_6 f_2''\,. \end{aligned} \tag{13}$$

Let L^σ_{ij} be the coefficients of the second quadratic form of F^3 with respect to n_σ and l^ν_{ij} be the next one for F^2 with respect to n_ν, $\nu = 1, 2$. Taking into attention (12) and (13), we find the matrices of the second fundamental forms II^1 and II^2 in the following form:

$$\| L^\nu_{ij} \| = \begin{pmatrix} f_1 l^\nu_{11} & f_1 l^\nu_{12} & 0 \\ f_1 l^\nu_{12} & f_1 l^\nu_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\| L^3_{ij} \|$ we can calculate in more concrete form. We have

$$\begin{aligned} L^3_{11} &= (r_{\alpha\alpha} n_3) = f_1(\rho_\alpha, \rho f_2' - e_6 f_1') = -3 f_1 f_2'\,,\\ L^3_{22} &= (r_{\beta\beta} n_3) = f_1(\rho_{\beta\beta}, \rho f_2' - e_6 f_1') = -3 f_1 f_2' \cos^2\alpha\,,\\ L^3_{33} &= (r_{tt} n_3) = (f_1''\rho + e_6 f_2'', \rho f_2' - e_6 f_1') = f_1'' f_2' - f_2'' f_1' \\ &= \frac{f_1}{\sqrt{1 - f_1'^2}}. \end{aligned}$$

All elements outside of the principal diagonal are equal to zero. Hence

$$\| L^3_{ij} \| = \begin{pmatrix} -3 f_1 f_2' & 0 & 0 \\ 0 & -3 f_1 f_2' \cos^2\alpha & 0 \\ 0 & 0 & \frac{f_1}{\sqrt{1 - f'^2}} \end{pmatrix}.$$

So, from the three matrices only this has the diagonal form. It is well known that the indicatrix of normal curvature of the Veronese surface $F^2 \subset S^4$ at every point $x \in F^2$ is a circle with the center at x. Therefore it is impossible to transform simultaneously all the quadratic forms of F^2 to the diagonal form. The same is true with respect to II^ν. Hence, *on the submanifold* $F^3 \subset E^6$ *do not exist principal directions.*

If $\tau = (\tau^1, \tau^2, \tau^3)$ is an asymptotical vector, then $II^\nu = 0$ only for two cases: either the vector $(\tau^1, \tau^2, 0)$ is asymptotical direction for $F^2 \subset S^4$ — it is impossible, or $f_1 = 0$ — it possible only at a singular point. Hence *at regular points of* F^3 *asymptotical directions do not exist.*

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Пример изометрического погружения области трехмерного пространства Лобачевского в E^6 с сечением в виде поверхности Веронезе

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Построен пример погружения области пространства Лобачевского L^3 в E^6 такой, что каждое сечение этого подмногообразия с гиперплоскостью $x^6 = const$ является поверхностью Веронезе. Подмногообразие не ориентируемо, допускает 2-параметрическое семейство движений по себе. На подмногообразии нет ни главных, ни асимптотических направлений. Доказано и более общее утверждение о возможности погружения некоторой области из L^3 в E^k, $k > 5$, в виде подмногообразия специальной формы.

Приклад ізометричного занурення області 3-вимірного простору Лобачевського в E^6 з перетином у вигляді поверхні Веронезе

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Побудовано приклад занурення області простору Лобачевського L^3 в E^6 такого, що кожний перетин цього підмноговиду з гіперплощиною $x^6 = const$ є поверхнею Веронезе. Підмноговид дозволяє 2-параметричне сімейство рухів по собі. На підмноговиді не існує ні головних, ні асимптотичних напрямків. Доведено і більш загальне твердження про можливість занурення деякої області з L^3 в E^k, $k > 5$, у вигляді підмноговиду спеціальної форми.