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Strongly parabolic timelike submanifolds of Minkowsky space

A. Borisenko, M.L. Rabelo[†], and K. Tenenblat[†]

Department of Mathematics and Mechanics Faculty, Kharkov State University, 4 Svobody Sq., 310077, Kharkov, Ukraine
E-mail: Alexander.A.Borisenko@univer.kharkov.ua

[†]*Departamento de Matemática Universidade de Brasília, 71910-900 Brasília, DF, Brasil* *
E-mail: rabelo@mat.unb.br
E-mail: keti@mat.unb.br

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R.P. Newman proved that a timelike geodesically complete pseudo-Riemannian manifold with nonnegative Ricci curvature for all vectors and admites a timelike line is isometric to the product of that line and a spacelike complete Riemannian manifold. This result gave a complete proof of a conjecture of Yau. In this paper we proof a cylinder type-theorem which corresponds to the extrinsic version of Newman's result. Moreover, we show that k-strongly parabolic geodesically complete submanifolds of a pseudo-Euclidean space with nonnegative Ricci curvature in the spacelike directions are also cylinders with k-dimensional generators.

According to a conjecture of Yau [12], a geodesically complete Lorentzian 4-manifold of nonnegative Ricci curvature in the timelike direction, which contains an absolutely maximizing timelike geodesic, is isometric to the cross product of that geodesic and a spacelike hypersurface.

Many results related to this conjecture were established. For example, Eschenburg [6] proved the conjecture in the globally hyperbolic case, assuming the manifold is timelike geodesically complete, satisfying the *strong energy condition* and contains a (complete) timelike line. Galloway [7] observed that the full assumption of timelike geodesic completeness is not needed. This property is derived as a consequence of global hyperbolicity. In 1990, Newman [10] complemented Galloway's result and completed Yau's conjecture. He proved that

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if (M^n, g) *is a timelike geodesically complete pseudo-Riemannian manifold with signature* $(-, +, \dots, +)$, *such that* $Ric(X, X) \geq 0$ *for all timelike vectors and also* M *admits a unit speed timelike line* $\alpha : \mathbb{R} \to M$, *then* (M^n, g) *is isometric to* $(\mathbb{R} \times \tilde{M}^{n-1}, -dt^2 \oplus h)$, *where* $(\tilde{M}, h)$ *is a complete Riemannian manifold and* α *is given by* $t \mapsto (t, p)$, *for some* $p \in \tilde{M}$.

In this paper, we prove extrinsic versions of this last result for timelike submanifolds of the pseudo-Euclidean space E_s^n with signature $(s, n-s)$, obtaining the so-called cylinder theorems. We denote by M_s^ℓ a manifold with signature $(s, \ell - s)$. Our main result is motivated by the classical Hartman–Nirenberg's theorem [9] which stablishes that *if* M^n *is a complete Riemannian manifold with nonnegative sectional curvature and* $f : M \to E^{n+p}$ *is an isometric immersion with index of nullity* $k > 0$ *then* $f(M)$ *is a* k*-cylinder, i.e., there exists a Riemannian complete manifold* $\overline{M}^{n-k}$ *such that* M *is isometric to* $\overline{M} \times E^k$, *and there exists an immersion* $\overline{f} : \overline{M} \to E^{n+p-k}$ *such that* $f(\overline{x}, r) = (\overline{f}(\overline{x}), r)$, *for all* $(\overline{x}, r) \in \overline{M} \times E^k = M$.

We formulate our results in the following theorems.

Theorem 1. *Let* M_1^ℓ *be a timelike geodesically complete submanifold of* E_1^n. *Assume that*

(i) $Ric(X, X) \geq 0$ *for all timelike vectors and*

(ii) the submanifold M_1^ℓ *has a timelike straight line of the ambient space* E_1^n.

Then M_1^ℓ *is a cylinder which splits as an euclidean product* $(E_1^1 \times \tilde{M}^{\ell-1}, -dt^2 \oplus h)$, *where* $\tilde{M}^{\ell-1} \subset E^{n-1}$, h *is the induced metric on* $\tilde{M}$ *and* E_1^1 *is a timelike straight line.*

Theorem 2. *Suppose* M_s^ℓ *is a geodesically complete* ℓ*-dimensional submanifold of signature* $(s, \ell - s)$ *in a pseudo-Euclidean space* E_s^n *of signature* $(s, n-s)$. *Assume that the nullity index* $\mu(q) = k = const$ *on the submanifold* $M_s^\ell, k \geq s$, *and* $Ric(X, X) \geq 0$ *for all spacelike vectors. If there exists a point* q_0 *such that the null space* $L^0(q_0)$ *is a pseudo-Euclidean plane* E_s^k *then* M_s^ℓ *is a cylinder with* k*-dimensional generators* E_s^k.

We observe that Abe and Magid in [2] studied the case where the nullity index $\mu(q) \geq \ell - 1$.

1. A cylinder theorem for a timelike submanifold of E_1^n

In this section we prove Theorem 1. We will need the following lemma.

Lemma 1. *Suppose M_1^2 is a simply connected surface of class C^1 in a pseudo-Euclidean space E_1^n, isometric to the Lorentzian plane E_1^2 with signature $(-,+)$. If there exists a straight timelike line of the ambient space which belongs to M_1^2 then this surface is a cylinder and splits as a product $(E_1^1 \times C, -dt^2 \oplus ds^2)$, where $C \subset E^{n-1}$ and E_1^1 is a timelike straight line.*

A similar result for the euclidean case was proved in [3].

P r o o f o f L e m m a 1. Let $\overline{\ell}$ be a straight line on the surface M_1^2 and let ℓ be the corresponding line on E_1^2, $\overline{\ell} = \phi(\ell)$, where ϕ is the isometry between E_1^2 and M_1^2. Parametrize ℓ by $\gamma(t)$, $t \in I\!R$, and take, for each t, λ_t the segment orthogonal to ℓ in the point $\gamma(t)$. Then $\phi(\lambda_t) = \overline{\lambda}_t$ lies on the subspace of E_1^n orthogonal to $\overline{\ell}$ in the pseudo-Euclidean metric of E_1^n. In fact, suppose this does not occur for some point $\gamma(t_0) = q_0$. Assuming $t_0 = 0$, there exists a point $p \in E_1^2$ such that the segment $q_0 p$ is orthogonal to ℓ and the corresponding segment $\overline{q}_0\overline{p}$ in E_1^n is not orthogonal to $\overline{\ell}$, where $\overline{q}_0 = \phi(q_0)$ and $\overline{p} = \phi(p)$.

Denote the length of the segment $q_0 p$ by h and $\overline{\gamma}(t) = \phi(\gamma(t))$. Assume that t is large enough such that the sides of the triangle $\gamma(-t)p\gamma(t)$ are timelike. If t measures the length in ℓ from $q_0 = \gamma(0)$ we have

$$|p\gamma(t)|^2 = |p\gamma(-t)|^2 = |-t^2 + h^2| = t^2 - h^2,$$

where in the last equality we use the assumption that the sides $p\gamma(t)$ and $p\gamma(-t)$ are timelike. Choose an orthonormal coordinate system in E_1^n such that the line $\overline{\ell}$ coincides with the x_0-axes and $\overline{q}_0$ is the origin. Assume that the point $\overline{p}$ belongs to the plane x_0, x_1. Then $\overline{p}$ has coordinates $(t_0, \overline{h}, 0, \ldots, 0)$ with $t_0 > 0$ (or $t_0 < 0$). Then

$$|\overline{p}\,\overline{\gamma}(t)|^2 = |-(t - t_0)^2 + \overline{h}^2| = (t - t_0)^2 - \overline{h}^2,$$

$$|\overline{p}\,\overline{\gamma}(-t)|^2 = |-(t + t_0)^2 + \overline{h}^2| = (t + t_0)^2 - \overline{h}^2,$$

where $\overline{\gamma}(t) = \phi(\gamma(t))$.

Since ϕ is an isometry, we obtain that $t^2 - h^2 = |\gamma(t)|^2 = d(\overline{p}, \overline{\gamma}(t))^2$, where d is the distance function on M_1^2. Now, using that straight timelike lines have maximal length among all timelike curves which connect two fixed points, it follows that

$$(t - t_0)^2 - \overline{h}^2 \geq t^2 - h^2,$$

$$(t + t_0)^2 - \overline{h}^2 \geq t^2 - h^2.$$

This implies that

$$-2tt_0 + t_0^2 \geq \overline{h}^2 - h^2,$$

$$2tt_0 + t_0^2 \geq \overline{h}^2 - h^2,$$

which is impossible for t large.

Hence we can consider a rectangle $q_0\gamma(t)p(t)p_0$ on the plane E_1^2 with $q_0 \in \ell$ and the segments q_0p_0 and $\gamma(t)p(t)$ orthogonal to ℓ. Let $\overline{q}_0\overline{\gamma}(t)\overline{p}(t)\overline{p}_0$ be the corresponding quadrangle in E_1^n. We take a system of coordinates as before, such that $\overline{p}_0 = (0, \overline{h}_0, 0, \ldots, 0)$ and $\overline{p}(t) = (t, \overline{h}_1 cos\, \alpha, \overline{h}_1 sin\, \alpha, 0, \ldots, 0)$, where α is the angle determined by $\overline{q}_0\overline{p}_0$ and $\overline{\gamma}(t)\overline{p}(t)$.

We know that

$$|\overline{p}_0\overline{p}(t)|^2 = t^2 - (\overline{h}_0 - \overline{h}_1 cos\, \alpha)^2 - (\overline{h}_1 sin\, \alpha)^2.$$

Again, using that ϕ is an isometry, we get

$$|\overline{p}_0\overline{p}(t)|^2 \geq |p_0p(t)|^2 = t^2.$$

Then

$$t^2 - (\overline{h}_0 - \overline{h}_1 cos\, \alpha)^2 - (\overline{h}_1 sin \alpha)^2 \geq t^2,$$

which gives that $\overline{h}_0 = \overline{h}_1$, $\alpha = 0$ and $|\overline{p}_0\overline{p}(t)| = t$.

Therefore the segment $\overline{p}_0\overline{p}(t)$ lies on the surface M_1^2 and is paralell to the line $\overline{\ell}$. This concludes the proof. ∎

Now the proof of Theorem 1 follows easily from Newman's result and Lemma 1.

P r o o f o f T h e o r e m 1. Using Newman's result it follows that M_1^ℓ is isometric to $(E_1^1 \times \tilde{M}^{\ell-1}, -dt^2 \oplus h)$, where $(\tilde{M}^{\ell-1}, h)$ is a complete Riemannian manifold and the straight timelike line $\alpha : E^1 \to M_1^\ell$ is given by $\alpha(t) = (t, p)$, for some $p \in \tilde{M}^{\ell-1}$. Let C be any curve in $\tilde{M}^{\ell-1}$, then $E_1^1 \times C$ is isometric to some strip in the Lorentzian plane and applying Lemma 1 it follows that M_1^ℓ is a cylinder in the pseudo-Euclidean space E_1^n. ∎

2. Strongly parabolic timelike submanifolds of E_s^n

In order to prove Theorem 2 we first recall some results and definitions.

If $f : M_r^n \to \overline{M}_s^{n+k}$ is an isometric immersion between indefinite Riemannian manifolds we define the relative null space at x, $L^0(x)$, by

$$L^0(x) = \{v \in T_xM : \ S_\xi(v) = 0, \ \forall\, \xi \in N(x)\},$$

where $N(x)$ is the normal space and S_ξ is the Weingarten map. The dimension $\mu(x)$ of $L^0(x)$ is called the index of relative nullity of the immersion at x. The minimum value of $\mu(x)$ on M is called the index of nullity and is denote by μ_0. M is called a *k-strongly parabolic submanifold* if $\mu(x) \geq k$, for all $x \in M$.

We observe that codimension-one isometric immersion between Lorentzian space has nullity index $\mu \geq n-1$, where n is the dimension of the hypersurface [8]. When the ambient manifold is the Lorentzian space form of positive curvature, the Lorentzian submanifolds with nullity index $\mu \neq 0$ were investigated in [1]. We need the following results proved in [1]:

Lemma 2. *Let $f : M_r^n \to \overline{M}_s^{n+p}(c)$ be an isometric immersion, where $\overline{M}$ is a space form with constant curvature c and let G be the set of points in M where $\mu(x) = \mu_0$. Then*

(i) G is an open subset of M;

(ii) $x \mapsto L^0(x)$, $x \in G$, is a differentiable and involutive distribution in G;

(iii) the foliation L^0 is totally geodesic in M;

(iv) each leaf of the foliation is immersed as a totally geodesic submanifold of $\overline{M}$.

Lemma 3. *With the above conditions and assuming M complete, we obtain that the relative null foliation is geodesically complete.*

In particular, we observe that when $\overline{M}$ is the pseudo-Euclidean space E_s^{n+p}, each leaf of the foliations is immersed as an affine subspace of E_s^{n+p}.

As a consequence of these lemmas and Theorem 1 we have

Proposition 1. *Suppose M_1^ℓ is a timelike complete submanifold in a pseudo-Euclidean space E_1^n, with nullity index $\mu_0 = \min_{x \in M_1^\ell} \mu(x) > 0$ and $Ric(X,X) \geq 0$ for all timelike vectors. If there exists a point $q_0 \in M_1^\ell$ such that $\mu(q_0) = \mu_0$ and $L^0(q_0)$ is Lorentzian, then the submanifold M_1^ℓ is a cylinder with 1-dimensional timelike generators.*

P r o o f. It follows from Lemma 2 that the leaf $L^0(q_0)$ is a Lorentzian μ_0-dimensional plane in E_1^n. Let $\overline{\ell} \subset L^0(q_0)$ be a timelike straight line of the ambient space. Then the submanifold M_1^ℓ satisfies the hypothesis of Theorem 1 which concludes the proof. ■

Let $X(u_1, \ldots, u_\ell)$ be a local parametrization of a pseudo-Riemannian ℓ-dimensional submanifold M_s^ℓ of the pseudo-Euclidean space E_s^{n+1}. We consider $\tilde{M}^\ell$ the submanifold of the euclidean space E^{n+1}, given locally by the same parametrization. Then each normal vector to M_s^ℓ corresponds a normal vector to $\tilde{M}^\ell$, and we can associated their second fundamental forms with respect to these normals as follows.

Lemma 4. *Let M_s^ℓ be an ℓ-dimensional pseudo-riemmanian submanifold of E_s^{n+1} and let $\tilde{M}^\ell$ be the corresponding ℓ-dimensional submanifold of the euclidean space E^{n+1}. Then their second fundamental forms with respect to corresponding normal vectors are proportional.*

P r o o f. Let $x_0, x_1, \ldots, x_n$ be an orthogonal system of coordinates for E_s^{n+1} and E^{n+1}. We also consider their respective metrics $ds^2 = -dx_0^2 - dx_1^2 - \ldots - dx_{s-1}^2 + dx_s^2 + \ldots + dx_n^2$ and $d\tilde{s}^2 = dx_0^2 + dx_1^2 + \ldots + dx_n^2$.

Let $X(u_1, \ldots, u_\ell)$ be a local parametrization of $M_s^\ell \subset E_s^{n+1}$ and we take $\tilde{M}^\ell$ the submanifold of E^{n+1} given by the same parametrization.

Let $\xi = (\xi^0, \ldots, \xi^n)$ be a unit normal vector to M_s^ℓ, then $< \dfrac{\partial X}{\partial u_j}, \xi >= 0$, $0 \le j \le n$, where $< \, , \, >$ is the inner product in E_s^{n+1}. We will show that there is a corresponding unit vector $\tilde{\xi} = (\tilde{\xi}^0, \ldots, \tilde{\xi}^n)$ normal to $\tilde{M}^\ell$ in the euclidean space with scalar product $(\, , \,)$. In fact, since $\tilde{\xi}$ must satisfy

$$\left(\frac{\partial X}{\partial u_j}, \tilde{\xi} \right) = 0, \quad 0 \le j \le n,$$

rewriting these equations in coordinates, we obtain

$$-\sum_{r=0}^{s-1} \frac{\partial x_r}{\partial u_j} \xi^r + \sum_{k=s}^{n} \frac{\partial x_k}{\partial u_j} \xi^k = 0,$$

$$\sum_{r=0}^{s-1} \frac{\partial x_r}{\partial u_j} \tilde{\xi}^r + \sum_{k=s}^{n} \frac{\partial x_k}{\partial u_j} \tilde{\xi}^k = 0,$$

for $0 \le j \le n$. Since ξ and $\tilde{\xi}$ are unit vectors we have

$$-\sum_{r=0}^{s-1} (\xi^r)^2 + \sum_{k=s}^{n} (\xi^k)^2 = 1 \qquad \text{and} \qquad \sum_{k=0}^{n} (\tilde{\xi}^k)^2 = 1.$$

It follows that $\tilde{\xi}$ is given by

$$\tilde{\xi} = \frac{1}{\sqrt{1 + 2\sum_{r=0}^{s-1} (\xi^r)^2}} (-\xi^0, -\xi^1, \ldots, -\xi^{s-1}, \xi^s, \ldots, \xi^n).$$

Now we calculate the coefficients of the second fundamental forms of M_s^ℓ and $\tilde{M}^\ell$ with respect to ξ and $\tilde{\xi}$, respectively. We obtain that

$$b_{ij}(\xi) =< \frac{\partial^2 X}{\partial u_i \partial u_j}, \; \xi >= -\sum_{r=0}^{s-1} \frac{\partial^2 x_r}{\partial u_i \partial u_j} \xi^r + \sum_{k=s}^{n} \frac{\partial^2 x_k}{\partial u_i \partial u_j} \xi^k$$

and

$$\tilde{b}_{ij}(\tilde{\xi}) = \left(\frac{\partial^2 X}{\partial u_i \partial u_j}, \tilde{\xi}\right) = \frac{1}{\sqrt{1 + 2\sum\limits_{r=0}^{s-1}(\xi^r)^2}} b_{ij}(\xi),$$

which completes the proof. ■

This result for the pseudo-Euclidean space E_1^3 was considered by Sokolov [11] and the analogous result for submanifolds in space forms was proved in [4].

Lemma 5. *Suppose that a submanifold M_s^ℓ of E_s^n has nonnegative Ricci curvature for all spacelike vectors at a point $q \in M_s^\ell$ where the null space $L^0(q)$ is a k-dimensional pseudo-Euclidean plane E_s^k, $k \geq s$. Then there exists a normal vector ξ at q such that the second fundamental form of M_s^ℓ with respect to ξ, restricted to the subspace of the tangent plane, orthogonal to $L^0(q)$, is positive definite.*

P r o o f. Let $I\!\!H$ be the mean curvature vector at q. We consider an orthonormal basis $\xi_1, \ldots, \xi_{n-\ell}$ normal to M_s^ℓ at q such that ξ_1 is in the direction of $I\!\!H$. Let $e_1, \ldots, e_\ell$ be an orthonormal basis tangent to M_s^ℓ at q which diagonalizes the second fundamental form in the direction of ξ_1 in such way that $e_1, \ldots, e_{\ell-k}$ are spacelike vectors.

We introduce the notation $b_{ij}^\alpha =< B(e_i, e_j),\ \xi_\alpha >,\ \ 1 \leq i, j \leq \ell, 1 \leq \alpha \leq n-\ell$, where B is the second fundamental form. From Gauss equation, the sectional curvature along the plane generated by e_i, e_j is given by

$$K(e_i, e_j) = \epsilon_i \epsilon_j \left[b_{ii}^1 b_{jj}^1 + \sum_{\alpha=2}^{n-\ell} \left(b_{ii}^\alpha b_{jj}^\alpha - (b_{ij}^\alpha)^2 \right) \right],$$

where $\epsilon_i =< e_i, e_i >$.

The Ricci curvature in the direction of e_i reduces to

$$Ric(e_i, e_i) = b_{ii}^1 (H - \epsilon_i b_{ii}^1) - \epsilon_i \sum_{\alpha=2}^{n-\ell} (b_{ii}^\alpha)^2 - \sum_{\substack{\alpha=2 \\ j \neq i}}^{n-\ell} \epsilon_j (b_{ij}^\alpha)^2,$$

where we use that

$$I\!\!H = H\xi_1 = \left(\sum_{j=1}^{\ell} \epsilon_j b_{jj}^1 \right) \xi_1 \qquad \text{and} \qquad \sum_{j=1}^{\ell} \epsilon_j b_{jj}^\alpha = 0, \ \text{ for } \ \alpha \neq 1.$$

Using the hyphotesis on the Ricci curvature and on the null space, we obtain that

$$b^1_{ii}(H - b^1_{ii}) \geq 0, \text{ for all } 1 \leq i \leq \ell - k.$$

It follows that $b^1_{ii} > 0$, for $1 \leq i \leq \ell - k$, which proves the lemma. ■

Now we can prove our main result.

P r o o f o f T h e o r e m 2. Let $q_0 \in M^\ell_s$ be a point where the null space $L^0(q_0)$ is pseudo-Euclidean. We consider a plane $E^{n-k} \subset E^n_s$ through q_0, which is orthogonal to the totally geodesic leaf $E^k_s(q_0)$ (see Lemma 2). In a neighbourhood of the point q_0 the submanifold $N^{\ell-k} = M^\ell_s \cap E^{n-k}$ can be parametrized by $\rho(u_1, \dots, u_{\ell-k})$. From Lemmas 3 and 4 it follows that the submanifold $\tilde{M}^\ell$ of the euclidean space E^n, which correspond to $M^\ell \subset E^n_s$, is a strongly parabolic submanifold with nullity index $\mu(q) = k = \text{const}$.

Let $E^k(q), q \in N$, be the totally geodesic leaf at q. We consider $e_{\ell-k+1}, \dots, e_\ell$ an orthonormal basis of $E^k_s(q_0)$. Let $\eta_r(u_1, \dots, u_{\ell-k})$, $\ell - k + 1 \leq r \leq \ell$, be a basis for the planes $E^k(q), q \in N$, such that their orthogonal projections on the plane $E^k_s(q_0)$ coincides with e_r. Let v_r, $\ell - k + 1 \leq r \leq \ell$, be coordinates in the plane $E^k(q)$ with respect to the basis η_r. Then, in a neighbourhood V of the point q_0, $\tilde{M}^\ell$ can be parametrized by

$$X(u_1, \dots, u_{\ell-k}, v_{\ell-k+1}, \dots, v_\ell)$$

$$= \rho(u_1, \dots, u_{\ell-k}) + \sum_{r=\ell-k+1}^{\ell} v_r \eta_r(u_1, \dots, u_{\ell-k}). \tag{1}$$

In V we have

$$\eta_r = e_r + \sum_{j=1}^{\ell-k} C^j_r \rho_{u_j} + \sum_{\alpha=1}^{n-\ell} S^\alpha_r \tilde{\xi}_\alpha, \tag{2}$$

where $\tilde{\xi}_\alpha$, $1 \leq \alpha \leq n - \ell$ is an orthonormal basis, normal to the submanifold $\tilde{M}^\ell$. Since $\dfrac{\partial X}{\partial v_r} = \eta_r$, it follows from Weingarten formulas that

$$\frac{\partial \eta_r}{\partial u_i} = \sum_{s=1}^{\ell-k} \tilde{\Gamma}^s_{ir} \frac{\partial X}{\partial u_s} + \sum_{t=\ell-k+1}^{\ell} \tilde{\Gamma}^t_{ir} \eta_t + \sum_{\alpha=1}^{n-\ell} b^\alpha_{ir} \tilde{\xi}_\alpha, \tag{3}$$

where $\tilde{\Gamma}^s_{ir}, \tilde{\Gamma}^t_{ir}$ are the Christofell simbols of the submanifold $\tilde{M}^\ell$. Since $\tilde{M}^\ell$ has constant nullity index and considering the choice of the basis η_r, it follows that $b^\alpha_{ir} = 0$, for $k + 1 \leq r \leq \ell$. On the other hand, we obtain from (2) that

$$\frac{\partial \eta_r}{\partial u_i} = \sum_{j=1}^{\ell-k} \frac{\partial C^j_r}{\partial u_i} \rho_{u_j} + \sum_{j=1}^{\ell-k} C^j_r \rho_{u_j u_i} + \sum_{\alpha=1}^{n-\ell} \frac{\partial S^\alpha_r}{\partial u_i} \tilde{\xi}_\alpha + \sum_{\alpha=1}^{n-\ell} S^\alpha_r \frac{\partial \tilde{\xi}_\alpha}{\partial u_i}. \tag{4}$$

Now, using Weingarten formulas for the submanifold $N^{\ell-k}$, we get

$$\begin{aligned} \frac{\partial\rho}{\partial u_j \partial u_i} &= \sum_{s=1}^{\ell-k} \Gamma^s_{ji} \frac{\partial \rho}{\partial u_s} + \sum_{\alpha=1}^{n-\ell} \delta^\alpha_{ji} \tilde{\xi}_\alpha, \\ \frac{\partial \tilde{\xi}_\alpha}{\partial u_i} &= -\sum_{m,s=1}^{\ell-k} \delta^\alpha_{mi} g^{ms} \frac{\partial \rho}{\partial u_s} + \sum_{\beta=1}^{n-\ell} \nu^\beta_{i\alpha} \tilde{\xi}_\beta, \end{aligned} \tag{5}$$

where g_{ij} are the components of the metric tensor, Γ^s_{ji} are Christofell simbols, δ^α_{ji} are the coefficients of the second fundamental forms of the submanifolds $N^{\ell-k} \subset E^{n-k}$ and $\nu^\beta_{i\alpha}$ are the components of the normal connection. Substituting (5) in (4), we obtain

$$\begin{aligned} \frac{\partial \eta_r}{\partial u_i} &= \sum_{s=1}^{\ell-k} \frac{\partial \rho}{\partial u_s} \left[\frac{\partial C^s_r}{\partial u_i} + \sum_{j=1}^{\ell-k} C^j_r \Gamma^s_{ji} - \sum_{m=1}^{\ell-k} \sum_{\alpha=1}^{n-\ell} S^\alpha_r \delta^\alpha_{mi} g^{ms} \right] \\ &\quad + \sum_{\alpha=1}^{n-\ell} \tilde{\xi}_\alpha \left[\frac{\partial S^\alpha_r}{\partial u_i} + \sum_{j=1}^{\ell-k} C^j_r \delta^\alpha_{ji} + \sum_{\beta=1}^{n-\ell} S^\beta_r \nu^\alpha_{i\beta} \right]. \end{aligned} \tag{6}$$

Since at a point q of the submanifold $\tilde{M}^\ell$ either $\{\frac{\partial X}{\partial u_s}, \eta_r\}$ or $\{\frac{\partial \rho}{\partial u_s}, \eta_r\}$ provides a basis of $T_q\tilde{M}$, it follows that (3) can be rewritten as

$$\frac{\partial \eta_r}{\partial u_i} = \sum_{s=1}^{\ell-k} \beta^s_{ir} \frac{\partial \rho}{\partial u_s} + \sum_{t=\ell-k+1}^{\ell} \beta^t_{ir} \eta_t. \tag{7}$$

Comparing (6) and (7), we obtain that $\beta^t_{ir} = 0$ for all t. Therefore

$$\frac{\partial \eta_r}{\partial u_i} = \sum_{s=1}^{\ell-k} \beta^s_{ir} \frac{\partial \rho}{\partial u_s}. \tag{8}$$

The integrability conditions for this system, taking into account (5), are

$$\begin{aligned} \frac{\partial \beta^s_{ir}}{\partial u_j} + \sum_{m=1}^{\ell-k} \Gamma^s_{mj} \beta^m_{ir} &= \frac{\partial \beta^s_{jr}}{\partial u_i} + \sum_{m=1}^{\ell-k} \Gamma^s_{mi} \beta^m_{jr}, \\ \sum_{s=1}^{\ell-k} \beta^s_{ir} \delta^\alpha_{sj} &= \sum_{s=1}^{\ell-k} \beta^s_{jr} \delta^\alpha_{si}. \end{aligned} \tag{9}$$

Lemma 5 and the hyphotesis on the Ricci curvature imply that there exists a normal ξ to M^ℓ_s at $q \in N^{\ell-k}$ such that the restriction of the second fundamental form of M^ℓ_s to the tangent space of $N^{\ell-k}$ at q is positive definite. From Lemma 4 it follows that there exists a vector $\tilde{\xi}$ normal to $N^{\ell-k}$ such that the second quadratic form of $N^{\ell-k} \subset E^{n-k}$ with respect to this normal is also positive definite. Denote by Δ the matrix of this fundamental form. Since the submanifold $\tilde{M}^{\ell} \subset E^n$ is complete along the leaves and

$$\begin{aligned} \frac{\partial X}{\partial u_i} &= \frac{\partial \rho}{\partial u_i} + \sum_{r=\ell-k+1}^{\ell} \frac{\partial \eta_r}{\partial u_i} v_r = \frac{\partial \rho}{\partial u_i} + \sum_{r=\ell-k+1}^{\ell} \sum_{s=1}^{\ell-k} \beta_{ir}^s v_r \frac{\partial \rho}{\partial u_s}, \\ \frac{\partial X}{\partial v_r} &= \eta_r, \end{aligned}$$

we conclude that

$$det\left(I + \sum_{r=\ell-k+1}^{\ell} \beta_r v_r\right) \neq 0,$$

for all values v_r, where β_r denotes the matrix with entries $\left(\beta_{ir}^j\right)$ and I is the identity matrix. Therefore we obtain that

$$det\left(\Delta + \sum_{r=\ell-k+1}^{\ell} v_r \beta_r \Delta\right) \neq 0,$$

for all values v_r.

Since the matrix $\beta_r \Delta$ is symmetric by (9), it follows that $\Delta + \sum_{r=\ell-k+1}^{\ell} v_r \beta_r \Delta$ is also symmetric. Using the continuity of the determinant and the fact that Δ is positive definite, we conclude that the last inequality holds for all values v_r, if and only if, $\beta_r \Delta$ is a null matrix. This implies that $\beta_r = 0$ at every point, i.e., $\beta_{ir}^j = 0$ for all indices i, j. Hence from (7) we get $\dfrac{\partial \eta_r}{\partial u_i} = 0, \quad \forall\, i$, and the expression (1) gives that $\tilde{M}^{\ell}$ is a cylinder with k-dimensional generators at some neighborhood of the point q_0. The hypothesis on the nullity index and the condition on the Ricci curvature imply that $\tilde{M}^{\ell}$ is globally a cylinder with k-dimensional generators. The same is true for the corresponding submanifold $M_s^{\ell} \subset E_s^n$. This concludes the proof. ∎

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Сильно параболические временеподобные подмногообразия пространства Минковского

А.А. Борисенко, М.Л. Рабело, К. Тененблат

Р.П. Ньюмен доказал, что геодезически полное во временеподобных направлениях лоренцево пространство с неотрицательной кривизной Риччи для временеподобных векторов, которое содержит временеподобную линию, является метрическим произведением этой прямой и риманового многообразия. В этой статье доказывается внешнегеометрический аналог этой теоремы для многомерных поверхностей пространства Минковского. Более того, показано что k — сильно параболические геодезически полные подмногообразия в псевдо-евклидовом пространстве с неотрицательной кривизной Риччи в пространственноподобных направлениях являются цилиндрами с k-мерными образующими.

Строго параболічні часоподібні підмноговиди простору Мінковського

О.А. Борисенко, М.Л. Рабело, К. Тененблат

Р.П. Н'юмен довів, що геодезично повний у часоподібних напрямках лоренців простір з невід'ємною кривиною Річчі для часоподібних напрямків, в якому існує часоподібна пряма, є метричним добутком прямої і простороподібного ріманового простору. В цій статті доводиться зовнішньогеометричний аналог цієї теореми для багатовимірних поверхонь простору Мінковського. Більш того, доводиться, що k — сильно параболічні геодезично повні підмноговиди в псевдо-евклідовому просторі з невід'ємною кривиною Річчі у простороподібних напрямках є циліндрами з k-вимірними твірними.