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Generatrix of catenoid of space 3-form

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Constant mean curvature surfaces of revolution in euclidean 3-space are known as surfaces of Ch. Delaunay. They possess one remarkable property: their profile curves (generatrices) are the trajectories of focuses of conic sections by its rolling along the straight line. Analogous construction is realized in the space forms H^3 and S^3 in the case of minimal surfaces of revolution and the following theorem is proved.

Theorem. *Generatrix of catenoid of revolution of space form $H^3(S^3)$ is the trajectory of focus of hyperbolic (spherical) parabola by its rolling along the geodesic ray.*

In differential geometry of euclidean 3-space are known Ch. Delaunay surfaces of constant mean curvature [1, 2]. Their analogs in multidimensional euclidean space were investigated by W.Y. Hsiang [3]. The surfaces of Delaunay are surfaces of revolution, whose generatrix has one remarkable property: it represents a trajectory of a line, which focus of conic section describes by rolling along the axis of rotation. In particular, the generatrix of catenoid of revolution in euclidien space (its mean curvature vanishes) is the trajectory (the chain line), which focus of parabola describes by rolling along the axis of revolution. In hyperbolic 3-space there is also catenoid of revolution and some of its properties are investigated [4]. In this work we point out the curves P_H and P_S which are located respectively in the hyperbolic plane and in the sphere and represent analogs of euclidean parabola. It is appeared, that generatrix of catenoid of revolution can be constructed by analogy with euclidean case, as a trajectory of a curve, which is obtained by rolling the "focus" of curve P_H or P_S along rotational axis. Curvature of space forms is assumed to be equal -1 (respectively 1), and the rolling process is represented as a curve in the group of proper isometries of $H^2(S^2)$.

Theorem. *Generatrix of catenoid of revolution of hyperbolic space H^3 (spherical space S^3) is the trajectory of focus of hyperbolic parabola P_H (spherical parabola P_S) obtained by its rolling along the axis of revolution.*

In the first section we consider hyperbolic case using Poincare model in the upper half-plane and its group of proper isometries $PSL(2,R)$. Short exposition of this section was published in joint work with E.G. Kaliniuk [8]. In the second section we deal with spherical case using standard embedding of unit sphere with the group of motions isomorphic to $SO(3)$.

1. Property of generatrix of hyperbolic catenoid

1.1. Cylindrical coordinates in H^3. We consider hyperbolic space H^3 of curvature -1 in the model of Poincare in the upper half-space R^3_+ $((x,y,z),\ z>0)$ with the metric $ds^2 = \dfrac{dx^2+dy^2+dz^2}{z^2}$. On the geodesic $(x=y=0)$, which we choose as an axis of cylindrical coordinates, we fix an origin $(0,0,1)$ and introduce cylindrical coordinates (t,r,ϕ) of arbitrary point $M(x,y,z)\in H^3$ according to the rule

$$t=\frac{1}{2}\ln(x^2+y^2+z^2),\quad t\in R\,,$$

$$\cosh r=\frac{\sqrt{x^2+y^2+z^2}}{z},\quad r\in R_+\,,$$

$$\tan\phi=\frac{y}{x},\quad \phi\in[0,2\pi]\,.$$

Geometric sense of sizes (t,r,ϕ) is the following: size r represents a length of an interval of geodesic, connecting a point $M(x,y,z)$ with its projection M' on the axis of cylindrical coordinates; size t is equal to the distance from the point M' to the origin $(0,0,1)$; the angle ϕ determines direction of the geodesic $M'M$ and is equal to the angle between the axis Ox and projection of geodesic to absolut $z=0$. The metric form of H^3 in cylindrical coordinates has the view

$$ds^2=dr^2+\cosh^2 r dt^2+\sinh^2 r d\phi^2\,.$$

It is easy to receive this expression using the formulas for x, y, z in terms of t, r, ϕ:

$$x=e^t\tanh r\cos\phi,\quad y=e^t\tanh r\sin\phi,\quad z=e^t/\cosh r\,.$$

Set of singular points is located on the axis $r=0$. Cristoffel symbols of cylindrical coordinates of H^3 are the following (indices 1, 2, 3 correspond to t, r, ϕ): the rest of symbols are equal to zero, $\Gamma^1_{22}=\Gamma^1_{33}=-\sinh r\cosh r$, $\Gamma^2_{12}=\tanh r$, $\Gamma^3_{13}=\coth r$.

1.2. Some facts of differential geometry of surfaces of revolution in H^3 and the catenoid in particular. In the plane $\phi = 0$ we consider a curve $\Gamma(t) := (t, r(t), 0)$, which is the profile curve of a surface of revolution in H^3 with an axis $r = 0$. The surface is formed by the set of orbits of points of generatrix $\Gamma(t)$ under the action of the group of isometries H^3, isomorphic to $SO(2)$. Coordinates on the surface are parameters t and ϕ.

Proposition 1. *The fundamental forms of a surface of revolution in H^3 have the following form*:

$$I = \left(\cosh^2 r(t) + r'(t)^2\right)\, dt^2 + \sinh^2 r(t)\, d\phi^2 \,,$$

$$II = \frac{\cosh r(t)}{(\cosh^2 r(t) + r'(t)^2)^{1/2}} \left(\left(r''(t) - 2\sinh r(t) r'(t)^2 - \sinh r(t)\cosh r(t)\right) dt^2 - \sinh r(t)\cosh r(t)\, d\phi^2\right) .$$

P r o o f. Let $r = r(t)$ be the equation of meridian in the plane (r, t), which generates the surface of revolution. We may regard (t, ϕ) as coordinates on the surface. Substituting $r = r(t)$ in the formula for metric of hyperbolic space in cylindrical coordinates, we obtain required expression for the first fundamental form. Surface in cylindrical coordinates has the equation $r = (r(t), t, \phi)$, therefore its tangent vectors are $r_t = (r', 1, 0)$ and $r_\phi = (0, 0, 1)$. Using expression of metric form H^3, we find the normal vector $n = (n^1, n^2, n^3)$ solving conditions of orthogonality $\langle n, r_t\rangle = 0$, $\langle n, r_\phi\rangle = 0$ and norming $\langle n, n\rangle = 1$ (in the metric of H^3):

$$n = \frac{\cosh r}{\left(r'(t)^2 + \cosh^2 r(t)\right)^{1/2}} \left(1, -\frac{r'(t)}{\cosh^2 r}, 0\right) .$$

The second fundamental form $II = (\Omega_{ij})$, i, $j = 1, 2$, of the surface is obtained by using the formula (43.8) [5]: $-\Omega_{ij} = g_{\alpha\beta} r_i^\alpha (n^\beta_{,j} + \Gamma^\beta_{\mu\nu} r^\mu_{,j} n^\nu) = \langle r_i, \nabla_j n\rangle_{H^3}$, where $g_{\alpha\beta}$ is the metric tensor of H^3 and $\Gamma^\beta_{\mu\nu}$ are its Cristoffel symbols, indices i, j take values 1 and 2, corresponding to coordinates t and ϕ on the surface. Now calculate $\nabla_1 n = \nabla_t n = (\nabla_1 n^1, \nabla_1 n^2, \nabla_1 n^3)$:

$$\begin{aligned}
\nabla_1 n^1 &= \frac{\partial n^1}{\partial t} + \Gamma^1_{\mu\nu} r^\mu_{,t} n^\nu = \frac{\partial}{\partial t}\left(\frac{\cosh r}{\sqrt{r'^2 + \cosh^2 r}}\right) + \frac{r' \sinh r}{\sqrt{r'^2 + \cosh^2 r}} \,, \\
\nabla_1 n^2 &= \frac{\partial n^2}{\partial t} + \Gamma^2_{\mu\nu} r^\mu_{,t} n^\nu = -\frac{\partial}{\partial t}\left(\frac{r'}{\cosh r (r'^2 + \cosh^2 r)^{1/2}}\right) \\
& \quad + \frac{\sinh r}{\cosh^2 r} \frac{\cosh^2 r - r'^2}{(r'^2 + \cosh^2 r)^{1/2}} \,, \\
\nabla_1 n^3 &= \frac{\partial n^3}{\partial t} + \Gamma^3_{\mu\nu} r^\mu_{,t} = 0 \,.
\end{aligned}$$

Similarly we find $\nabla_2 n = \nabla_\phi n = (\nabla_2 n^1, \nabla_2 n^2, \nabla_2 n^3)$:

$$\begin{aligned}
\nabla_2 n^1 &= \frac{\partial n^1}{\partial \phi} + \Gamma^1_{\mu\nu} r^\mu_{,\phi} n^\nu = 0\,, \\
\nabla_2 n^2 &= \frac{\partial n^2}{\partial \phi} + \Gamma^2_{\mu\nu} r^\mu_{,\phi} n^\nu = 0\,, \\
\nabla_3 n^3 &= \frac{\partial n^3}{\partial \phi} + \Gamma^3_{\mu\nu} r^\mu_{,\phi} n^\nu = \frac{\cosh^2 r}{\sinh r (r'^2 + \cosh^2 r)^{1/2}}\,.
\end{aligned}$$

Then we find coefficients of the second fundamental form:

$$-\Omega_{11} = \langle r_t, \nabla_1 n \rangle = r' \frac{\partial}{\partial t} \frac{\cosh r}{(r'^2 + \cosh^2 r)^{1/2}} + \frac{r'^2 \sinh r}{(r'^2 + \cosh^2 r)^{1/2}}$$

$$+ \cosh^2 r \left(-\frac{\partial}{\partial t} \frac{r'}{\cosh r (r'^2 + \cosh^2 r)^{1/2}} + \frac{\sinh r}{\cosh^2 r} \frac{\cosh^2 r - r'^2}{(r'^2 + \cosh^2 r)^{1/2}} \right)$$

$$= \frac{-r'' \cosh r + \sinh r \cosh^2 r + 2 r'^2 \sinh r}{(r'^2 + \cosh^2 r)^{1/2}}\,,$$

$$-\Omega_{12} = \langle r_\phi, \nabla_1 n \rangle = 0\,,$$

$$-\Omega_{22} = \langle r_\phi, \nabla_2 n \rangle = \frac{\sinh r \cosh^2 r}{(r'^2 + \cosh^2 r)^{1/2}}\,.$$

Corollary 1. *Differential equation of catenoid of revolution in H^3 with generatrix $\Gamma(t) = (t, r(t), 0)$ is the following:*

$$r'' \sinh r \cosh r - (2 \sinh^2 r + \cosh^2 r)(r')^2 - \cosh^2 r (\sinh^2 r + \cosh^2 r) = 0\,.$$

Really, since the fundamental forms of catenoid are diagonal it is easy to find its principal curvatures: $k_1 = \dfrac{\Omega_{11}}{g_{11}}$, $k_2 = \dfrac{\Omega_{22}}{g_{22}}$. We find

$$k_1 = \frac{\cosh r r'' - 2 r'^2 \sinh r - \sinh r \cosh^2 r}{(r'^2 + \cosh^2 r)^{3/2}}\,,$$

$$k_2 = -\frac{\cosh^2 r}{\sinh r (r'^2 + \cosh^2 r)^{1/2}}\,.$$

Using the condition of minimality $k_1 + k_2 = 0$, we receive required differential equation.

It is possible to reduce the order of the found equation.

Corollary 2 [4]. *Generatrix of catenoid of revolution is determined by initial condition* $(r(t_0), t_0)$ *and satisfies the first order differential equation*

$$(r')^2 = a^2 \sinh^2 r \cosh^4 r - \cosh^2 r, \quad a = const\,. \tag{1}$$

In the paper [4] this statement is shown by a variational method, without calculation principal curvatures of catenoid. For the sake of completeness we give here this derivation.

The area element of catenoid has the form

$$dS = \sqrt{g_{11}g_{22} - g_{12}^2}\, dtd\phi = \sinh r(t)\sqrt{r'^2 + \cosh^2 r}\, dtd\phi\,.$$

Therefore in variational problem for minimum of the area of the surface of revolution the Lagrangian $L(r(t), r'(t)) = \sinh r(t)(r'^2 + \cosh^2 r)^{1/2}$ doesn't depend explicitly on t and the energy $E = r'\dfrac{\partial L}{\partial r'} - L$ is conserved along the extremal [6]. Calculating, we obtain

$$E = r'\frac{r' \sinh r}{(r'^2 + \cosh^2 r)^{1/2}} - \sinh r(r'^2 + \cosh^2 r)^{1/2} = c = const\,,$$

$$r'^2 \sinh r - \sinh r(r'^2 + \cosh^2 r)^{1/2} = c(r'^2 + \cosh^2 r)^{1/2}\,,$$

$$c^{-2} \sinh^2 r \cosh^4 r = r'^2 + \cosh^2 r\,,$$

that coincides with the first integral, pointed out in Corollary 2.

1.3. An analog of parabolic curve in H^2. In the upper half-plane $w = u + iv$, $v > 0$, with the Poincare metric $dw^2 = \dfrac{du^2 + dv^2}{v^2}$ we define the set of points P_H, satisfying the condition $d(w, e^{\theta_0}) = d(w, l)$, $(0 < \theta_0 < \pi/2)$, where d is hyperbolic distance in H^2, l is a straight line in H^2, given by equation $\mathrm{Re} w = 0$. Obviously, given curve is analog of parabola in euclidien plane (fig. 1). It is natural to call the point $e^{i\theta_0}$ the focus of parabola and the straight line l the directrix, the point $\left(\dfrac{1 - \sin\theta_0}{\cos\theta_0}, \dfrac{2\sqrt{\tan\theta_0/2}}{1 + \tan\theta_0/2}\right)$ corresponds to the vertex.

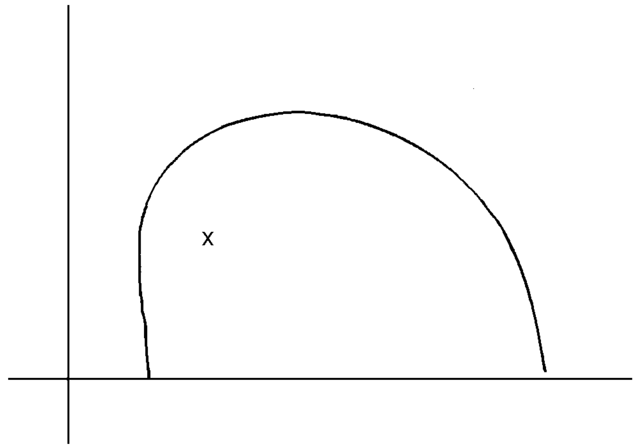

Fig. 1. Hyperbolic parabola (directrix $\mathrm{Re}\,\zeta = 0$, focus $e^{i\frac{\pi}{4}}$).

Lemma 1. 1) *Equation of parabola P_H with the focus e^{θ_0} and the directrix $\mathrm{Re}\,w = 0$ is the following:*

$$2\sinh\theta_0|w| = |w|^2 + 1 - 2u\cosh\theta_0\,. \tag{2}$$

In cylindrical coordinates (t, r), where $\mathrm{Re}\,w = e^t\tanh r$, $\mathrm{Im}\,w = \dfrac{e^t}{\cosh r}$ the same equation has the form

$$\tanh r = \frac{\cosh t - \sin\theta_0}{\cos\theta_0} = f(t, \theta_0) \tag{2'}$$

P r o o f. We use formulas for the distance between two points in hyperbolic plane: $\cosh d(z, w) = 1 + \dfrac{|z-w|^2}{2\mathrm{Im}\,z\,\mathrm{Im}\,w}$ [7, §7.2] and for the distance between a point and a straight line: $\cosh d(l, w) = \dfrac{|w|}{\mathrm{Im}\,w}$ [7, §7.20]. Substituting $z = e^{i\theta_0}$, $w = u + iv$, we obtain $1 + \dfrac{(u-\cos\theta_0)^2 + (v - \sin\theta_0)^2}{2v\sin\theta_0} = \dfrac{\sqrt{u^2+v^2}}{v}$. Simplifying, we obtain equation (2). To rewrite this equation we substitute in (2) $|w| = e^t$, $u = e^t\tanh r$ and obtain the same equation recorded in cylindrical coordinates.

Lemma 2. *a) The arclength of hyperbolic parabola P_H from the vertex $t = 0$ to the point $t = t_0$ is equal to*

$$s(t_0) = \int\limits_0^{t_0} \frac{\sqrt{2\tan\theta_0 f(t,\theta_0)}}{1 - f^2(t,\theta_0)}\,dt\,. \tag{3}$$

b) Cosine of the angle θ between normal to the parabola and the geodesic line, projecting the point $(r(t),t)$ to the axis l is the following:

$$\cos\theta = (\sinh 2r(t)\tan\theta_0)^{-1/2}. \qquad (4)$$

P r o o f. We use equation (2) and write it down in unexplicit form: $F(u,v) = (u^2+v^2+1-2u\cos\theta_0)^2 - 4\sin^2\theta_0(u^2+v^2) = 0$. Because of conformality Poincare and euclidien metrics vector of normal is parallel to vector (F_u, F_v), where

$$F_u = 2(u^2+v^2+1-2u\cos\theta_0)(2u-2\cos\theta_0) - 8\sin^2\theta_0\,,$$

$$F_v = 2(u^2+v^2+1-2u\cos\theta_0)2v - 8\sin^2\theta_0\,.$$

After reducing we can count that normal to parabola at the point (u,v) has the same direction as vector

$$\bar{N} = \left(-\sin\theta_0 u + \sqrt{u^2+v^2}(u-\cos\theta_0), -\sin\theta_0 v + \sqrt{u^2+v^2}v\right)\,.$$

It is clear, that geodesic which projects the point of parabola (u,v) to the l axis has just the same direction as vector $\bar{\tau} = (-v,u)$ at this point. Because of conformality between hyperbolic and euclidien metrics we can count the angle θ in the usual way:

$$\cos\theta = \frac{\langle\bar{N},\bar{\tau}\rangle}{|\bar{N}||\bar{\tau}|} = \frac{\sin\theta_0 uv - v(u-\cos\theta_0)\sqrt{u^2+v^2} - uv\sin\theta_0 + u\sqrt{u^2+v^2}}{\sqrt{u\sin 2\theta_0\sqrt{u^2+v^2}}\sqrt{u^2+v^2}}$$

$$= \frac{v\cos\theta_0}{\sqrt{u\sqrt{u^2+v^2}\sin 2\theta_0}} = (\sinh 2r(t)\tan\theta_0)^{-1/2}\,.$$

The length of arc of parabola of hyperbolic plane can be computed in cylindrical coordinates in the following way: $ds^2 = dr^2 + \cosh^2 r dt^2 = (r'^2 + \cosh^2 r)dt^2$. By differentiating equation (2') we obtain $(\tanh r)' = \dfrac{r'}{\cosh^2 r} = \dfrac{\sinh t}{\cos\theta_0}$. On the other hand,

$$\cosh^{-2} r = 1 - \tanh^2 r = \frac{\cos^2\theta_0 - (\cosh t - \sin\theta_0)^2}{\cos^2\theta_0}\,.$$

Hence

$$ds = \sqrt{r'^2 + \cosh^2 r}dt = \sqrt{\frac{\cosh^4 r\sinh^2 t}{\cos^2\theta_0} + \cosh^2 r}dt$$

$$= \cosh r\sqrt{\frac{\cosh^2 r\sinh^2 t}{\cos^2\theta_0} + 1}\,dt = \frac{\cos\theta_0\sqrt{2\sin\theta_0(\cosh t - \sin\theta_0)}dt}{\cos^2\theta_0 - (\cosh t - \sin\theta_0)^2}\,.$$

And Lemma 2 is proved.

1.4. The rolling of parabola along the straight line in H^2. We represent a process of rolling of parabola along geodesic in H^2 with help of a curve in the group $PSL(2,R)$ of proper isometries of H^2: $g_t(\zeta) = \dfrac{a(t)\zeta + b(t)}{c(t)\zeta + d(t)}$, where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are functions of real variable t; $ad - bc > 0$. The following lemma is central in the chain of arguments.

Lemma 3. *Curve $g_t(\zeta)$ for every $t \in [0,\infty)$ transforms parabola P_H with the focus $w_0 = e^{i\theta_0}$ and the directrix l $(\mathrm{Re}\, w = 0)$ by proper motion into parabola, tangent to axis l at the point, which is distanced on the length $s(t)$ from the point $w = i$. Required curve is determined by the equation*

$$g_t(\zeta) = e^{s(t)} \frac{a(t)\zeta + b(t)}{c(t)\zeta + d(t)}\,, \tag{5}$$

where

$$a(t) = \cos\frac{\theta}{2}(|w| - u + v) + \sin\frac{\theta}{2}(|w| - u - v)\,,$$

$$b(t) = \left(\cos\frac{\theta}{2}(|w| - u - v) + \sin\frac{\theta}{2}(|w| - u + v)\right)|w|\,,$$

$$c(t) = \cos\frac{\theta}{2}(|w| - u - v) + \sin\frac{\theta}{2}(-|w| + u - v)\,,$$

$$d(t) = \left(\cos\frac{\theta}{2}(|w| - u + v) + \sin\frac{\theta}{2}(-|w| + u + v)\right)|w|\,,$$

where $u(t)$, $v(t)$, $|w(t)| = |u(t) + iv(t)|$; $\theta(t)$ (the angle mentioned in Lemma 2) represent parameters of the point of parabola P_H distanced at the length $s(t)$ along the arc apart its vertex.

P r o o f. Required motion of parabola we realize in four steps, each of whose have evident geometric meaning.

1) The first motion $g_1(\zeta) = \dfrac{1}{|w|}\zeta$ represents homothety with center at origin. By this motion the point w passes to a point of unit circle $|\zeta| = 1$ and the angle θ of normal with projecting geodesic is preserved.

2) The second motion represents parallel translation of parabola along the geodesic $|\zeta| = 1$ from the point $w/|w|$ to the point i. Analytically it may be written so:

$$g_2(\zeta) = \frac{(1 + \tan\frac{\phi}{2})\zeta + \tan\frac{\phi}{2} - 1}{(-1 + \tan\frac{\phi}{2})\zeta + \tan\frac{\phi}{2} + 1}\,,$$

where $\phi = \arg w$. It is needful to check, that $g_2(e^{i\phi}) = i$; really,

$$g_2(e^{i\phi}) = \frac{\left(1 + \frac{1-\cos\phi}{\sin\phi}\right)e^{i\phi} + \frac{1-\cos\phi}{\sin\phi} - 1}{\left(-1 + \frac{1-\cos\phi}{\sin\phi}\right)e^{i\phi} + \frac{1-\cos\phi}{\sin\phi} + 1}$$

$$= \frac{(1+\sin\phi-\cos\phi)(\cos\phi+i\sin\phi)+1-\cos\phi-\sin\phi}{(1-\sin\phi-\cos\phi)(\cos\phi+i\sin\phi)+1-\cos\phi+\sin\phi}$$

$$= \frac{(1+\sin\phi-\cos\phi)i\sin\phi+\cos\phi+\sin\phi\cos\phi-\cos^2\phi+1-\cos\phi-\sin\phi}{(1-\sin\phi-\cos\phi)i\sin\phi+\cos\phi-\sin\phi\cos\phi-\cos^2\phi+1-\cos\phi+\sin\phi}$$

$$= \frac{(1+\sin\phi-\cos\phi)i-1+\cos\phi+\sin\phi}{(1-\sin\phi-\cos\phi)i+1-\cos\phi+\sin\phi} = i\,.$$

Then we have to verify, that vector $\tau' = -\sin\phi + i\cos\phi$, tangent to the geodesic $|\zeta| = 1$ at the point $e^{i\phi}$ under the action of differential of mapping g_2 passes to the vector $-\lambda \in R_-$, parallel to the real axis. Really,

$$g_2^*(\tau') = \frac{1}{(cz+d)^2}\tau' = \frac{(c\bar{z}+d)^2}{|cz+d|^4}\tau' = \left(\left(\tan\frac{\phi}{2}-1\right)e^{-i\phi}+1+\tan\frac{\phi}{2}\right)^2\frac{\tau'}{|cz+d|^4}$$

$$= \left(\left(\frac{1-\cos\phi}{\sin\phi}-1\right)(\cos\phi-i\sin\phi)+1+\frac{1-\cos\phi}{\sin\phi}\right)^2\frac{\tau'}{|cz+d|^4}$$

$$= (\sin\phi-\cos\phi+1+i(\sin\phi+\cos\phi-1))^2\frac{\tau'}{\sin^2\phi|cz+d|^4}$$

$$= (\sin^2\phi+\cos^2\phi+1+4\sin\phi-4\sin\phi\cos\phi-\sin^2\phi-\cos^2\phi-1$$

$$+2i(\sin^2\phi-(\cos\phi-1)^2)\frac{\tau'}{\sin^2\phi|cz+d|^4} = \frac{4(1-\cos\phi)(\sin\phi+\cos\phi)\tau'}{\sin^2\phi|cz+d|^4}$$

$$= \frac{4(1-\cos\phi)(\sin\phi+i\cos\phi)(-\sin\phi+i\cos\phi)}{\sin^2\phi|cz+d|^4} = \frac{4(1-\cos\phi)}{\sin^2\phi|cz+d|^4} \in R_-\,.$$

3) The third motion represents an anticlockwise rotation about the angle θ around the point i. It has the following view [7, §7.33.]: $g_3(\zeta) = \dfrac{\cos\frac{\theta}{2}\zeta+\sin\frac{\theta}{2}}{-\sin\frac{\theta}{2}\zeta+\cos\frac{\theta}{2}}$. After accomplishment third motion we obtain parabola contacting to axis l at the point i.

4) The fourth motion transforms parabola P_H to parabola touching the line l at the point $e^{is(t)}$ at hyperbolic distance $s(t)$ apart the point i. It is easy to see that $g_4(\zeta) = e^{s(t)}\zeta$. Composition $g(\zeta) = g_4\circ g_3\circ g_2\circ g_1(\zeta)$ has the following view:

$$g(\zeta) = e^s\frac{\cos\frac{\theta}{2}\frac{(1+\tan\frac{\phi}{2})\frac{\zeta}{|w|}+\tan\frac{\phi}{2}-1}{(-1+\tan\frac{\phi}{2})\frac{\zeta}{|w|}+\tan\frac{\phi}{2}+1}+\sin\frac{\theta}{2}}{-\sin\frac{\theta}{2}\frac{(1+\tan\frac{\phi}{2})\frac{\zeta}{|w|}+\tan\frac{\phi}{2}-1}{(-1+\tan\frac{\phi}{2})\frac{\zeta}{|w|}+\tan\frac{\phi}{2}+1}+\cos\frac{\theta}{2}},$$

where $\tan\dfrac{\phi}{2} = \dfrac{1-\cos\phi}{\sin\phi} = \dfrac{|w|-u}{v}$. Simplifying, we obtain required expression for $g_t(\zeta)$ in the following view:

$$g(\zeta) = e^s \frac{\cos\frac{\theta}{2}(a\zeta + b|w|) + \sin\frac{\theta}{2}(b\zeta + a|w|)}{-\sin\frac{\theta}{2}(a\zeta + b|w|) + \cos\frac{\theta}{2}(b\zeta + a|w|)}$$

(where $a = |w| + v - u$, $b = |w| - u - v$), which coincide with (5), q.e.d.

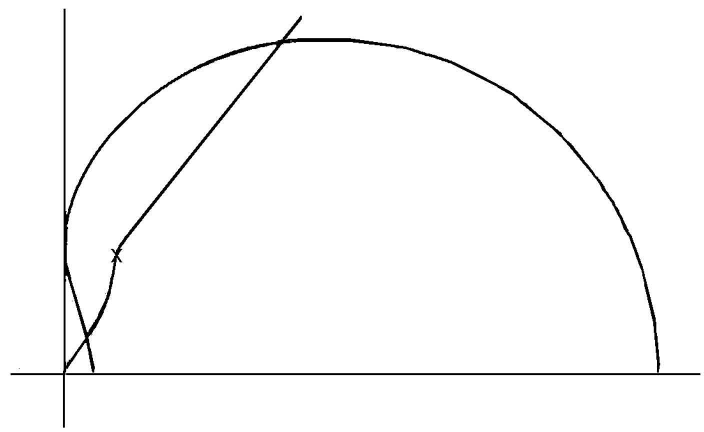

Fig. 2. Moved parabola touching axis $\mathrm{Re}\,\zeta = 0$ at the point i and corresponding generatrix, passing through its focus.

Lemma 4. *Trajectory which the focus $e^{i\theta_0}$ of parabola P_H describes during its rolling along the line l has the following equation in the complex plane w (fig. 2):*

$$\Gamma(t) = g_t(e^{i\theta_0}) = e^{s(t)} \frac{\left(\cos\theta_0\sqrt{f(t,\theta_0)} + i\sqrt{\sin 2\theta_0}\right)\sqrt{1-f^2(t,\theta_0)}}{\sinh t\sqrt{f(t,\theta_0)} + i\sqrt{\sin 2\theta_0}}. \qquad (6)$$

P r o o f. If we substitute in formula (5) instead of ζ the focus $e^{i\theta_0}$ and assume for the sake of brevity $|w| + v - u = a$, $|w| - u - v = b$, then we get equation of the trajectory described by the focus of parabola during its rolling along the line:

$$l : \Gamma(t) = e^{s(t)} \frac{\left(a\cos\frac{\theta}{2} + b\sin\frac{\theta}{2}\right)e^{i\theta_0} + |w|\left(a\sin\frac{\theta}{2} + b\cos\frac{\theta}{2}\right)}{\left(-a\sin\frac{\theta}{2} + b\cos\frac{\theta}{2}\right)e^{i\theta_0} + |w|\left(a\cos\frac{\theta}{2} - b\sin\frac{\theta}{2}\right)}.$$

Now we calculate the fraction multiplying numerator and denominator by expression, conjugate to the denominator. Then the numerator is equal to

$$\left(\left(a\cos\frac{\theta}{2}+b\sin\frac{\theta}{2}\right)e^{i\theta_0}+|w|\left(a\sin\frac{\theta}{2}+b\cos\frac{\theta}{2}\right)\right)\left(\left(-a\sin\frac{\theta}{2}+b\cos\frac{\theta}{2}\right)e^{-i\theta_0}\right.$$

$$\left.+|w|\left(a\cos\frac{\theta}{2}-b\sin\frac{\theta}{2}\right)\right)=ab\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)+(b^2-a^2)\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$+e^{i\theta_0}|w|\left(a^2\cos^2\frac{\theta}{2}-b^2\sin^2\frac{\theta}{2}\right)+e^{-i\theta_0}|w|\left(-a^2\cos^2\frac{\theta}{2}+b^2\sin^2\frac{\theta}{2}\right)$$

$$-|w|^2\left(ab\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)+(a^2-b^2)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)=ab\cos\theta+\frac{1}{2}\sin\theta(b^2-a^2)$$

$$+|w|^2\left(ab\cos\theta+\frac{1}{2}(a^2-b^2)\sin\theta\right)+|w|\cos\theta_0\left(\cos^2\frac{\theta}{2}(a^2+b^2)-\sin^2\frac{\theta}{2}(a^2+b^2)\right)$$

$$=\cos\theta ab(|w|^2+1)+\frac{1}{2}\sin\theta(a^2-b^2)(|w|^2+1)+|w|\cos\theta_0\cos\theta_0(a^2+b^2)$$

$$+|w|\sin\theta_0(a^2-b^2)i\,.$$

Calculating the denominator, we obtain

$$\left(\left(-a\sin\frac{\theta}{2}+b\cos\frac{\theta}{2}\right)e^{i\theta_0}+|w|\left(a\cos\frac{\theta}{2}-b\sin\frac{\theta}{2}\right)\right)$$

$$\times\left(\left(-a\sin\frac{\theta}{2}+b\cos\frac{\theta}{2}\right)e^{-i\theta_0}+|w|\left(a\cos\frac{\theta}{2}-b\sin\frac{\theta}{2}\right)\right)$$

$$=\left(b\cos\frac{\theta}{2}-a\sin\frac{\theta}{2}\right)^2+|w|^2\left(a\cos\frac{\theta}{2}-b\sin\frac{\theta}{2}\right)^2$$

$$+2\cos\theta_0|w|\left(\cos^2\frac{\theta}{2}ab+\sin^2\frac{\theta}{2}ab-\sin\frac{\theta}{2}\cos\frac{\theta}{2}(a^2+b^2)\right)$$

$$=\cos^2\frac{\theta}{2}(b^2+a^2|w|^2)+\sin^2\frac{\theta}{2}(a^2+b^2|w|^2)-\sin\theta ab(1+|w|^2)$$

$$+2\cos\theta_0|w|\left(ab-\frac{1}{2}\sin\theta(a^2+b^2)\right)=\frac{\cos\theta+1}{2}(b^2+a^2|w|^2)$$

$$+\frac{1-\cos\theta}{2}(a^2+b^2|w|^2)-\sin\theta ab(1+|w|^2)+|w|\cos\theta_0(2ab-\sin\theta(a^2+b^2))$$

$$=\frac{1}{2}\cos\theta(a^2-b^2)(|w|^2-1)+\frac{1}{2}(a^2+b^2)(|w|^2+1)-ab(1+|w|^2)\sin\theta$$

$$+|w|\cos\theta_0(2ab-\sin\theta_0(a^2+b^2))\,.$$

Since $ab=(|w|-u)^2-v^2=2u(u-|w|)$, $a^2+b^2=4|w|(|w|-u)$, $a^2-b^2=4v(|w|-u)$, we can reduce numerator and denominator on $2(|w|-u)$. Then

$$\Gamma(t) = e^{s(t)}$$
$$\times \frac{-u(|w|^2+1)\cos\theta + v(|w|^2-1)\sin\theta + 2|w|^2\cos\theta_0\cos\theta + 2|w|v\sin\theta_0 i}{v(|w|^2-1)\cos\theta + u(1+|w|^2)\sin\theta + |w|(|w|^2+1) - 2|w|\cos\theta_0(u+|w|\sin\theta)}\,.$$

It is convenient to make subsequent simplifications in cylindrical coordinates. We remind that $u = e^t\tanh r$, $v = \dfrac{e^t}{\cosh r}$, $|w| = e^t$. Then the numerator of the last fraction is equal to

$$-e^t\tanh r(e^{2t+1})\frac{\sqrt{1-f^2(t,\theta_0)}}{\sqrt{2\tan\theta_0 f(t,\theta_0)}} + e^t\sqrt{1-f^2(t,\theta_0)}\frac{(e^{2t}-1)\sinh t}{\cos\theta_0\sqrt{2\tan\theta_0 f(t,\theta_0)}}$$
$$+2e^{2t}\cos\theta_0\frac{\sqrt{1-f^2(t,\theta_0)}}{\sqrt{2\tan\theta_0 f(t,\theta_0)}} + 2e^{2t}\sqrt{1-f^2(t,\theta_0)}\sin\theta_0 i$$
$$= 2e^{2t}\sqrt{1-f^2}\left(\frac{-f\cosh t\cos\theta_0 + \sinh^2 t + \cos^2\theta_0}{\cos\theta_0\sqrt{2\tan\theta_0 f}} + i\sin\theta_0\right)$$
$$= 2e^{2t}\sqrt{1-f^2}\left(\frac{\sin\theta_0(\cosh t - \sin\theta_0)}{\sqrt{2\tan\theta_0 f}\cos\theta_0} + i\sin\theta_0\right)$$
$$= 2e^{2t}\sqrt{1-f^2}\sin\theta_0\left(\sqrt{\frac{f}{2\tan\theta_0}} + i\right)\,.$$

Denominator is equal to

$$e^t\sqrt{1-f^2}(e^{2t}-1)\sqrt{\frac{1-f^2}{2\tan\theta_0 f}} + e^t f(1+e^{2t})\frac{\sinh t}{\cos\theta_0\sqrt{2\tan\theta_0 f}}$$
$$+2\sin\theta_0 e^{2t} - 2e^{2t}\cos\theta_0\frac{\sinh t}{\cos\theta_0\sqrt{2\tan\theta_0 f}}$$
$$= 2e^{2t}\left(\frac{\sinh t(1-f^2)\cos\theta_0 + \sin t\cosh t f - \cos\theta_0\sinh t}{\cos\theta_0\sqrt{2\tan\theta_0 f}} + \sin\theta_0\right)$$
$$= 2e^{2t}\left(\frac{\sinh t\cos\theta_0 - f^2\cos\theta_0\sinh t + \sinh t\cosh t f - \cos\theta_0\sinh t}{\cos\theta_0\sqrt{2\tan\theta_0 f}} + \sin\theta_0\right)$$
$$= 2e^{2t}\sin\theta_0\left(\frac{\sinh t\sqrt{f}}{\sqrt{2\sin\theta_0\cos\theta_0}} + 1\right)\,.$$

From here follows formula (6), q.e.d.

We denote by $X(t) = \operatorname{Re}\Gamma(t)$, $Y(t) = \operatorname{Im}\Gamma(t)$ and pass to cylindrical coordinates (R, T) using the formulas $\cosh R = \dfrac{\sqrt{X^2+Y^2}}{Y}$, $T = \frac{1}{2}\log(X^2+Y^2)$. According to Lemma 4, we can represent variable T in the form

$$T = s(t)\frac{1}{2}\log(1-f^2(t,\theta_0)) + \frac{1}{2}\log\left(\frac{f(t,\theta_0)}{2\tan\theta_0}+1\right) - \log\left(\sinh t\sqrt{\frac{f(t,\theta_0)}{2\sin\theta_0}}+1\right)$$

Remind that function $f(t,\theta_0)$ is defined by equation $(2')$.

Lemma 5. *The derivatives of functions $R(t)$, $T(t)$ with respect to parameter t are*

$$\frac{dR}{dt}\frac{\sinh t}{2\sqrt{\cosh^2 t - \sin^2\theta_0}}, \quad \frac{dT}{dt} = \frac{\sqrt{2\tan\theta_0}}{2\sqrt{f(t,\theta_0)}(f(t,\theta_0)+2\tan\theta_0)}.$$

P r o o f. From the definition of R it follows that $\sinh R = \dfrac{X}{Y} = \dfrac{\sqrt{f(t,\theta_0)}}{\sqrt{2\tan\theta_0}}$. Differentiating with respect to t, we have $dR = \dfrac{d(\sinh R)}{\cosh R}$. On the other hand,

$$d(\sinh R) = d\left(\sqrt{\frac{f(t,\theta_0)}{2\tan\theta_0}}\right) = \frac{1}{2}\frac{1}{\sqrt{f(t,\theta_0)}}\frac{\sinh t}{\cos\theta_0}\frac{dt}{\sqrt{2\tan\theta_0}},$$

$$\cosh R = \frac{\sqrt{X^2+Y^2}}{Y} = \left(\frac{f(t,\theta_0)}{2\tan\theta_0}+1\right)^{1/2}. \tag{7}$$

Hence

$$dR = \frac{1}{\sqrt{\frac{f}{2\tan\theta_0}+1}}\frac{1}{\sqrt{2\tan\theta_0}}\frac{1}{2\sqrt{f}}\frac{\sinh t}{\cos\theta_0}\,dt = \frac{1}{f+\sqrt{2\tan\theta_0}}\frac{1}{2\cos\theta_0}\frac{\sinh t}{\sqrt{f}}$$

$$= \frac{1}{2}\frac{1}{\sqrt{\cosh t+\sin\theta_0}}\frac{\sinh t}{\sqrt{\cosh t-\sin\theta_0}}\,dt = \frac{\sinh t\,dt}{\sqrt{\cosh^2 t-\sin^2\theta_0}}.$$

So, we have proved the first formula. Let us prove the formula for $\dfrac{dT}{dt}$. We have

$$\frac{dT}{dt} = \frac{\sqrt{2\tan\theta_0}}{1-f^2} - \frac{f\frac{\sinh t}{\cos\theta_0}}{1-f^2} + \frac{\frac{1}{4\tan\theta_0}\frac{\sinh t}{\cos\theta_0}}{\frac{f}{2\tan\theta_0}+1} - \frac{1}{\sqrt{2\sin\theta_0}}\frac{\cosh t\sqrt{f}+\frac{1}{2\sqrt{f}}\frac{\sinh^2 t}{\cos\theta_0}}{\frac{\sinh t\sqrt{f}}{\sqrt{\sin 2\theta_0}}+1}$$

$$= \frac{\sqrt{f}\left(\sqrt{2\tan\theta_0}-\frac{\sqrt{f}\sinh t}{\cos\theta_0}\right)}{1-f^2} + \frac{\sinh t}{2(\cosh t+\sin\theta_0)} - \frac{2f\cosh t\cos\theta_0+\sinh^2 t}{2\sqrt{f}\cos\theta_0(\sinh t\sqrt{f}+\sqrt{\sin 2\theta_0})}$$

$$= \frac{2f(\sin 2\theta_0 - f\sinh^2 t) - (1-f^2)(2\cosh t\cos\theta_0 + \sinh^2 t)}{2\sqrt{f}\cos\theta_0(1-f^2)(\sinh t\sqrt{f} + \sqrt{\sin 2\theta_0})} + \frac{\sinh t}{2(\cosh t + \sin\theta_0)}$$

$$= \frac{\cos^2\theta_0(f^2-1)}{2\sqrt{f}\cos\theta_0(1-f^2)(\sinh t\sqrt{f} + \sqrt{\sin 2\theta_0})} + \frac{\sinh t}{2\cos\theta_0(f + 2\tan\theta_0)}$$

$$= \frac{\cos^2\theta_0(1-f^2)(f+2\tan\theta_0) + f\sinh^2 t + \sqrt{f}\sinh t\sqrt{\sin 2\theta_0}}{2\cos\theta_0(f\sinh t + \sqrt{f\sin 2\theta_0})(f+2\tan\theta_0)}$$

$$= \frac{\sqrt{f\sin 2\theta_0}}{2\cos\theta_0\sqrt{f}(f+2\tan\theta_0)} = \frac{\sqrt{2\tan\theta_0}}{2\sqrt{f}(f+2\tan\theta_0)},$$

q.e.d.

Lemma 6. *Derivative* $\frac{dR}{dT}$ *satisfies the following differential equation:*

$$\left(\frac{dR}{dT}\right)^2 + \cosh^2 R = 4\tan^2\theta_0 \sinh^2 R\cosh^4 R.$$

P r o o f. From the previous lemma we can obtain expression for $\frac{dR}{dT}$:

$$\frac{dR}{dT} = \frac{\sinh t}{2\cos\theta_0\sqrt{f}\sqrt{f+2\tan\theta_0}} \frac{2\sqrt{f}(f+2\tan\theta_0)}{\sqrt{2\tan\theta_0}} = \frac{\sinh t\sqrt{f+2\tan\theta_0}}{\sqrt{\sin 2\theta_0}}.$$

Now we compute the left hand side of stated formula, using (7):

$$\left(\frac{dR}{dT}\right)^2 + \cosh^2 r = \frac{\sinh^2 t(f+2\tan\theta_0)}{\sin 2\theta_0} + \frac{f}{2\tan\theta_0} + 1$$

$$= (f+2\tan\theta_0)\left(\frac{\sinh^2 t}{\sin 2\theta_0} + \frac{1}{2\tan\theta_0}\right) = \frac{f+2\tan\theta_0}{2\tan\theta_0}\frac{(-1+\cosh^2 t + \cos^2\theta_0)}{\cos^2\theta_0}$$

$$= \frac{f+2\tan\theta_0}{2\tan\theta_0}\frac{\cosh t - \sin\theta_0}{\cos\theta_0}\frac{\cosh t + \sin\theta_0}{\cos\theta_0} = \frac{f+2\tan\theta_0}{2\tan\theta_0} f(f+2\tan\theta_0)$$

$$= \frac{f+2\tan\theta_0}{2\tan\theta_0}\frac{f}{2\tan\theta_0}\frac{f+2\tan\theta_0}{2\tan\theta_0} 4\tan^2\theta_0 = 4\tan^2\theta_0\cosh^4 R\sinh^2 R,$$

q.e.d.

1.5. Proof of the theorem for hyperbolic catenoid. It follows from Lemma 6 and Corollary 2 that curve $\Gamma(t)$ satisfies the differential equation characterizing generatrix of catenoid of revolution of hyperbolic space. It is easy to verify that origin of the curve $\Gamma(0)$ (see Lemma 4) coincides with position of the focus of parabola touching at its vertex the straight line l at the moment $t = 0$. By virtue of uniqueness of solution of first order differential equation with given initial condition focus of parabola during its rolling describes the curve $\Gamma(t)$.

2. Property of generatrix of spherical catenoid

2.1. Cylindrical coordinates in S^3. Now we consider the unite sphere S^3 with the metric $ds^2 = d\phi^2 + \cos^2\phi\, d\varphi^2 + \sin^2\phi\, d\lambda^2$. The parameters ϕ, φ, λ have the following geometric meaning: $0 \leq \varphi \leq 2\pi$ measures arclength along the chosen geodesic (the axis of spherical cylindrical coordinates), $0 \leq \phi \leq \pi$ is the distance from the axis $\phi = 0$ to the point (φ, ϕ, λ), and the parameter λ determines the direction in the totally geodesic sphere $\varphi = const$. Since the method of proof of spherical version of theorem is similar to hyperbolic one we try to contract our exposition.

2.2. Differential equation of generatrix of spherical catenoid. In the sphere $\lambda = 0$ we choose meridian $(\varphi, \phi(\varphi), 0)$ that generates spherical catenoid of revolution. Arguing as in 1.2, we may state the following assertion.

Lemma 7. *Generatrix of spherical catenoid of revolution is determined by initial condition $(\varphi_0, \phi(\varphi_0))$ and satisfies the differential equation*

$$(\phi_\varphi)^2 = \cos^2\phi(a^2\sin^2\phi\cos^2\phi - 1), \ a = const.$$

2.3. Analog of the spherical parabola. We determine set of points on the sphere by the rule: $d(F, M) = d(l, M)$, where $F(\varphi = 0, \phi = \phi_0)$ is the focus and equator $l(\phi = 0)$ is the directrix of spherical parabola. The definition may be restated in the following way: $d(F, M) + d(l^*, M) = \pi/2$, where l^* is the pole of the geodesic l.

Lemma 8. *Equation of spherical parabola with the focus $F(\varphi = 0, \phi = \phi_0)$ and the directrix $l(\phi = 0)$ has the form*

$$\tan\phi = \frac{1 - \cos\varphi\cos\phi_0}{\sin\phi_0} = g(\varphi, \phi_0)\,. \tag{8}$$

P r o o f. For spherical distance d between the points F and $M(\varphi, \phi)$ we have $\cos d(F, M) = \cos\varphi\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0$. On the other hand, spherical distance from $M(\varphi, \phi)$ to equator l is ϕ. From here the desired equation follows.

Lemma 9. *a) The differential of arclength of spherical parabola is equal to*

$$ds = \sqrt{\frac{2g(\varphi, \phi_0)}{\sin\phi_0}}\frac{d\varphi}{1 + g(\varphi, \phi_0)^2}\,.$$

b) The angle θ between normal to spherical parabola and the geodesic that projects the point of parabola to l is equal to

$$\cos\theta = \sqrt{\frac{\sin\phi_0}{\sin 2\phi}} = \sqrt{\frac{\sin\varphi_0(1 + g^2)}{2g}}\,.$$

P r o o f. To obtain expression for the differential of arclength we use (8) and metric form of the sphere $ds^2 = d\phi^2 + \cos^2\phi\, d\varphi^2$.

b) The mentioned angle θ is equal to the angle between tangent vector to parabola (ϕ', φ') and vector $(0,1)$ tangent to the parallel ($\phi = const$). Since $1 = (\phi')^2 + \cos^2\phi(\varphi')^2$ we have $\cos\theta = \cos\phi\varphi'$. On the other hand, differentiating (8), we get $\dfrac{\phi'}{\cos^2\phi} = \cot\phi_0 \sin\varphi\varphi'$ and the statement follows.

2.4. Process of rolling of spherical parabola P_s along the equator l. By $R_z(s)$ we denote the rotation about the angle s around Oz axis (anticlockwise in xOy plane). The next statement is clear from visual geometric reasons.

Lemma 10. *Motion $A(s) = R_z(s)R_x(-\theta)R_y(-\phi)R_z(-\varphi)$ is the motion of the sphere that transforms the spherical parabola P_s to parabola touching the geodesic l (equator) at the point $(s, 0)$.*

Successively multiplying four matrices, we receive expression for $A(s)$. The following statement may be verified by direct calculations.

Lemma 11. *Trajectory which the focus $F(\cos\phi_0, 0, \sin\phi_0)^t$ of spherical parabola P_s describes during its rolling along the equator l has the following view:*

$$\Gamma(s) = A(s)F = (X(s), Y(s), Z(s))^t,$$

where

$$X(s) = (\cos s\cos\phi\cos\varphi + \sin s\cos\theta\sin\varphi + \sin s\sin\theta\sin\phi\cos\varphi)\cos\phi_0$$

$$+(\cos s\sin\phi - \sin s\sin\theta\cos\phi)\sin\phi_0\,,$$

$$Y(s) = (\sin s\cos\phi\cos\varphi - \cos s\cos\theta\sin\varphi - \cos s\sin\theta\sin\phi\cos\varphi)\cos\phi_0$$

$$+(\sin s\sin\phi + \cos s\sin\theta\cos\phi)\sin\phi_0\,,$$

$$Z(s) = (\sin\theta\cos\varphi - \sin\phi\cos\varphi\cos\theta)\cos\phi_0 + \cos\theta\cos\phi\sin\phi_0\,.$$

2.5. Proof of the spherical version of theorem. We have to check that spherical curve $\Gamma(s)$ satisfies to differential equation cited in Lemma 7. Therefore we have to rewrite this equation in terms of coordinates of $\Gamma(s)$ introduced by the rule $\sin\phi = Z$, $\tan\varphi = \frac{Y}{X}$. From here we deduce

$$\phi' = \frac{Z'}{\cos\phi} = \frac{Z'}{\sqrt{1-Z^2}},\ \varphi' = \frac{Y'X - X'Y}{X^2+Y^2}\,,\ \frac{d\phi}{d\varphi} = \frac{Z'\sqrt{1-Z^2}}{Y'X - X'Y}\,.$$

Substituting in the equation cited in Lemma 7 instead of $\dfrac{d\phi}{d\varphi}$ its expression in terms of functions $X(s)$, $Y(s)$, $Z(s)$, we receive the following differential equation to be verified $\left(\frac{Z'}{Y'X-X'Y}\right)^2 = a^2Z^2(1-Z^2)-1$. First of all we calculate $X(s)$, $Y(s)$, $Z(s)$:

$$Z(s) = \left(\frac{\cos\phi_0}{\sqrt{2\sin\phi_0}}\sin\varphi\frac{1}{\sqrt{g}}\sin\varphi - \frac{g}{\sqrt{1+g^2}}\cos\varphi\sqrt{\frac{\sin\phi_0}{2}}\sqrt{\frac{1+g^2}{g}}\right)\cos\phi_0$$

$$+\sqrt{\frac{\sin\phi_0}{2}}\sqrt{\frac{1+g^2}{g}}\frac{1}{\sqrt{1+g^2}}\sin\phi_0$$

$$= \frac{1}{\sqrt{g}}\sqrt{\frac{\sin\phi_0}{2}}\left(\frac{\cos^2\phi_0}{\sin\phi_0}\sin^2\varphi - g\cos\varphi\cos\phi_0 + \sin\phi_0\right)$$

$$= \sqrt{\frac{\sin^3\phi_0}{2g}}\left(\cot^2\phi_0 + 1 - \frac{\cos\varphi\cos\phi_0}{\sin^2\phi_0}\right) = \sqrt{\frac{g\sin\phi_0}{2}}.$$

Here we used formulas:

$$g = \frac{1-\cos\varphi\cos\phi_0}{\sin\phi_0},\ \ \varphi = \sqrt{\frac{\sin\phi_0}{2}}\frac{1+g^2}{\sqrt{g}},$$

$$\phi' = \cos^2\phi\cot\phi_0\sin\varphi\varphi' = \frac{\cos\phi_0}{\sqrt{2\sin\phi_0}}\sin\varphi\frac{1}{\sqrt{g}},$$

$$\cos\theta = \cos\phi\varphi' = \sqrt{\frac{\sin\phi_0}{2}}\sqrt{\frac{1+g^2}{g}},\ \ \sin\theta = \phi',\ \ \sin\phi = \frac{g}{\sqrt{1+g^2}}.$$

Similarly we calculate

$$X(s) = \frac{1}{\sqrt{1+g^2}}(\cos s + F(s)\sin s),\ \ Y(s) = \frac{1}{\sqrt{1+g^2}}(\sin s - F(s)\cos s),$$

where $F(s) = \cos\phi_0\sin\varphi\sqrt{\dfrac{g}{2\sin\phi_0}}$.

Lemma 12. *a)* $Z' = \frac{1}{4}\cos\phi_0\sin\varphi\dfrac{1+g^2}{g}$, *b)* $Y'X - X'Y = \sin\phi_0\dfrac{1+g^2}{4g}$.

P r o o f. a)

$$Z' = \sqrt{\frac{\sin\phi_0}{2}}\frac{d}{ds}\sqrt{g} = \sqrt{\frac{\sin\phi_0}{2}}\frac{1}{2\sqrt{g}}\frac{\cos\phi_0}{\sqrt{2\sin\phi_0}}\sin\varphi\frac{1+g^2}{\sqrt{g}} = \frac{\cos\phi_0\sin\varphi(1+g^2)}{4g}.$$

b) Since $Y'X - X'Y = \left(\frac{Y}{X}\right)' X^2$, we have

$$Y'X - X'Y = \frac{d}{ds}\left(\frac{\sin s - F(s)\cos s}{\cos s - F(s)\sin s}\right)\frac{(\cos s + F(s)\sin s)^2}{1+g^2} = \frac{1 + F^2(s) - F'(s)}{1+g^2}.$$

Calculation of the numerator yields $\dfrac{\sin\phi_0(1+g^2)^2}{4g}$, therefore formula b) is valid.

Now we are able to point out explicit value of constant a mentioned in Lemma 7.

Lemma 13. $\left(\dfrac{Z'}{Y'X - X'Y}\right)^2 = \left(\dfrac{2}{\sin\phi_0}\right)^2 Z^2(1-Z^2) - 1.$

P r o o f. First, using previous lemma, we have

$$\left(\frac{Z'}{Y'X - X'Y}\right)^2 + 1 = \cot^2\phi_0 \sin^2\varphi + 1$$

$$= \frac{2g\sin\phi_0 - \sin^2\phi_0(1+g^2)}{\sin^2\phi_0} + 1 = g\left(\frac{2}{\sin\phi_0} - g\right).$$

On the other hand,

$$a^2Z^2(1-Z^2) = a^2\frac{g}{2}\sin\phi_0\left(1 - \frac{g}{2}\sin\phi_0\right) = a^2\frac{\sin^2\phi_0}{4}g\left(\frac{2}{\sin\phi_0} - g\right).$$

It is clear that if we take $a = \dfrac{2}{\sin\phi_0}$, both expressions coincide, q.e.d.

Finally, to prove spherical version of the theorem we may repeat without changes arguments given in 1.5.

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Генератриса катеноида пространственной 3-формы

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Поверхности вращения постоянной средней кривизны в евклидовом трехмерном пространстве известны как поверхности Ш. Делоне. Они обладают замечательным свойством : их профильные кривые (генератрисы) есть траектории фокусов конических сечений при их качении вдоль прямой. Реализована аналогичная конструкция в пространственных формах H^3 и S^3 . Изложение ограничивается случаем минимальных поверхностей и доказательством следующей теоремы.

Теорема. *Генератрисой катеноида вращения пространственной формы $H^3(S^3)$ является траектория фокуса гиперболической (сферической) параболы при ее качении вдоль геодезического луча.*

Генератриса катеноїда просторової 3-форми

Л.О. Масальцев

Поверхні обертання постійної середньої кривини в евклідовому тривимірному просторі відомі як поверхні Ш. Делоне. Вони мають видатну властивість: їх профільні криві (генератриси) є траєкторії фокусів конічних перерізів при їх котінні вздовж прямої. Аналогічну конструкцію реалізовано в просторових формах H^3 і S^3. Виклад обмежено випадком мінімальних поверхонь і доведенням наступної теореми.

Теорема. *Генератрисою катеноїда обертання просторової форми $H^3(S^3)$ є траєкторія фокуса гіперболічної (сферичної) параболи при її котінні вздовж геодезічного променя.*