Matematicheskaya fizika, analiz, geometriya
1999, v. 6, No. 1/2, p. 124–129

Transitive actions have funny rank one

Alexandre M. Sokhet

State Institute for Labour and Social Economic Researches,
1 Sumskaya Str., 310003, Kharkov, Ukraine

Received November 3, 1998

Here we prove that any transitive action of a locally compact separable group has funny rank one. This fact has been already proved for the special case of a solvable group. Our theorem is proved, primarily, for the Lie group action case, and then we generalize it by using the approximation of topological groups by Lie groups. We obtain as a natural corollary that the discrete spectrum implies funny rank one.

We prove in this article that any transitive action of a locally compact separable (l.c.s.) group has funny rank one. This fact has been already proved for the special case of an Abelian group in [1] and for the special case of a solvable group in [2] and [5], but for the general case the question was open and the methods used in the solvable case were inapplicable in the general one. Our theorem is proved, primarily, for a l.c.s. Lie group action case, and then we generalize it by using the approximation of topological groups by Lie groups. We obtain as a natural corollary that discrete spectrum implies funny rank one.

A G-action on a Lebesgue space $(\Omega, \mathfrak{B}, \mu)$ is called *transitive*, when almost all the points of Ω belong to the same orbit. It is clear that transitivity implies ergodicity. It is also well known that any transitive action is isomorphic to the translation action of G on the homogeneous space with respect to a suitable closed subgroup.

Definition. *An action α of a group G on a Lebesgue space $(\Omega, \mathfrak{B}, \mu)$ is said to have funny rank one, if for each $A_1, A_2, ..., A_n \in \mathfrak{B}$, $\mu(A_i) < \infty$, and every $\varepsilon > 0$ one can find $E_0 \in \mathfrak{B}$ and a finite set $\Lambda \subset G$, and subsets $\Lambda_i \subset \Lambda$, $i = 1, ..., n$, and numbers $r(g) \in \mathbb{R}_+$, $g \in \Lambda$, such that the sets $g \cdot E_0$ are disjoint for $g \in \Lambda$, and*

$$\mu(A_i \bigtriangleup \bigcup_{g\in\Lambda_i} g \cdot E_0) < \varepsilon, \tag{1}$$

$$\sum_{g\in\Lambda}\int_{E_0} \mid \frac{d\mu\circ g}{d\mu} - r(g) \mid d\mu(\omega) < \varepsilon. \tag{2}$$

The collection $\{g\cdot E_0 : g\in\Lambda\}$ is usually referred as a stack or a Rohlin tower, and E_0 as its base.

It is evident that funny rank one implies ergodicity. In the measure-preserving case, the second condition vanishes; in the general case, it provides the invariance of the class of funny rank one actions under the choice of an equivalent measure [5].

Theorem 1. *A transitive action of a l.c.s. Lie group has funny rank one.*

P r o o f. Let G be a l.c.s. Lie group over $\mathbb{R}$, H its closed Lie subgroup, $X = G/H$ the left co-set space that will be considered as a n-dimensional manifold over $\mathbb{R}$. Let G act on X by translations: $g\cdot g_0H = (gg_0)H$. This action is smooth. We choose some smooth G-quasiinvariant measure on X and denote it by μ.

Let $\varepsilon > 0$ and $A_1, A_2, ..., A_m$ be any given finite collection of measurable subsets of X. Our purpose is to approximate them by G-stack and to satisfy the conditions (1), (2). We may assume that these sets are contained in some compact set K and have null boundary.

Let now K_1 be a compact subset of G, such that for any $x\in K\subset X$ there exists $g\in K_1\subset G$, $gH = x$.

Fix some $\varepsilon', \sigma > 0$ such that for each $i = 1,\ldots,m$:

$$\mu(A_i)\cdot\left(\frac{1+\varepsilon'}{1-\varepsilon'} - (1-\varepsilon')^n\right) < \frac{\varepsilon}{6}, \tag{3}$$

$$\sigma\frac{1+\varepsilon'}{1-\varepsilon'} < 1, \quad \sigma\frac{1+\varepsilon'}{1-\varepsilon'}\mu(A_i) < \varepsilon/4, \tag{4}$$

$$(1+\varepsilon')^2(1+\varepsilon') < 1+\sigma, \quad (1-\varepsilon')^2(1-\varepsilon') > 1-\sigma. \tag{5}$$

Choose two neighbourhoods, U of $e\in H$, and V of $eH\in X$, so that
(i) there exist a local map of V, $\varphi : V\to\mathbb{R}^n$;
(ii) for any $x_1, x_2\in V$ there exists $g\in U$ such that $g\cdot x_1 = x_2$;
(iii) $\varphi(V)$ is a zero-centered cube in $\mathbb{R}^n$;
(iv) let ρ be the usual measure on $\mathbb{R}^n$; for any measurable set $A\subset V$,

$$(1-\varepsilon')\rho(\varphi(A)) < \mu(A) < (1+\varepsilon')\rho(\varphi(A));$$

(v) for any $g\in U$, consider the mapping $x\mapsto g\cdot x$ for those of $x\in V$ that $gx\in V$ also. Then, this mapping is such that the Jacoby matrix

$$\Delta_g(x) = \left(\frac{\partial_j \varphi(g\cdot x)}{\partial_i \varphi(x)}\right)_{i,j=1}^{n}$$

is close to the unit matrix I, i.e., $\| \Delta_g(x) - I \| < \varepsilon'$ (here $\| \cdot \|$ means the operator norm), and simultaneously $|\det\Delta_g(x) - 1| < \varepsilon'$;

(vi) for any $g \in K_1$ there exists $r \in \mathbb{R}_+$ such that $x \in V$ implies

$$\left| \frac{d\mu \circ g}{d\mu}(x) - r \right| < \frac{\varepsilon}{4\mu(K)}.$$

There exists a finite covering of K by the copies of V having the form $V_k' = g_k \cdot V,\ g_k \in G,\ k = 1, ..., M$. Now let $V_1 = V_1',\ V_2 = V_2' \setminus V_1, \ldots, V_M = V_M' \setminus \bigcup\limits_{k=1}^{M-1} V_k$. The covering of K by V_k is disjoint, and each V_k has null boundary, and $g_k^{-1} \cdot V_k \subset V$. We may assume that $g_k \in K_1$, and hence there exist now $r_k \in \mathbb{R}$ determined by these g_k according to (vi).

Note that (vi) implies

$$\sum_{k=1}^{M} \int\limits_{g_k^{-1} V_k} \left| \frac{d\mu \circ g_k}{d\mu}(x) - r_k \right| d\mu(x) < \frac{\varepsilon}{4}. \tag{6}$$

For each k, $\partial V_k = Cl(V_k) \setminus Int(V_k)$ is contained in a finite union of submanifolds of dimension $n-1$.

Our next purpose is to consider, for each fixed $k = 1, ..., M$, the sets $A_{ik} = g_k^{-1} \cdot (A_i \bigcap V_k),\ i = 1, ..., m$ (note that $A_{ik} \subset V$) and to approximate them by G-stacks up to some $\delta(k) > 0$ so that the diameter of each element of these stack would be less than some $\varepsilon_0 > 0$, while the base element of all these stacks is common. When this is done, one can drop out the elements of the stack which g_k-images intersect with ∂V_k. It is evident that the sum of their measures, for each k, is $\underline{O}(\varepsilon_0)$. Now consider the elements of the stacks shifted by the corresponding g_k, and one will obtain the desired approximation of the initial sets A_i up to $\underline{O}(\varepsilon_0) \cdot M + \sum\limits_{k=1}^{M} \delta(k)$. To make this expression less than the pre-given ε, it suffices to take some appropriate ε_0 and to provide that $\sum\limits_{k=1}^{M} \delta(k) < \frac{\varepsilon}{2}$.

Let $B_{ik} = \varphi(A_{ik}) \subset \mathbb{R}^n$. Fix k; these subsets of the zero-centered cube $\varphi(V)$ can be approximated by the action of $\mathbb{R}^n$ on itself so that the base of the stacks will be common for each k (and, in fact, will be a zero-centered cube). We see that there exists $B_0 \subset \varphi(V)$ with null boundary, and some $r_{ikj} \in \varphi(V),\ j = 1, ..., L_{ik},\ k = 1, ..., M,\ i = 1, ..., m$, such that $r_{ikj} + B_0$ form a stack, and

$$\rho\Big(B_{ik} \bigtriangleup \bigcup_{j=1}^{L_{ik}} (r_{ikj} + B_0)\Big) < \varepsilon_1, \tag{7}$$

where $\varepsilon_1 > 0$ can be chosen under the condition

$$\varepsilon_1 < \frac{\varepsilon}{6M(1+\varepsilon')}, \quad \varepsilon_1 < \rho(B_{ik})/4. \tag{8}$$

Consider now $\varphi^{-1}(r_{ikj} + B_0)$. Note that (7) and (iv) imply

$$\mu\Big(A_{ik} \bigtriangleup \bigcup_{j=1}^{L_{ik}} \varphi^{-1}(r_{ikj} + B_0)\Big) < \varepsilon_1(1+\varepsilon'), \tag{9}$$

and (7) implies also that

$$L_{ik}\rho(B_0) < \rho(B_{ik}) + \varepsilon_1. \tag{10}$$

The sets $\varphi^{-1}(r_{ikj} + B_0)$ are disjoint subsets of V when k is fixed; but, in the general case, they do not form any stack because they must not be g-images of one another. Nevertheless, let $x_0 = \varphi^{-1}(0) = eH \in \varphi^{-1}B_0$ and $x_{ikj} = \varphi^{-1}(r_{ikj}) \in \varphi^{-1}(r_{ikj} + B_0)$. Find $g_{ikj} \in U$ such that $g_{ikj} \cdot x_0 = x_{ikj}$. Now let us consider the sets $g_{ikj} \cdot \varphi^{-1}(B_0)$; they are g-images of one another, but it is not clear whether they form a stack or not because they may intersect.

Now let $E_0 = \varphi^{-1}((1-\varepsilon')B_0)$.

Lemma. $g_{ikj} \cdot E_0 \subset \varphi^{-1}(r_{ikj} + B_0)$.

We see that the sets $g_{ikj} \cdot E_0 \subset \varphi^{-1}(r_{ikj} + B_0)$ can not intersect with one another and, hence, form a stack. Note that

$$\rho(B_0 \setminus \varphi(E_0)) = \Big(1 - (1-\varepsilon')^n\Big) \cdot \rho(B_0)$$

and

$$\rho\Big((r_{ikj} + B_0) \setminus \varphi(g_{ikj} \cdot E_0)\Big) \le \rho(B_0) \cdot \Big(1 - \frac{1-\varepsilon'}{1+\varepsilon'} \cdot (1-\varepsilon')^n\Big).$$

This allows us to estimate how does this stack approximate the sets A_{ik}. Use (9) and (10):

$$\mu(A_{ik} \bigtriangleup \bigcup_{j=1}^{L_{ik}} g_{ikj} \cdot E_0) < (1+\varepsilon') \cdot \Big(\varepsilon_1 + (\rho(B_{ik}) + \varepsilon_1) \cdot \Big(1 - \frac{1-\varepsilon'}{1+\varepsilon'} \cdot (1-\varepsilon')^n\Big)\Big) = \delta(k).$$

Now we have to deal with sum of these expression by k and to provide that it would be less than $\varepsilon/2$. This is easy. The requirement (1) is hence satisfied.

It remains only to take care on the requirement (2). Our stack consists of the elements having the form $g_k g_{ikj} \cdot E_0$, and we must show that

$$\sum_{kj} \int_{E_0} \left| \frac{d\mu \circ g_k g_{ikj}}{d\mu}(x) - r_k \right| d\mu(x) < \varepsilon \, .$$

But the value under the integral sign is less than or equal to

$$\left| \frac{d\mu \circ g_k g_{ikj}}{d\mu}(x) - r_k \frac{d\mu \circ g_{ikj}}{d\mu}(x) \right| + r_k \left| \frac{d\mu \circ g_{ikj}}{d\mu}(x) - 1 \right| ,$$

and the sum of the integrals by E_0 of the first addends is less than $\varepsilon/4$ due to (6). To deal with the second addends, note that ($x \in E_0$)

$$\frac{d\mu \circ g_{ikj}}{d\mu}(x) = \frac{d\mu \circ g_{ikj}}{d\rho \circ \varphi \circ g_{ikj}}(x) \cdot \frac{d\rho \circ \varphi \circ g_{ikj}}{d\rho \circ \varphi}(x) \cdot \frac{d\rho \circ \varphi}{d\mu}(x).$$

Here the first and the third multipliers are close to 1 due to (iv), while the second one is close to 1 due to (v). ■

Theorem 2. *A transitive action of any l.c.s. group has funny rank one.*

This follows from Theorem 1, theorem on funny rank one for induced actions [5], from the fact that transitive actions of totally disconnected groups have funny rank one (see [5], Proposition 13, or [2]) and from the fact that it is possible to approximate a connected l.c.s. group by Lie groups [4].

Definition (see [3]). *A measure-preserving action of a group G on a Lebesgue space $(\Omega, \mathfrak{B}, \mu)$ has discrete spectrum, if the unitary representation $U_g f(\omega) = f(g \cdot \omega)$, $f \in L^2(\Omega)$, is a discrete direct sum of finite-dimensional irreducible representations of G.*

Corollary. *Any action of a l.c.s. group with discrete spectrum has funny rank one.*

Acknowledgement. I am deeply grateful to Professor Georges Skandalis who suggested me the idea of this proof.

References

[1] *V. Ya. Golodets and A.M. Sokhet,* Ergodic actions of an Abelian group with discrete spectrum, and approximate transitivity. — J. Soviet Math. (1990), v. 52, No. 6., p. 3530–3533; transl. from Teor. Funkts., Funkts. Anal., Prilozhen. (1989), v. 51, p. 117–122.

[2] *V. Ya. Golodets and A.M. Sokhet,* Transitive actions of solvable groups and approximate transitivity. — Preprint of Institute for Low Temperature Physics and Engineering, Kharkov (1989), No. 19, 26 p.

[3] *G.W. Mackey*, Ergodic transformation groups with a pure point spectrum. — Ill. J. Math. (1964), v. 8, No. 2, p. 593–600.

[4] *D. Montgomery and L. Zippin*, Topological transformation groups. Interscience, New York, London (1965), 289 p.

[5] *A.M. Sokhet*, Funny rank one and the approximate transitivity for induced actions. — Monatshefte für Mathematik (to appear).

Транзитивные действия имеют странный ранг один

Александр М. Сохет

В работе доказывается, что транзитивное действие локально компактной сепарабельной группы имеет странный ранг один. Этот факт был ранее доказан только для действий разрешимых групп. Теорему доказываем сначала для действий групп Ли, а затем обобщаем при помощи аппроксимации топологических групп группами Ли. Как естественное следствие отсюда получаем, что из дискретного спектра следует странный ранг один.

Транзитивні дії мають дивний ранґ один

Олександр М. Сохет

В роботі доводиться, що транзитивна дія локально компактної сепарабельної групи має кумедний ранґ один. Цей факт раніше було доведено тільки для дій розв'язних груп. Теорему доводимо спочатку для дій груп Лі, а потім узагальнюємо з використанням апроксимації топоґічних груп групами Лі. Як природний наслідок одержуємо, що з дискретного спектру витікає кумедний ранґ один.