Matematicheskaya fizika, analiz, geometriya
1999, v. 6, No. 1/2, p. 158–181

Upper semicontinuity of attractors of semilinear parabolic equations with asymptotically degenerating coefficients

I.D. Chueshov and L.S. Pankratov*

Department of Mechanics and Mathematics, Kharkov State University, 4 Svobody Sqr., 310077, Kharkov, Ukraine
E-mail: chueshov@ilt.kharkov.ua

**Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering, National Academy of Sciences of Ukraine, 47 Lenin Ave., 310164, Kharkov, Ukraine*
E-mail: pankratov@ilt.kharkov.ua

Received June 12, 1997

The initial boundary value problem for semilinear parabolic equation

$$\frac{\partial u^\varepsilon}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), \quad x \in \Omega, t \in (0,T);$$

with the coefficients $a_{ij}^\varepsilon(x)$ depending on a small parameter ε is considered. We suppose that $a_{ij}^\varepsilon(x)$ have an order $\varepsilon^{3+\gamma}$ $(0 \le \gamma < 1)$ on a set of spherical annuli G_ε^α having the thickness $d_\varepsilon = d\varepsilon^{2+\gamma}$. The annuli are periodically (with a period ε) distributed in Ω. On the remaining part of the domain these coefficients are constants. The asymptotical behavior of the global attractor $\mathcal{A}_\varepsilon$ of the problem as $\varepsilon \to 0$ is studied. It is shown that the global attractors $\mathcal{A}_\varepsilon$ tend in a appropriate sense to a weak global attractor $\mathcal{A}$ of the homogenized model as $\varepsilon \to 0$. This model is a system of a parabolic p.d.e. coupled with an o.d.e.

1. Introduction

We consider a semilinear initial boundary value problem

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), & x \in \Omega, t > 0; \\ \dfrac{\partial u^\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, t > 0; \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), & x \in \Omega. \end{cases} \tag{1.1}$$

We suppose that $a^{\varepsilon}_{ij}(x)$ depend on a parameter ε and for any ε these coefficients satisfy the following condition:

$$\alpha^{(\varepsilon)}(x)|\xi|^2 \le \sum_{i,j=1}^{n} a^{\varepsilon}_{ij}(x)\xi_i\xi_k \le \beta^{(\varepsilon)}(x)|\xi|^2, \tag{1.2}$$

where

$$0 \le \alpha^{(\varepsilon)}(x) \le \beta^{(\varepsilon)}(x) < \infty, \quad x \in \Omega.$$

Under hypothesis (1.2) and natural assumptions on $u^{\varepsilon}_0(x), h^{\varepsilon}(x)$ and $f(u)$ the existence and uniqueness of the generalized solution of problem (1.1) in the appropriate classes follow from standard parabolic theory (see Theorem 2.1 below).

In this paper we suppose that $a^{\varepsilon}_{ij}(x)$ are of the order $\varepsilon^{3+\gamma}$ $(0 \le \gamma < 1)$ on the union $\mathcal{G}_\varepsilon$ of spherical annuli G^{α}_{ε} having the thickness $d_\varepsilon = d\varepsilon^{2+\gamma}$. These annuli are periodically, with a period ε, distributed along the directions of the axes in Ω. On the set $\Omega \setminus \bigcup_\alpha G^{\alpha}_{\varepsilon}$ these coefficients are equal to the Kronecker symbol δ_{ij}.

This paper deals with the study of asymptotic behavior of the global attractor $\mathcal{A}_\varepsilon$ of problem (1.1) as $\varepsilon \to 0$. The main goal of the present paper is to learn how the transition to homogenized system reflects on the long-time dynamics.

The asymptotic behavior of the solutions $u^{\varepsilon}(x,t)$ of problem (1.1) as $\varepsilon \to 0$ is studied in paper [1] for a finite time interval. It is shown (see [1]) that the homogenization of this problem leads to a system of a semilinear parabolic equation coupled with an ordinary differential equation with respect to the variable t:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum\limits_{i,j=1}^{n} b_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j} + b_1(u-v) + f(u) = h_1(x), & x \in \Omega,\ t > 0; \\ \dfrac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0; \quad u(x,0) = u_0(x), & x \in \Omega; \\ \dfrac{\partial v}{\partial t} + b_2(v-u) + f(v) = h_2(x), & x \in \Omega,\ t > 0; \\ v(x,0) = v_0(x), & x \in \Omega; \end{cases} \tag{1.3}$$

where the coefficients $b_{ij}(i,j = 1,2,...,n)$ and $b_k(k = 1,2)$ are calculated from the solutions of cellular problems and the parameters of the structure.

We consider the long-time dynamics of homogenized system (1.3) and show that it possesses a finite-dimensional global attractor $\mathcal{A}$ (for the definitions and basic facts see, e.g., [2–5]). We investigate the properties of $\mathcal{A}$ and prove that global attractors $\mathcal{A}_\varepsilon$ tend to $\mathcal{A}$ in a suitable sense as $\varepsilon \to 0$. For the first time the homogenization problem for global attractors corresponding to nonlinear evolutionary equations was considered in [6]. Here we use some ideas and methods developed in [6].

A problem that is very close to above mentioned problem is the problem of homogenization of the nonlinear parabolic equations in the domains with "traps", considered by L. Boutet de Monvel, I. Chueshov, and E. Khruslov [6] and by A. Bourgeat and L. Pankratov [7]. Problem (1.1) with $f(u) \equiv 0$ was considered by E. Khruslov (see, for example, [8]). We also note that the homogenization problem for (1.1) with uniformly nondegenerating elliptic operator was studied by a number of authors (see, e.g., [9–11] and the bibliography cited there).

The paper is organized as follows. In the Section 2 we formulate some preliminary theorems and also our main result on the upper semicontinuity of $\mathcal{A}_\varepsilon$. Namely we formulate the existence and uniqueness theorem for problem (1.1) and the theorem on the existence of the global attractor of the dynamical system corresponding to (1.1). The last one is proved in the Section 3. We also formulate the homogenization result from paper [1] and the theorems concerning the properties of the homogenized system (1.3). These theorems are proved in the Section 4. The main result of the paper is proved in the Section 5.

2. Preliminaries and statement of main result

Let Ω be a smooth bounded domain from $\mathbf{R}^n$ $(n \geq 2)$. Let us introduce the notation

$$G_\varepsilon^\alpha = \{x \in \Omega : r_\varepsilon - d_\varepsilon < |x - x^\alpha| < r_\varepsilon\}; \quad B_\alpha(r_\varepsilon - d_\varepsilon) = \{x \in \Omega : |x - x^\alpha| < r_\varepsilon - d_\varepsilon\};$$

$$\mathcal{G}_\varepsilon = \bigcup_{\alpha \in N_\varepsilon} G_\varepsilon^\alpha; \quad \mathcal{B}_\varepsilon = \bigcup_{\alpha \in N_\varepsilon} B_\alpha(r_\varepsilon - d_\varepsilon); \quad \Omega_\varepsilon = \Omega \setminus (\mathcal{G}_\varepsilon \bigcup \mathcal{B}_\varepsilon);$$

where $x^\alpha = \alpha\varepsilon$ $(\alpha \in \mathbf{Z}^n)$ and N_ε is a set of multi–indices such that $G_\varepsilon^\alpha \subset \Omega$; $r_\varepsilon = r\varepsilon$ $(r < 1/4)$; $d_\varepsilon = d\varepsilon^{2+\gamma}$ $(0 \leq \gamma < 1)$.

In the domain Ω we consider the boundary value problem (1.1). The coefficients $a_{ij}^\varepsilon(x)$ in (1.1) are defined as follows:

$$a_{ij}^\varepsilon(x) = \begin{cases} \delta_{ij}, & x \in \Omega \setminus \mathcal{G}_\varepsilon; \\ a_\varepsilon \delta_{ij} \equiv a\delta_{ij}\varepsilon^{3+\gamma}, (a > 0), & x \in \mathcal{G}_\varepsilon. \end{cases} \tag{2.1}$$

This structure of the diffusion matrix allows us to interpret the problem as a reaction-diffusion problem in a medium with traps $B_\alpha(r_\varepsilon - d_\varepsilon)$.

We assume that the function $f(u) \in C^2(\mathbf{R})$ has the following properties:

$$\sup\{|f'(u)| : u \in \mathbf{R}\} < \infty; \tag{2.2}$$

and there exist positive constants B_1, B_2, B_3 such that

$$uf(u) \geq B_1 u^2 - B_2\,; \tag{2.3}$$

$$\mathcal{F}(u) \equiv \int_0^u f(\xi)d\xi \geq B_1 u^2 - B_3. \tag{2.4}$$

Let us introduce the notation

$$J_\varepsilon(u^\varepsilon) = \frac{1}{2}\left\{\|\nabla u^\varepsilon\|^2_{2,\Omega_\varepsilon} + a_\varepsilon \|\nabla u^\varepsilon\|^2_{2,\mathcal{G}_\varepsilon} + \|\nabla u^\varepsilon\|^2_{2,\mathcal{B}_\varepsilon}\right\}; \tag{2.5a}$$

$$\Lambda_\varepsilon(u^\varepsilon) = J_\varepsilon(u^\varepsilon) + \int_\Omega \{\mathcal{F}(u^\varepsilon) - h^\varepsilon u^\varepsilon\}\, dx. \tag{2.5b}$$

Here and below $(\cdot,\cdot)_{2,\mathcal{O}}$ and $\|\cdot\|_{2,\mathcal{O}}$ are the scalar product and the norm in the space $L^2(\mathcal{O})$, $\mathcal{O}$ is a subdomain of Ω.

By standard way (see, e.g., [12, 13]), we can prove the following existence and uniqueness theorem:

Theorem 2.1. *i) Let $h^\varepsilon(x), u_0^\varepsilon(x) \in L^2(\Omega)$. Then for any time interval $[0,T]$ problem (1.1) has a unique generalized solution $u^\varepsilon(x,t)$ such that*

$$u^\varepsilon(x,t) \in C(0,T;L^2(\Omega)) \bigcap L^2(0,T;W_2^1(\Omega)),$$

and, moreover, this solution depends continuously in the L^2-norm on the initial datum and it satisfies the equality

$$\frac{1}{2}\|u^\varepsilon(t)\|^2_{2,\Omega} + \int_0^t J_\varepsilon(u^\varepsilon(\tau))d\tau + \int_0^t (f(u^\varepsilon(\tau)) - h^\varepsilon, u^\varepsilon(\tau))_{2,\Omega} d\tau = \frac{1}{2}\|u_0^\varepsilon\|^2_{2,\Omega} \tag{2.6}$$

for any $t > 0$.

ii) Let $h^\varepsilon(x), u_0^\varepsilon(x) \in W_2^1(\Omega)$. Then problem (1.1) has a unique generalized solution such that

$$u^\varepsilon(x,t) \in C(0,T;W_2^1(\Omega)); \quad u_t^\varepsilon(x,t) \in L^2(0,T;L^2(\Omega)),$$

and we have the equality

$$\Lambda_\varepsilon(u^\varepsilon) + \int_0^t \|u_t^\varepsilon(x,\tau)\|^2_{2,\Omega} d\tau = \Lambda_\varepsilon(u_0^\varepsilon). \tag{2.7}$$

Theorem 2.1 makes it possible to define an evolution operator S_t^ε on the space $L^2(\Omega)$ by the formula $S_t^\varepsilon u_0^\varepsilon = u^\varepsilon(t)$, where $u^\varepsilon(t) = u^\varepsilon(x,t)$ is the solution of problem (1.1). This evolutionary operator is a continuous mapping of $L^2(\Omega)$ into itself. It is also strongly continuous with respect to the time variable. The following assertion gives a description of long–time behavior of the dynamical system $(S_t^\varepsilon, L^2(\Omega))$ generated by the operator S_t^ε in the space $L^2(\Omega)$.

Theorem 2.2. *The dynamical system* $(S_t^\varepsilon, L^2(\Omega))$ *for every* $\varepsilon > 0$ *has compact global attractor, i.e., there exists a compact set* $\mathcal{A}_\varepsilon$ *in* $L^2(\Omega)$ *such that* $S_t^\varepsilon \mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon$ *for* $t \geq 0$ *and*

$$\lim_{t\to+\infty} \sup\{\mathrm{dist}_{L^2(\Omega)}(S_t^\varepsilon v, \mathcal{A}_\varepsilon) : v \in B\} = 0$$

for any bounded set B *in* Ω. *This attractor* $\mathcal{A}_\varepsilon$ *has the finite Hausdorff dimension and it belongs to the space* $H^2(\varepsilon, \Omega)$ *consisting of functions* $u(x) \in W_2^1(\Omega)$ *that locally lie in* $W_2^2(\Omega_\varepsilon)$, $W_2^2(\mathcal{G}_\varepsilon)$, $W_2^2(\mathcal{B}_\varepsilon)$ *and possess the properties:*

$$\frac{\partial u^\varepsilon}{\partial \nu} = 0, \quad x \in \partial\Omega;$$

and

$$\begin{cases} (u^\varepsilon)^+ = (u^\varepsilon)^-; & x \in \partial B_\alpha(r_\varepsilon); \\ \left(\dfrac{\partial u^\varepsilon}{\partial \nu}\right)^+ = a_\varepsilon \left(\dfrac{\partial u^\varepsilon}{\partial \nu}\right)^-; & x \in \partial B_\alpha(r_\varepsilon); \end{cases}$$

$$\begin{cases} (u^\varepsilon)^+ = (u^\varepsilon)^-; & x \in \partial B_\alpha(r_\varepsilon - d_\varepsilon); \\ a_\varepsilon \left(\dfrac{\partial u^\varepsilon}{\partial \nu}\right)^+ = \left(\dfrac{\partial u^\varepsilon}{\partial \nu}\right)^-; & x \in \partial B_\alpha(r_\varepsilon - d_\varepsilon); \end{cases}$$

where $\alpha = 1, ..., N_\varepsilon$, *and we denote by* "$\pm$" *the values of the function* $u^\varepsilon(x,t)$ *and its normal derivative on the external (internal) surface of* $\partial B_\alpha(r_\varepsilon)$ *(or* $\partial B_\alpha(r_\varepsilon - d_\varepsilon)$*).*

Let us recall that the Hausdorff dimension is defined as follows. Let M be a compact set in the space H. For $d, \varepsilon > 0$ define the value

$$\mu(M, d, \varepsilon) = \inf \sum r_j^d,$$

where inf is taken over all coverings of M by the balls of radius $r_j \leq \varepsilon$. It is evident that $\mu(M, d, \varepsilon)$ is a monotone function of ε. Therefore there exists

$$\mu(M, d) = \lim_{\varepsilon\to 0} \mu(M, d, \varepsilon) = \sup_{\varepsilon > 0} \mu(M, d, \varepsilon).$$

The value

$$\dim_H M = \inf\{d : \mu(M, d) = 0\}$$

is called the Hausdorff dimension of the set M.

We also note that $H^2(\varepsilon, \Omega)$ is a Hilbert space with the inner product generated by the norm

$$\|u\|^2_{H^2(\varepsilon,\Omega)} = \|u\|^2_{2,\Omega} + J_\varepsilon(u) + \left\| \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}^\varepsilon \frac{\partial u}{\partial x_j} \right) \right\|^2_{2,\Omega}.$$

We suppose for sake of simplicity that $0 \in \Omega$. Let K be a cube in $\mathbf{R}^n$:

$$K = \{x \in \mathbf{R}^n;\ |x_i| < \frac{1}{2r};\ i = 1, 2, ..., n\};$$

and B is the unit ball in $\mathbf{R}^n$:

$$B = \{x \in K; \sum_{i=1}^{n} x_i^2 < 1\}.$$

Define now in $P = K \setminus B$ the functions $v_i(x)$, $i = 1, 2, ..., n$, that are the solutions of the following auxiliary problem:

$$\begin{cases} \Delta v_i = 0, & x \in P = K \setminus \bar{B}; \\ \dfrac{\partial v_i}{\partial \nu} = (x_i, n), & x \in \partial B; \\ v_i(x), Dv_i(x) & \text{are } K - \text{periodic}; \end{cases} \tag{2.8}$$

where ν is the external normal vector to B. It is known that this problem has a unique solution $v_i(x)$ up to a constant (see, e.g. [11]).

Let $\{x^\alpha = \alpha\varepsilon, \alpha \in \mathbf{Z}^n\}$ be a lattice in $\mathbf{R}^n$. Let Q_ε be a linear interpolation operator that is defined as follows. For each node of the sublattice $\{x^\alpha = \alpha\varepsilon,\ \alpha \in N_\varepsilon\}$ we set

$$(Q_\varepsilon u)(x^\alpha) = \frac{1}{m_\varepsilon} \int\limits_{B_\alpha(r_\varepsilon - d_\varepsilon)} u(x)dx, \quad \alpha \in N_\varepsilon,$$

where m_ε is the volume of the ball $B_\alpha(r_\varepsilon - d_\varepsilon)$ of radius $(r_\varepsilon - d_\varepsilon)$ centered at x^α. For each node $\{x^\alpha = \alpha\varepsilon,\ \alpha \notin N_\varepsilon\}$ we set $(Q_\varepsilon u)(x^\alpha)$ being equal to a mean between the values of $(Q_\varepsilon u)$ in the nearest nodes of the lattice. In the whole $(Q_\varepsilon u)$ is a polylinear spline, i.e.,

$$(Q_\varepsilon u)(x) = \sum_\alpha (Q_\varepsilon u)(x^\alpha) \prod_{j=1}^{n} \chi\left(\frac{x_j}{\varepsilon} - \alpha_j\right),$$

where $x = (x_1, ..., x_n) \in \Omega$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{Z}^n$ and $\chi(\tau) = 0$ for $|\tau| > 1$; $\chi(\tau) = 1 - |\tau|$ when $|\tau| \le 1$. It is clear that Q_ε is linear bounded operator from $L^2(\Omega)$ into $W_2^1(\Omega)$ for every $\varepsilon > 0$ and

$$||Q_\varepsilon u||_{2,\Omega} \le C||u||_{2,\Omega}, \quad u \in L^2(\Omega), \tag{2.9}$$

with a constant $C > 0$ independent of ε.

We involve the following result of the paper [1]:

Theorem 2.3. *Let $u^\varepsilon(x,t)$ be the solution of problem (1.1). We assume that*
i) for any $\varepsilon \in (0,\varepsilon_0)$

$$\|u_0^\varepsilon\|_{2,\Omega}^2 + J_\varepsilon(u_0^\varepsilon) + \|\nabla Q_\varepsilon u_0^\varepsilon\|_{2,\Omega}^2 + (\|h^\varepsilon\|_{2,\Omega}^{(1)})^2 + \|\nabla Q_\varepsilon h^\varepsilon\|_{2,\Omega}^2 \le C,$$

where C denotes any constant independent of ε and $\|\cdot\|_{2,\Omega}^{(1)}$ denotes a norm in the space $W_2^1(\Omega)$;
ii) there exist the functions u_0, v_0, h_1, h_2 from $L^2(\Omega)$ such that

$$\|u_0^\varepsilon - u_0\|_{2,\Omega_\varepsilon} \to 0, \quad \|h^\varepsilon - h_1\|_{2,\Omega_\varepsilon} \to 0;$$

and

$$\|u_0^\varepsilon - v_0\|_{2,\mathcal{B}_\varepsilon} \to 0, \quad \|h^\varepsilon - h_2\|_{2,\mathcal{B}_\varepsilon} \to 0$$

as $\varepsilon \to 0$. Then we have

$$\lim_{\varepsilon\to 0}\{\max_{[0,T]}\|u^\varepsilon(x,t) - u(x,t)\|_{2,\Omega_\varepsilon}^2 + \max_{[0,T]}\|u^\varepsilon(x,t) - v(x,t)\|_{2,\mathcal{B}_\varepsilon}^2\} = 0,$$

where the pair of functions $U(x,t) = (u(x,t), v(x,t))$ is the solution of problem (1.3). The coefficients b_{ij} and b_k in (1.3) are calculated from cellular problem (2.8) solutions and the structure parameters as follows:

$$b_{ij} = \delta_{ij}\left[1 - \frac{r^n}{1-\mu}\int_P (\nabla v_i, \nabla v_j)dx\right],$$

$$b_1 = \frac{b_2\mu}{1-\mu}, \quad \mu = \frac{\pi^{n/2}r^n}{\Gamma\left(\frac{n}{2}+1\right)}, \quad b_2 = \frac{an}{rd}.$$

Here δ_{ij} is the Kronecker symbol and Γ is the Gamma function.

The following theorem explains how to understand solutions of problem (1.3).

Theorem 2.4. *Assume that (2.2)–(2.4) are satisfied and $U_0 = (u_0, v_0) \in \mathcal{F}_0 = L^2(\Omega) \times L^2(\Omega)$. Then the problem (1.3) has unique generalized solution $U(t) = (u(t), v(t))$ belonging to the space $C(\mathbf{R}_+, \mathcal{F}_0)$. Moreover, if $U_0 \in \mathcal{F}_1 = W_2^1(\Omega) \times L^2(\Omega)$ then*

$$U(t) \in C(\mathbf{R}_+, \mathcal{F}_1) \quad \text{and} \quad \frac{d}{dt}U(t) \in L^2(\mathbf{R}_+, \mathcal{F}_0); \tag{2.10}$$

if $U_0 \in \mathcal{F}_2 = W_2^1(\Omega) \times W_2^1(\Omega)$ and $h_2 \in W_2^1(\Omega)$ then

$$U(t) \in C(\mathbf{R}_+, \mathcal{F}_2) \quad \text{and} \quad \frac{d}{dt}U(t) \in L^2_{loc}(\mathbf{R}_+, L^2(\Omega) \times W_2^1(\Omega)). \tag{2.11}$$

The proof of this Theorem is of standard character and relies on the methods presented in [13].

Theorem 2.4 allows us to define the evolutionary semigroup S_t in each of the spaces $\mathcal{F}_i$ by the formula $S_t U_0 = U(t)$, where $U(t) = (u(x,t), v(x,t))$ is the solution of the problem (1.3) and $U_0 = (u_0, v_0)$. We prove the following assertion on the existence of finite dimensional weak global attractor for the semigroup S_t in the space $\mathcal{F}_2$.

Theorem 2.5. *Assume that (2.2)–(2.4) are satisfied and*

$$b_2 + \inf\{f'(u) : u \in \mathbf{R}\} > 0, \quad h_2(x) \in W_2^1(\Omega). \tag{2.12}$$

Then the dynamical system $(S_t, \mathcal{F}_2)$ has a weak global attractor $\mathcal{A}$. This attractor has the finite Hausdorff dimension as a compact set in $\mathcal{F}_0$.

Recall (see [2, 3]) that weak global attractor $\mathcal{A}$ is a bounded weakly closed set in $\mathcal{F}_2$ such that (i) $S_t\mathcal{A} = \mathcal{A}$ for any $t > 0$ and (ii) for any weak neighbourhood $\mathcal{O}$ of $\mathcal{A}$ and for any bounded set $B \subset \mathcal{F}_2$ we have $S_t B \subset \mathcal{O}$, when $t \geq t_0(B, \mathcal{O})$.

In fact Theorem 2.5 for $b_{ij} = \delta_{ij}$ was proved in [6]. Here (see Section 4) we repeat main points of the proof in more details. We also note that assumption (2.12) is of prime importance for the existence of finite dimensional attractor $\mathcal{A}$ (see Remark 6.2 in [6]).

At last using the methods developed in [6] and some estimates borrowed from [1], we prove the main result of the paper.

Theorem 2.6. *Assume that (2.2)–(2.4), (2.12) are satisfied and assumptions of Theorem 2.2 that deal with $h^\varepsilon(x)$ are satisfied. Then we have*

$$\lim_{\varepsilon\to 0} \sup_{u^\varepsilon \in \mathcal{A}_\varepsilon} \{ \inf_{(u,v)\in\mathcal{A}} \left(\|P_\varepsilon u^\varepsilon - u\|_{2,\Omega}^2 + \|Q_\varepsilon u^\varepsilon - v\|_{2,\Omega}^2 \right) \} = 0.$$

This theorem means that global attractor $\mathcal{A}_\varepsilon$ of the problem (1.1) tends to a weak global attractor $\mathcal{A}$ of homogenized system (1.3).

3. Properties of the semigroup S_t^ε

The goal of this section is to prove the existence of the global attractor of the dynamical system $(S_t^\varepsilon, L^2(\Omega))$. We also obtain uniform estimates (with respect to ε) for the trajectories lying on the attractor. We rely on the following assertion (see, e.g., [2–5]):

Theorem 3.1. *Suppose that a dynamical system* (S_t, H) *is compact, i.e., there exists a compact set* K *such that for any bounded set* $B \subset H$ *we have* $S_t B \subset K$ *for* $t \geq t_0(B)$. *Then* ω*–limiting set*

$$\mathcal{A} = \omega(K) \equiv \bigcap_{t>0} \operatorname{cl}\left\{ \bigcup_{\tau>t} S_\tau(K) \right\}$$

is not empty and represents a global attractor. It is unambiguously defined connected set in H. *Moreover,* $\mathcal{A} = \omega(K) \subset K$.

We first prove the dissipativeness of our dynamical system.

Lemma 3.1. *There exists* $R_\varepsilon > 0$ *such that for any bounded set* B_ε *in* $L^2(\Omega)$ *we have*

$$\|S_t^\varepsilon u_0^\varepsilon\|_{2,\Omega} \leq R_\varepsilon$$

for all $u_0^\varepsilon \in B_\varepsilon$, $t \geq t_0(B_\varepsilon)$. *The radius of dissipativity* R_ε *satisfies the estimate*

$$R_\varepsilon \leq C(1 + \|h^\varepsilon\|_{2,\Omega}),$$

where C *is a constant independent of* ε.

P r o o f of L e m m a 3.1. Using the differential form of (2.6) and assumption (2.3), we have

$$\frac{d}{dt}\|u^\varepsilon(t)\|_{2,\Omega}^2 + B_1\|u^\varepsilon(t)\|_{2,\Omega}^2 \leq B_1^{-1}(C_1 + \|h^\varepsilon\|_{2,\Omega})^2,$$

where $C_1 = 2B_1B_2|\Omega|$.

Therefore Gronwall's lemma gives

$$\|S_t^\varepsilon u_0^\varepsilon\|_{2,\Omega}^2 \leq e^{-B_1 t}\|u_0^\varepsilon\|_{2,\Omega}^2 + B_1^{-2}(1 - e^{-B_1 t})(C_1 + \|h^\varepsilon\|_{2,\Omega})^2. \tag{3.1}$$

Consequently we can choose

$$R_\varepsilon^2 = 1 + B_1^{-2}(C_1 + \|h^\varepsilon\|_{2,\Omega})^2.$$

In this case if $\|u_0^\varepsilon\|_{2,\Omega} \leq \rho$ we have that $\|S_t^\varepsilon u_0^\varepsilon\|_{2,\Omega} \leq R_\varepsilon$ for $t \geq T = 2B_1^{-1}\ln\rho$ and Lemma 3.1 is proved.

In order to prove the existence of a compact absorbing set for the dynamical system $(S_t^\varepsilon, L^2(\Omega))$ we use the following Lemmas.

Lemma 3.2. *Let* $u_0^\varepsilon \in H^2(\varepsilon, \Omega)$. *Then the solution* $u^\varepsilon(x,t)$ *of problem (1.1) possesses the properties*

$$u^\varepsilon(t) \in C(0, T; H^2(\varepsilon, \Omega)); \tag{3.2}$$

$$u_t^\varepsilon(t) \in C(0,T;L^2(\Omega)) \bigcap L^2(0,T;W_2^1(\Omega)). \tag{3.3}$$

Moreover, if $\|h^\varepsilon\|_{2,\Omega} \le C$ *we have the estimate*

$$\|u_t^\varepsilon\|_{2,\Omega}^2 + \int_0^t J_\varepsilon(u_t^\varepsilon)d\tau \le A_1[1 + J_\varepsilon(u_0^\varepsilon)] + \|u_1^\varepsilon\|_{2,\Omega}^2, \tag{3.4}$$

with a constant A_1 *independent of* ε, t. *Here in (3.4)*

$$u_1^\varepsilon(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u_0^\varepsilon}{\partial x_j}\right) - f(u_0^\varepsilon) + h^\varepsilon(x) \tag{3.5}$$

P r o o f of L e m m a 3.2. Since $u_1^\varepsilon(x) \in L^2(\Omega)$, using the second part of Theorem 2.1, it is easy to see that $w^\varepsilon(x,t) = u_t^\varepsilon(x,t)$ is the generalized solution of the following problem:

$$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t} - \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial w^\varepsilon}{\partial x_j}\right) + f'(u^\varepsilon)w^\varepsilon = 0, & x \in \Omega, t > 0; \\ \dfrac{\partial w^\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, t > 0; \\ w^\varepsilon(x,0) = u_1^\varepsilon(x), & x \in \Omega. \end{cases} \tag{3.6}$$

Therefore standard methods (see, e.g., [12]) give (3.3). As for (3.2), it follows from the first equation of (1.1), from the definition of the norm in the space $H^2(\varepsilon,\Omega)$ and from (3.3).

Now we prove (3.4). It follows from (2.2) and (3.6) that

$$\frac{1}{2}\frac{d}{dt}\|w^\varepsilon\|_{2,\Omega}^2 + J_\varepsilon(w^\varepsilon) \le C\|w^\varepsilon\|_{2,\Omega}^2, \tag{3.7}$$

where C is a constant independent of ε, t. Integrating (3.7) over an interval $(0,t)$, we get

$$\|w^\varepsilon\|_{2,\Omega}^2 + 2\int_0^t J_\varepsilon(w^\varepsilon)d\tau \le C\int_0^t \|w^\varepsilon\|_{2,\Omega}^2 d\tau + \|u_1^\varepsilon\|_{2,\Omega}^2. \tag{3.8}$$

It is easy to see that

$$\Lambda_\varepsilon(u^\varepsilon) \ge -(B_3|\Omega| + \frac{1}{4B_1}\|h^\varepsilon\|_{2,\Omega}^2), \tag{3.9}$$

where the constants B_1, B_3 are defined in (2.3)–(2.4). Hence from (2.7) and (3.6) we obtain

$$\int_0^t \|w^\varepsilon\|^2_{2,\Omega} d\tau \le \Lambda_\varepsilon(u_0^\varepsilon) + B_3|\Omega| + \frac{1}{4B_1}\|h^\varepsilon\|^2_{2,\Omega}. \tag{3.10}$$

Now (3.8), (3.10) and the definition of $\Lambda_\varepsilon(u_0^\varepsilon)$ imply (3.4) and Lemma 3.2 is proved.

Lemma 3.3. *Let $u^\varepsilon(x,t)$ be the solution of problem (1.1) with $u_0^\varepsilon \in W_2^1(\Omega)$ and $\|h^\varepsilon\|_{2,\Omega} \le C$. Then we have*

$$t\left\|\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}(t)\right)\right\|^2_{2,\Omega} \le e^{A_1 t}A_2[1 + \|u_0^\varepsilon\|^2_{2,\Omega} + J_\varepsilon(u_0^\varepsilon)] \tag{3.11}$$

with constants A_2, A_3 independent of ε, t.

P r o o f o f L e m m a 3.3. Let $u_0^\varepsilon \in H^2(\varepsilon,\Omega)$. It follows from (3.7) and Gronwall's lemma that

$$t\|u_t^\varepsilon\|^2_{2,\Omega} \le e^{C_1 t}\int_0^t \|u_t^\varepsilon\|^2_{2,\Omega} d\tau. \tag{3.12}$$

Therefore from (3.10) and (3.12) we get

$$t\|u_t^\varepsilon\|^2_{2,\Omega} \le e^{C_1 t}\{\Lambda_\varepsilon(u_0^\varepsilon) + C_2\}. \tag{3.13}$$

Here in (3.13) C_1 and C_2 are constants independent of ε, t.

It follows from (1.1)

$$t\left\|\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right)\right\|^2_{2,\Omega} \le C_3 t\{\|h^\varepsilon\|^2_{2,\Omega} + \|f(u^\varepsilon)\|^2_{2,\Omega} + \|u_t^\varepsilon\|^2_{2,\Omega}\}. \tag{3.14}$$

Now the statement of the lemma with $u_0^\varepsilon \in H^2(\varepsilon,\Omega)$ follows from (3.1), (3.13) and (3.14). In order to complete the proof we only need to note that $H^2(\varepsilon,\Omega)$ is dense in $W_2^1(\Omega)$. Lemma 3.3 is proved.

The following assertion together with Lemma 3.1 means that the dynamical system $(S_t^\varepsilon, L^2(\Omega))$ is compact.

Lemma 3.4. *Let*

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \|h^\varepsilon\|_{2,\Omega} < \infty.$$

Then there exist constants R_1 and R_2 independent of ε such that the sets

$$B_1 = \{u \in W_2^1(\Omega) : J_\varepsilon(u) \le R_1^2\}$$

and

$$B_2 = \left\{ u \in H^2(\varepsilon, \Omega) : \left\| \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}^{\varepsilon}(x) \frac{\partial u}{\partial x_j} \right) \right\|_{2,\Omega} \leq R_2 \right\}$$

are absorbing sets for the dynamical system $(S_t^{\varepsilon}, L^2(\Omega))$.

P r o o f o f L e m m a 3.4. In the proof of the lemma we use the standard approach presented for example in [2, 4]. Let $u_0^{\varepsilon} \in H^2(\varepsilon, \Omega)$ and let $u^{\varepsilon}(x,t)$ be a solution of (1.1). According to Lemma 3.2 this solution possesses the properties (3.2), (3.3). Let us multiply the first equation from (1.1) by the function

$$-t \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}^{\varepsilon}(x) \frac{\partial u^{\varepsilon}}{\partial x_j} \right) \equiv t L_{\varepsilon} u^{\varepsilon}(t).$$

We get

$$t(u_t^{\varepsilon}, L_{\varepsilon} u^{\varepsilon})_{2,\Omega} + t \|L_{\varepsilon} u^{\varepsilon}\|_{2,\Omega}^2 = t(f(u^{\varepsilon}) - h^{\varepsilon}, L_{\varepsilon} u^{\varepsilon})_{2,\Omega}.$$

Consequently

$$t \frac{d}{dt} J_{\varepsilon}(u^{\varepsilon}(t)) + t \|L_{\varepsilon} u^{\varepsilon}\|_{2,\Omega}^2 \leq t \|f(u^{\varepsilon}) - h^{\varepsilon}\|_{2,\Omega}^2$$

and therefore

$$\frac{d}{dt} \left(t J_{\varepsilon}(u^{\varepsilon}(t)) \right) + t \|L_{\varepsilon} u^{\varepsilon}\|_{2,\Omega}^2 \leq J_{\varepsilon}(u^{\varepsilon}(t)) + Ct \left(\|u^{\varepsilon}(t)\|_{2,\Omega}^2 + \|h^{\varepsilon}\|_{2,\Omega}^2 \right). \tag{3.15}$$

It follows from (2.6) that

$$\int_0^t J_{\varepsilon}(u^{\varepsilon}(\tau)) d\tau \leq \|u_0^{\varepsilon}\|_{2,\Omega}^2 + C \int_0^t \left(\|u^{\varepsilon}(\tau)\|_{2,\Omega}^2 + \|h^{\varepsilon}\|_{2,\Omega}^2 \right) d\tau.$$

Therefore (3.15) and (3.1) imply

$$t J_{\varepsilon}(u^{\varepsilon}(t)) + \int_0^t \tau \|L_{\varepsilon} u^{\varepsilon}(\tau)\|_{2,\Omega}^2 d\tau \leq C_T (1 + \|h^{\varepsilon}\|_{2,\Omega}^2 + \|u_0^{\varepsilon}\|_{2,\Omega}^2) \tag{3.16}$$

for $u_0^{\varepsilon} \in H^2(\varepsilon, \Omega)$ and $t \in [0, T]$, where T is an arbitrary positive number. Since $H^2(\varepsilon, \Omega)$ is dense in $L^2(\Omega)$, this inequality remains true for any generalized solution $u^{\varepsilon}(t)$ with an initial datum u_0^{ε} from $L^2(\Omega)$.

If we shift in (1.1) an initial moment from 0 to $s > 0$, then (3.16) implies

$$\tau J_{\varepsilon}(u^{\varepsilon}(\tau + s)) \leq C_T (1 + \|u^{\varepsilon}(s)\|_{2,\Omega}^2), \quad \tau \in [0, T].$$

Therefore, using Lemma 3.1, we find that for any bounded set B_ε from $L^2(\Omega)$ there exists $s_0 = s_0(B_\varepsilon) > 0$ such that

$$\tau J_\varepsilon(u^\varepsilon(\tau+s)) \le C_T(1+R_\varepsilon^2), \quad u_0^\varepsilon \in B_\varepsilon, s \ge s_0 = s_0(B_\varepsilon), \tau \in [0,T].$$

Setting now $\tau = T = 1$, we obtain for $t \ge s_0(B)+1$ that $J_\varepsilon(u^\varepsilon(t)) \le C_R$, where $R = \sup_{\varepsilon>0} R_\varepsilon < R_0$. This means that B_1 is an absorbing set. This fact and Lemma 3.3 allow us to prove in a similar way that B_2 is an absorbing set, too. This completes the proof of Lemma 3.4.

Now the existence of the global attractor $\mathcal{A}_\varepsilon$ follows from Theorem 3.1. The finiteness of the Hausdorff dimension of $\mathcal{A}_\varepsilon$ one can prove by the same way as in [2, 5]. We note that Lemmas 3.1 and 3.4 imply that

$$\mathcal{A}_\varepsilon \subset \left\{ u \in H^2(\varepsilon,\Omega) : \|u\|_{2,\Omega}^2 + J_\varepsilon(u) + \left\| \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u}{\partial x_j}\right)\right\|_{2,\Omega}^2 < R_3^2 \right\} \quad (3.17)$$

provided

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \|h^\varepsilon\|_{2,\Omega} < C.$$

Here R_3 is independent of ε. Besides it follows from (3.4) that

$$\sup_{-\infty<t<\infty} \|u_t^\varepsilon(t)\|_{2,\Omega} < C \quad (3.18)$$

for any trajectory $u^\varepsilon(t)$ belonging to the attractor $\mathcal{A}_\varepsilon$, where C is independent of ε.

Lemma 3.5. *Let $L + b_2 > 0$, where $L = \inf_{u\in\mathbf{R}} f'(u)$ then*

$$\|\nabla Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega} \le C, \quad -\infty < t < \infty, \quad (3.19)$$

for any trajectory $u^\varepsilon(t)$ lying in $\mathcal{A}_\varepsilon$ and ε small enough. Here C is a constant independent of ε, t.

P r o o f o f L e m m a 3.5. Using the method developed in [6] and modifying slightly the proof of Lemma 4.1 from paper [1], we get

$$\|\nabla Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega} \le C_1[\|u^\varepsilon\|_{2,\Omega}^2 + J_\varepsilon(u^\varepsilon)] + C_2 e^{-\omega_\varepsilon t}[\|u_0^\varepsilon\|_{2,\Omega}^2 + J_\varepsilon(u_0^\varepsilon) + \|\nabla Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega}]$$

$$+ C_3(\delta)\int_0^t e^{-\omega_\varepsilon(t-\tau)}\{\|u^\varepsilon\|_{2,\Omega}^2 + J_\varepsilon(u^\varepsilon) + \|\nabla Q_\varepsilon h^\varepsilon(t)\|_{2,\Omega} + (\|h^\varepsilon\|_{2,\Omega}^{(1)})^2\}d\tau,$$

where for ε and δ small enough

$$\omega_\varepsilon = (L(1+\rho(\varepsilon)) - 2n\lambda_2^\varepsilon - \delta) > 0.$$

Here above $\rho(\varepsilon) = o(1)$ as $\varepsilon \to 0$ and

$$\lim_{\varepsilon\to 0} \lambda_2^\varepsilon = -\frac{a}{2rd}.$$

Therefore for any trajectory $u^\varepsilon(t) \in \mathcal{A}_\varepsilon$ from (3.17), we obtain

$$\|\nabla Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega} \leq C_4 + C_5 e^{-\omega_\varepsilon(t-s)}(1 + \|\nabla Q_\varepsilon u^\varepsilon(s)\|_{2,\Omega}) \qquad (3.20)$$

for all $t \geq s$. Since, due to the definition of the polylinear spline Q_ε and (3.17), we have

$$\|\nabla Q_\varepsilon u^\varepsilon(s)\|_{2,\Omega} \leq C_1(\varepsilon)\|u^\varepsilon(s)\|_{2,\Omega} \leq C_2(\varepsilon), \quad -\infty < s < \infty,$$

then letting $s \to -\infty$ in (3.20), we obtain the statement of the lemma.

Lemma 3.5 and properties (2.9), (3.17), (3.18) give that for any trajectory $\{u^\varepsilon(t) : t \in \mathbf{R}\}$ lying in the attractor $\mathcal{A}_\varepsilon$ we have the following uniform estimate:

$$\|u^\varepsilon(t)\|_{2,\Omega}^2 + J_\varepsilon(u^\varepsilon(t)) + \|u_t^\varepsilon(t)\|_{2,\Omega}^2 + \|Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega}^2$$
$$+\|\nabla Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega}^2 + \|Q_\varepsilon u_t^\varepsilon(t)\|_{2,\Omega}^2 \leq C. \qquad (3.21)$$

for $-\infty < t < \infty$, where C is a constant independent of ε.

4. Weak global attractor of the homogenized system

In this section we prove Theorems 2.4 and 2.5. We act here along the line of the arguments given in [6].

We rewrite system (1.3) as the following first order evolution equation in the space $\mathcal{F}_0 = L^2(\Omega) \times L^2(\Omega)$:

$$\frac{d}{dt}U + AU = B(U), \quad t > 0, \quad U\Big|_{t=0} = U_0, \qquad (4.1)$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} -\sum\limits_{i,j=1}^n b_{ij}\dfrac{\partial^2}{\partial x_i \partial x_j} + b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad B(U) = \begin{pmatrix} b_1 v - f(u) + h_1 \\ b_2 u - f(v) + h_2 \end{pmatrix}.$$

It is easy to see that A is a positive self–adjoint operator in $\mathcal{F}_0$ such that

$$(AU, U)_{\mathcal{F}_0} \geq C(r,n)\|\nabla u\|_{2,\Omega}^2 + \gamma\|U\|_{\mathcal{F}_0}^2, \quad U \in \mathcal{D}(A^{1/2}), \qquad (4.2)$$

where $\gamma = min(b_1, b_2)$, and

$$\|B(U)\|_{\mathcal{F}_0} \leq M_1(1 + \|U\|_{\mathcal{F}_0}), \quad \|B(U_1) - B(U_2)\|_{\mathcal{F}_0} \leq M_2\|U_1 - U_2\|_{\mathcal{F}_0}, \quad (4.3)$$

If we consider the equation (4.1) in the integral form

$$U(t) = e^{-At}U_0 + \int_0^t e^{-A(t-\tau)}B(U(\tau))d\tau, \qquad (4.4)$$

then, using fixed point method in the space $C(0,T;\mathcal{F}_0)$, we can easily prove (see, e.g., [13]) the existence and uniqueness of solutions for $T < T_0$, where T_0 is small enough. It is clear that the function $U(t)$ gives a generalized solution of the system (1.3) on the interval $[0,T]$, $T < T_0$. If we multiply (4.1) in the space $\mathcal{F}_0$ by $\tilde{U} = (b_2 u, b_1 v)$ then using the properties of the function $f(u)$ we easily get the differential inequality

$$\frac{d}{dt}\Upsilon(t) + B_1\Upsilon(t) \le 2B_2(b_1+b_2)|\Omega| + B_1^{-1}(b_2\|h_1\|^2_{2,\Omega} + b_1\|h_2\|^2_{2,\Omega}),$$

where

$$\Upsilon(t) = b_2\|u\|^2_{2,\Omega} + b_1\|v\|^2_{2,\Omega}.$$

Now from this inequality we obtain

$$\|U(t)\|_{\mathcal{F}_0} \le C_1\|U_0\|_{\mathcal{F}_0}e^{-\omega t} + C_2(1-e^{-\omega t}), \qquad (4.5)$$

where ω, C_1 and C_2 are positive constants. This estimate allow us to continue the solution $U(t)$ on the whole $\mathbf{R}_+$. The proofs of the properties (2.10) and (2.11) are also of standard character. This proves Theorem 3.2.

Let S_t be the evolutionary semigroup defined by the formula $S_tU_0 = U(t)$, where $U_0 = (u_0, v_0)$ and $U(t)$ is the solution of the problem (4.1). Since

$$\|A^\beta e^{-tA}\| \le Ct^{-\beta}e^{-\gamma t}, \quad t > 0,\ 0 < \beta < 1,$$

and $\mathcal{D}(A^{1/2}) = \mathcal{F}_1 = W_2^1(\Omega)\times L^2(\Omega)$, we have from (4.4) and (4.5) the following dissipativity property of S_t : there exists a constant $R > 0$ such that for any bounded set B in $\mathcal{F}_0$ we have

$$\|S_tU_0\|_{\mathcal{F}_1} \le R \quad \text{for all} \quad U_0 \in B \quad \text{and} \quad t \ge t_0(B). \qquad (4.6)$$

Now we rely on the following assertion (see [3]) that follows from general considerations of [2].

Theorem 4.1. *Let a dynamical system (H, S_t) be dissipative and the evolutionary operator S_t be weakly closed, i.e., for any $t > 0$ the conditions: $u_n \to u$ and $S_tu_n \to v$ weakly in H as $n \to \infty$ imply that $v = S_tu$. Then this system possesses a weak global attractor.*

In order to prove this theorem in our case we need in the following lemmas.

Lemma 4.1. *Assume that (2.12) is satisfied. Then S_t is a dissipative semigroup in the space $\mathcal{F}_2 = W_2^1(\Omega) \times W_2^1(\Omega)$, i.e., there exists a constant $R^\star > 0$ such that for any bounded set B in $\mathcal{F}_2$ we have*

$$\|S_t U_0\|_{\mathcal{F}_2} \le R^\star \quad \text{for all} \quad U_0 \in B \quad \text{and} \quad t \ge t_0(B). \tag{4.7}$$

Proof of Lemma 4.1. Because of (4.6) it is sufficient to prove the dissipativity in v–direction only. Using (1.3) we have that the function $w_k(x,t) = \partial_{x_k} v(x,t)$ satisfies the equation

$$\frac{d}{dt} w_k(t) + (b_2 + f'(v(t)))w_k = b_2 \partial_{x_k} u + \partial_{x_k} h_2.$$

Therefore from (2.12) we have

$$\frac{1}{2}\frac{d}{dt}\|w_k(t)\|_{2,\Omega}^2 + \delta\|w_k(t)\|_{2,\Omega}^2 \le C[(\|u\|_{2,\Omega}^{(1)})^2 + (\|h_2\|_{2,\Omega}^{(1)})^2],$$

where $\delta > 0$ and $\|\cdot\|_{2,\Omega}^{(1)}$ is a norm in the space $W_2^1(\Omega)$. Consequently, using (4.6), we obtain

$$\|\partial_{x_k} v(t)\|_{2,\Omega}^2 \le (\|v(s)\|_{2,\Omega}^{(1)})^2 e^{-2\delta(t-s)} + C_R, \quad t \ge s \ge t_0(B).$$

This estimate and (4.6) imply (4.7), and Lemma 4.1 is proved.

The weak closeness of the semigroup S_t follows from the following

Lemma 4.2. *The semigroup S_t is weakly closed in the space $\mathcal{F}_2$.*

Proof of Lemma 4.2. Let $U_{0N} \to U_0$ and $S_{t_1} U_{0N} \to W$ weakly in the space $\mathcal{F}_2$ as $N \to +\infty$. We denote the solution of (4.1) with the initial datum U_{0N} by $U_N(t)$. Let $[0,T]$ be an interval that contains t_1. It follows from (4.5) and (4.7) that

$$\max_{[0,T]} \|U_N(t)\|_{\mathcal{F}_2} < C.$$

It is also easy to see that

$$\int_0^T \|\frac{\partial}{\partial t} U_N(t)\|_{\mathcal{F}_0}^2 dt < C.$$

These estimates and the Dubinsky theorem [16] imply that for some subsequence $\{m\} \subset \{N\}$ we have that $U_m(t) \to U(t)$ in the space $C(0,T;\mathcal{F}_0)$ and the function $U(t)$ satisfies equation (4.1) with the initial datum U_0, i.e., $U(t) = S_t U_0$. Moreover, $S_{t_1} U_m \to U(t_1) = S_{t_1} U_0$. So we have $S_{t_1} U_0 = W$ and Lemma 4.2 is proved.

Now the existence of the weak global attractor $\mathcal{A}$ follows from Theorem 4.1. This attractor is a bounded weakly closed set in $\mathcal{F}_2$.

As in [6] (see Remark 6.1) we can also prove that $\mathcal{A}$ coincides with unstable manifold $\mathcal{M}_+(\mathcal{N})$ of the set $\mathcal{N}$ of stationary points of system (1.3).

It is also easy to see that initial data from $\mathcal{F}_2$ is C^1 with respect to the semigroup S_t. Therefore in order to prove the finiteness of the Hausdorff dimension of $\mathcal{A}$ we can use the approach presented in [5] (see also [3]).

First we consider some general facts concerning this approach and then formulate the theorem that allow us to complete the proof of Theorem 2.5. Let us suppose that a dynamical system (H, S_t) is generated by a differential equation

$$\frac{\partial u}{\partial t} = F(u), \quad u(0) = u_0 \in H. \tag{4.9}$$

Then considering the first variation equation corresponding to (4.9):

$$\frac{\partial U}{\partial t} = F'(S_t u_0)U, \quad U(0) = \xi \tag{4.10}$$

as in a finite-dimensional case it is possible to prove the Liouville formula that describes the evolution of N-dimensional volumes (see, e.g., [3, 5]):

$$\mathrm{Vol}(U_1(t), ..., U_N(t)) = \mathrm{Vol}(\xi_1, ..., \xi_N) \cdot \exp\left\{\int_0^t \mathrm{Tr}(F'(S_t u_0) \cdot Q_N(\tau))d\tau\right\}.$$

Here above $U_k(t)$ is a solution of (4.10) with an initial condition ξ_k, $Q_N(t) = Q_N(t, u, \xi)$ is an orthoprojector in the space H on a subspace generated by the elements $U_1(t), ..., U_N(t)$,

$$\mathrm{Vol}(h_1, ..., h_N) = |\det\{(h_i, h_j)\}_{i,j=1}^N|.$$

Let us introduce the contraction coefficient of N–dimensional volumes at the point $u_0 \in H$ by setting

$$\omega_N(t, u_0) = \sup\{\mathrm{Vol}(U_1(t), ..., U_N(t)) : \xi_j \in H, \|\xi_j\| \le 1, j = 1, 2, ..., N\}.$$

Then for the contraction coefficient on the attractor the following estimate is valid:

$$\omega_N(t) \equiv \sup\{\omega_N(t, u_0) : u_0 \in \mathcal{A}\} \le \exp\{t q_N(t)\}, \tag{4.11}$$

where

$$q_N(t) = \sup_{u_0 \in \mathcal{A}} \sup\left\{\frac{1}{t}\int_0^t \mathrm{Tr}(F'(S_t u_0) \cdot Q_N(\tau))d\tau : \xi_j \in H, \|\xi_j\| \le 1\right\}.$$

Let us consider now the values

$$\Pi_N = \lim_{t\to\infty} [\omega_N(t)]^{1/t}$$

and introduce the uniform Lyapunov exponents by the formulae

$$\mu_1 = \ln \Pi_1, \quad \mu_j = \ln \Pi_j - \ln \Pi_{j-1}, j \geq 2.$$

As in [6] in the proof of the finiteness of the Hausdorff dimension of the attractor $\mathcal{A}$ for the dynamical system corresponding to homogenized problem (1.3) we rely on the following assertion [5] (see also [3]).

Theorem 4.2. *Suppose that the uniform Lyapunov exponents possess the property* $\mu_1 + ... + \mu_{N+1} < 0$ *for some* $N \geq 1$. *Let* $\mu_+ = \max(\mu, 0)$. *Then* $\mu_{N+1} < 0$ *and the estimate is valid*

$$\dim_H \mathcal{A} \leq N + |\mu_{N+1}|^{-1}(\mu_1 + ... + \mu_N)_+ < N + 1. \tag{4.12}$$

Let us consider now the first variation equation corresponded to (4.1):

$$\frac{d}{dt}W = -(A - B'(U(t)))W$$

for trajectory $U(t)$ lying in the attractor $\mathcal{A}$. As in [5] it is necessary to estimate the quantity

$$\sigma_N(t) = -\mathrm{tr}\,\{(A - B'(U(t)))Q_N\}$$

for any N dimensional orthoprojector Q_N in the space $\mathcal{F}_0$ such that $Q_N\mathcal{F}_0 \subset \mathcal{F}_1$. We can do it in the way similar to one presented in [6]. It is clear that for $W = (w_1, w_2) \in \mathcal{F}_1$ we have

$$([A - B'(U(t))]W, W)_{\mathcal{F}_0} \geq C\|\nabla w_1\|^2 + \alpha_\delta\|w_1\|^2 + \beta_\delta\|\nabla w_2\|^2,$$

where

$$\alpha_\delta = b_1 - \frac{(b_1 + b_2)^2}{4\delta} + \inf_u f'(u)$$

and

$$\beta_\delta = b_2 + \inf_u f'(u) - \delta.$$

Here δ is any positive number such that $\beta_\delta > 0$. Let $\{W^k = (w_1^k, w_2^k)\}_{k=1}^N$ be orthonormal basis in $Q_N\mathcal{F}_0$. Using the equality

$$\sum_{k=1}^N \|w_2^k\|^2 = N - \sum_{k=1}^N \|w_1^k\|^2,$$

we have

$$\sigma_N(t) \leq -\beta_\delta N - C \sum_{k=1}^{N} \|\nabla w_1^k\|^2 - (\alpha_\delta - \beta_\delta) \sum_{k=1}^{N} \|w_2^k\|^2.$$

Now we use the following version of the Sobolev–Lieb–Thirring inequality

$$k_1 \sum_{k=1}^{N} \|\nabla w_1^k\|^2 + \frac{k_2}{[d(\Omega)]^2} \int\limits_{\Omega} \rho(x) dx \geq \int\limits_{\Omega} \rho(x)^{1+2/n} dx \, ,$$

which follows from [14, Theorem 2.1]. Here above

$$\rho(x) = \sum_{k=1}^{N} [w_1^k]^2,$$

k_1 and k_2 are constants dependent on n and the shape of Ω, $d(\Omega)$ is a diameter of Ω. We obtain

$$-\sigma_N(t) \geq \beta_\delta N + \int\limits_{\Omega} \left\{ \frac{C}{k_1} \rho(x)^{1+2/n} - \lambda \rho(x) \right\} dx \, ,$$

where

$$\lambda = \frac{Ck_2}{k_1} [d(\Omega)]^{-2} + b_2 + \frac{(b_1 + b_2)^2}{4\delta}.$$

Since

$$z^{1+2/n} - \lambda k_1 C^{-1} z \geq -\frac{2}{n} \left(\frac{\lambda k_1 n}{C(n+2)} \right)^{1+n/2}$$

for any $z > 0$, we have

$$-\sigma_N(t) \geq \beta_\delta N - \frac{C|\Omega|}{k_1} \frac{2}{n} \left(\frac{\lambda k_1 n}{C(n+2)} \right)^{1+n/2} .$$

Let N_0 be an integer such that

$$N_0 - 1 \leq \frac{2C|\Omega|}{nk_1\beta_\delta} \left(\frac{\lambda k_1 n}{C(n+2)} \right)^{1+n/2} < N_0 \, .$$

Then for $N \geq N_0$:

$$\mu_1 + ... + \mu_N \leq \overline{\lim_{t\to\infty}} \frac{1}{t} \int\limits_0^t \sigma_N(\tau) d\tau < 0$$

and according to Theorem 4.2 $\dim_H \mathcal{A} < N_0$. Thus Theorem 3.3 is proved.

5. Upper semicontinuity of the family $\{\mathcal{A}_\varepsilon\}$

Now we prove Theorem 2.6 on upper semicontinuity of the family $\{\mathcal{A}_\varepsilon : \varepsilon > 0\}$ of attractors for the problem (1.1), as $\varepsilon \to 0$.

Let P_ε be a linear continuation operator from Ω_ε to Ω (see, for example, [15]) having the properties:

i) $P_\varepsilon : W_2^l(\Omega_\varepsilon) \to W_2^l(\Omega)$ for $l = 0, 1$ such that

$$\|P_\varepsilon u\|_{2,\Omega}^{(l)} \le C \|u\|_{2,\Omega_\varepsilon}^{(1)}, \quad l = 0, 1,$$

where C is a constant independent of ε. It follows from (3.18) that for any trajectory $\{u^\varepsilon(t) \ -\infty < t < \infty\}$, belonging to attractor $\mathcal{A}_\varepsilon$, we have the uniform estimates

$$\left\| P_\varepsilon \frac{\partial u^\varepsilon}{\partial t} \right\|_{2,\Omega}^2 + \|\nabla P_\varepsilon u^\varepsilon(t)\|_{2,\Omega}^2 + \|P_\varepsilon u^\varepsilon(t)\|_{2,\Omega}^2 \le C_1, \tag{5.1}$$

and

$$\left\| Q_\varepsilon \frac{\partial u^\varepsilon}{\partial t} \right\|_{2,\Omega}^2 + \|\nabla Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega}^2 + \|Q_\varepsilon u^\varepsilon(t)\|_{2,\Omega}^2 \le C_2, \tag{5.2}$$

for all $t \in (-\infty, \infty)$ and ε small enough.

Let

$$\bar{\mathcal{A}}_\varepsilon = \left\{ (\bar{u}_\varepsilon, \bar{v}_\varepsilon) \equiv (P_\varepsilon u^\varepsilon, Q_\varepsilon u^\varepsilon), u^\varepsilon \in \mathcal{A}_\varepsilon \right\},$$

and let us suppose that Theorem 2.6 is not valid. This means that there exists a subsequence $\{u_0^\varepsilon, \varepsilon = \varepsilon_n \to 0\}$ such that $u_0^\varepsilon \in \mathcal{A}$ and

$$\operatorname{dist}_{\mathcal{F}_2}\{(P_\varepsilon u_0^\varepsilon, Q_\varepsilon u_0^\varepsilon), \mathcal{A}\} \ge \delta > 0, \quad \varepsilon = \varepsilon_n. \tag{5.3}$$

Let $\gamma_\varepsilon(t) = \{u^\varepsilon(t), -\infty < t < \infty\}$ be the full trajectory of the system $(S_t^\varepsilon, L^2(\Omega))$ such that $u^\varepsilon(0) = u_0^\varepsilon$. Let us consider now an arbitrary interval $[a, b]$, then $u^\varepsilon(t)$ is a solution of problem (1.1) on this interval such that

$$u^\varepsilon \Big|_{t=a} = u^\varepsilon(a) \in \mathcal{A}_\varepsilon.$$

It follows from (3.21) that there exists a subsequence $\varepsilon = \varepsilon_n$ such that (5.3) remains true and $u^\varepsilon(a)$ satisfies the first condition of Theorem 2.3. For this subsequence we also have that $u^\varepsilon(a) \to u(a)$ in $L^2(\Omega_\varepsilon)$ and $u^\varepsilon(a) \to v(a)$ in $L^2(\mathcal{B}_\varepsilon)$. Applying now Theorem 2.3, we obtain that for any interval $[a, b]$ $(P_\varepsilon u^\varepsilon, Q_\varepsilon u^\varepsilon)$ converges weakly in $C(a, b; \mathcal{F}_0)$ to a solution $(u(t), v(t))$ of (1.3), and according to the Dubinsky theorem [16], we get

$$\max_{[a,b]} \|\bar{u}_\varepsilon(t) - u(t)\|_{2,\Omega} + \max_{[a,b]} \|\bar{v}_\varepsilon(t) - v(t)\|_{2,\Omega} \to 0, \tag{5.4}$$

as $\varepsilon = \varepsilon_n \to 0$. For this solution (u, v), according to (5.1), (5.2), we have

$$\left\| \frac{\partial u}{\partial t} \right\|_{2,\Omega}^2 + \|\nabla u\|_{2,\Omega}^2 + \|u\|_{2,\Omega}^2 \le C_1,$$

and

$$\left\| \frac{\partial v}{\partial t} \right\|_{2,\Omega}^2 + \|\nabla v\|_{2,\Omega}^2 + \|v\|_{2,\Omega}^2 \le C_2,$$

for all $-\infty < t < \infty$. Consequently $U(t) = (u(t), v(t))$ belongs to weak global attractor $\mathcal{A}$. We also have from (5.4) that

$$P_\varepsilon u^\varepsilon(0) \to u_0, \quad Q_\varepsilon u^\varepsilon(0) \to v_0$$

as $\varepsilon = \varepsilon_n \to 0$, where $(u_0, v_0) \in \mathcal{A}$. This fact contradicts to (5.3), so Theorem 2.6 is proved.

Theorem 2.6 gives the following

Corollary 5.1. *Let*

$$\sup_{\varepsilon > 0} \|h^\varepsilon\|_{2,\Omega} < \infty\,,$$

and let $u^\varepsilon(x)$ be the stationary solution to (1.1). Then $u^\varepsilon \in H^2(\varepsilon, \Omega)$ and verifies the estimate

$$\|u^\varepsilon\|_{2,\Omega}^2 + J_\varepsilon(u^\varepsilon) + \|Q_\varepsilon u^\varepsilon\|_{2,\Omega}^2 + \|\nabla Q_\varepsilon u^\varepsilon\|_{2,\Omega}^2 + \left\| \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_j} \right) \right\|_{2,\Omega}^2 \le C \tag{5.5}$$

with a constant C independent of ε. Moreover,

$$\lim_{\varepsilon \to 0} \sup_{u^\varepsilon \in Z_\varepsilon} \left\{ \inf_{(u,v) \in Z} \left(\|P_\varepsilon u^\varepsilon - u\|_{2,\Omega}^2 + \|Q_\varepsilon u^\varepsilon - v\|_{2,\Omega}^2 \right) \right\} = 0. \tag{5.6}$$

Here Z_ε (respectively Z) is the set of stationary solutions to (1.1) (respectively to (1.3)).

P r o o f o f C o r o l l a r y 5.1. Since any stationary point u^ε belongs to the attractor $\mathcal{A}_\varepsilon$ we have estimate (5.5) from (3.17) and (3.21). Let $U = (u, v)$ be a certain limit point of the family $\{(P_\varepsilon u^\varepsilon; Q_\varepsilon u^\varepsilon)\}$ as $\varepsilon \to 0$. From (5.5) and from the compactness of the imbedding of $W_2^1(\Omega)$ into $L^2(\Omega)$ it follows that

$$\|P_\varepsilon u^\varepsilon - u\|_{2,\Omega}^2 + \|Q_\varepsilon u^\varepsilon - v\|_{2,\Omega}^2 \to 0 \tag{5.7}$$

along the subsequence $\varepsilon = \varepsilon_n \to 0$. It is also clear (cf.(5.4)) that there exists a bounded in $\mathcal{F}_0$ full trajectory $U(t) = (u(t), v(t))$ such that $U(0) = U = (u, v)$.

This trajectory belongs to $\mathcal{A}$. Since $u^\varepsilon, \varepsilon = \varepsilon_n$ is a stationary point for (1.1), using (5.7), it is easy to see that $U(t)$ is independent of t. So $U = (u, v)$ is a stationary point of the system (1.3) and therefore (5.7) and the contradiction argument imply (5.6).

References

[1] *L.S. Pankratov*, Homogenization of a semilinear parabolic equations with asymptotically degenerating coefficients. — Math. Fiz., Analiz, Geom. (1998), v. 5, No 3/4, p. 250–273.

[2] *A.V. Babin and M.J. Vishik*, Attractors of evolution equation. North–Holland, Amsterdam (1992).

[3] *I.D. Chueshov*, Global attractors for nonlinear problems of mathematical physics. — Russ. Math. Surveys (1993), v. 48:3, p. 133–161.

[4] *J.K. Hale*, Asymptotic behavior of dissipative systems. Amer. Math. Soc., Providence, RI (1988).

[5] *R. Temam*, Infinite dimensional systems in mechanics and physics. Springer–Verlag, New York (1988).

[6] *L. Boutet de Monvel, I.D. Chueshov, and E.Ya. Khruslov*, Homogenization of attractors. — Annali di Mat. Pura ed Applicata (IV), (1997), v. CLXXII, p. 297–322.

[7] *A. Bourgeat and L.S. Pankratov*, Homogenization of a semilinear parabolic equations in domain with spherical traps. — Appl. Analysis (1997), v. 64, p. 303–317.

[8] *E. Khruslov*, Homogenized models of strongly inhomogeneous media. Proc. Intern. Congr. of Mathematicians, Zürich, Switzerland (1994), Birkhäuser–Verlag, Basel (1995), p. 1270–1277.

[9] *A. Bensoussan, J.-L. Lions, and G. Papanicolaou*, Asymptotic analysis for periodic structures. — Studies in Mathematics and its Applications, V. 5, North–Holland, Amsterdam (1978).

[10] *E. Sanchez-Palencia*, Nonhomogeneous media and vibration theory. Lectures notes in physics, No. 127. Springer–Verlag, New York, (1980).

[11] *V.V. Zhikov, S.M. Kozlov, and O.E. Oleinik*, Homogenization of differential operators and integral functionals. Springer–Verlag, New York (1994).

[12] *O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva*, Linear and quasilinear equations of parabolic type. Amer. Math. Soc., Providence, RI (1968).

[13] *D. Henry*, Geometric theory of semilinear parabolic equations. Springer–Verlag, New York (1981).

[14] *J.-M. Ghidaglia, M. Marion, and R. Temam*, Generalization of the Sobolev–Lieb–Thirring inequalities and applications to the dimension of attractors. — Diff. and Int. Eqs., (1988), v. 1, p. 67–92.

[15] *E. Acerbi, V. Chiado Piat, G.Dal Maso, and D. Perciuale,* An extension theorem from connected sets and homogenization in general periodic domains. Nonlinear analysis. — Theory, Methods, Appl., (1992), v. 18, p. 481–496.

[16] *Yu.A. Dubinski,* Weak convergence in nonlinear elliptic and parabolic equations. — Mat. Sb., (1965), v. 67(109), p. 609–642.

Полунепрерывность сверху аттракторов нелинейных параболических уравнений с асимптотически вырождающимися коэффициентами

И.Д. Чуешов, Л.С. Панкратов

Рассматривается начально–краевая задача для нелинейного параболического уравнения вида

$$\frac{\partial u^\varepsilon}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), \quad x \in \Omega, t \in (0,T),$$

коэффициенты $a_{ij}^\varepsilon(x)$ которого зависят от малого параметра ε, так что $a_{ij}^\varepsilon(x)$ имеют порядок $\varepsilon^{3+\gamma}$ $(0 \le \gamma < 1)$ на множестве сферических колец G_ε^α толщины $d_\varepsilon = d\varepsilon^{2+\gamma}$. Кольца периодически (с периодом ε) распределены в области Ω. На множестве $\Omega \setminus \bigcup_\alpha G_\varepsilon^\alpha$ эти коэффициенты равны постоянной величине. Изучается асимптотическое поведение глобального аттрактора $\mathcal{A}_\varepsilon$ этой задачи при $\varepsilon \to 0$. Показано, что глобальные аттракторы $\mathcal{A}_\varepsilon$ сходятся в соответствующем смысле к слабому глобальному аттрактору $\mathcal{A}$ усредненной модели, которая представляет собой систему, состоящую из параболического уравнения в частных производных и связанного с ним обыкновенного дифференциального уравнения.

Напівнеперервність зверху аттракторів нелінійних параболічних рівнянь з коефіцієнтами, що асимптотично вироджуються

І.Д. Чуєшов, Л.С. Панкратов

Розглядається початково–крайова задача для нелінійного параболічного рівняння

$$\frac{\partial u^\varepsilon}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x_j}\right) + f(u^\varepsilon) = h^\varepsilon(x), \quad x \in \Omega, t \in (0,T),$$

коэфіцієнти $a_{ij}^\varepsilon(x)$ якого залежать від малого параметру ε, так що $a_{ij}^\varepsilon(x)$ мають порядок $\varepsilon^{3+\gamma}$ $(0 \le \gamma < 1)$ на множині сферичних колець G_ε^α товщини $d_\varepsilon = d\varepsilon^{2+\gamma}$. Кільця періодично (з періодом ε) розподілено в області Ω. На множині $\Omega \setminus \bigcup_\alpha G_\varepsilon^\alpha$ ці коефіцієнти дорівнюють сталій величині. Вивчається асимптотична поведінка глобального аттрактора $\mathcal{A}_\varepsilon$ цієї задачі, коли $\varepsilon \to 0$. Показано, що глобальні аттрактори $\mathcal{A}_\varepsilon$ збігаються у відповідному сенсі до слабкого глобального аттрактора $\mathcal{A}$ усередненої моделі, яка є системою, що складається з параболічного рівняння у частинних похідних та звичайного диференціального рівняння.