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Completions with respect to total nonnorming subspaces

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The structure of completions of Banach spaces with respect to total nonnorming subspaces of dual spaces is studied. The obtained results imply, in particular, that such completions can be non-isomorphic to quotients of the space. In a separable case any one of the completions is isomorphic to a completion of l_1.

Let X be a Banach space and X^* be its dual space. A subspace M of X^* is said to be *total* if for every $x \in X, x \neq 0$ there is an $f \in M$ such that $f(x) \neq 0$. Let M be a total subspace of X^*. Define the *completion of X with respect to M* as a completion of X under the norm

$$||x||_M = \sup\{|f(x)| : \ f \in M, ||f|| \leq 1\}.$$

We shall denote this completion by X_M. Each element of X may be considered as an element of X_M. We denote this canonical mapping of X to X_M by C_M.

Such completions are of interest in connections with the questions of regularizability of superpositions of regularizable operators (see [2] and [8]) and for Fréchet space theory (see [6]).

If the norm $||\cdot||_M$ is equivalent to the initial norm of X then subspace M is said to be *norming*. It is clear that if M is norming then C_M is an isomorphism onto. Much more interesting is the case when M is nonnorming. Banach spaces which are isomorphic to X_M for some Banach space X and some total nonnorming subspace $M \subset X^*$ were characterized in [10]. The present paper is devoted to the question: how closely related are the structures of the spaces X and X_M? In particular, we study the properties of the canonical embedding $C_M : X \to X_M$ and find conditions onto a pair (X, Z) of Banach spaces under which Z is isomorphic to X_M for some total nonnorming $M \subset X^*$.

Our sources for basic concepts and results of Banach space theory are [5] and [11]. The unit ball of a Banach space X will be denoted by $B(X)$.

Let X be a Banach space and let M be a subspace of its dual. Every element of X may be considered as a functional on M. So there is a natural mapping of X into M^*. We shall denote this mapping by H_M.

Proposition 1. *Let X be a separable Banach space and $M \subset X^*$ be a total subspace. Then there exists an infinite dimensional Banach space Y and a quotient mapping $Q : X_M \to Y$ such that QC_M is surjective.*

If $\mathrm{cl}H_M(X)$ has infinite codimension in M^*, then the space Y may be chosen to be non-quasi-reflexive.

P r o o f. By separability of X_M there exists a normalized weak* null sequence $\{y_n^*\}_{n=1}^\infty \subset M \subset (X_M)^*$. By Theorem III.1 of [4] (see also [5, p. 11]) there exists a subsequence $\{y_{n(i)}^*\}_{i=1}^\infty \subset \{y_n^*\}_{n=1}^\infty$ such that both $\{y_n^*\}_{n=1}^\infty$ and $\{C_M^* y_n^*\}_{n=1}^\infty$ are w^* basic. Let

$$Y = (X_M)/((\{y_{n(i)}^*\}_{n=1}^\infty)^\top)$$

and $Q : X_M \to Y$ be the natural quotient mapping. It follows from the definition of the w^* basic sequence (see also Proposition 1.b.9 of [5]) that $\mathrm{im}(QC_M)^*$ is weak* closed and that $(QC_M)^*$ is an isomorphic embedding. Hence QC_M is a quotient mapping.

Now let $\mathrm{cl}H_M(X)$ be of infinite codimension in M^*. By Theorem 1 of [1] there exists a weak* null (we mean weak* topology of $(X_M)^*$) sequence $\{y_n^*\}_{n=1}^\infty \subset M$ and a partition $\{I_k\}_{k=1}^\infty$ of $\mathbb{N}$ onto a collection of pairwise disjoint infinite subsets such that for some sequence $\{m_k\}_{k=1}^\infty \subset M^*$ we have

$$m_k(y_n^*) = \begin{cases} 1, & \text{if } n \in I_k, \\ 0, & \text{if } n \notin I_k. \end{cases}$$

By Theorem III.1 of [4] (see also [5, p. 11]) there exists a subsequence $\{y_{n(i)}^*\}_{i=1}^\infty \subset \{y_n^*\}_{n=1}^\infty$ such that

(a) $\{y_{n(i)}^*\}_{i=1}^\infty$ is a w^* basic sequence in $(X_M)^*$;

(b) $\{C^* y_{n(i)}^*\}_{i=1}^\infty$ is a w^* basic sequence in X^*;

(c) the sequence $\{n(i)\}_{i=1}^\infty$ has infinite intersections with every I_k $(k \in \mathbb{N})$.

Condition (c) implies that the space

$$Y = (X_M)/((\{y_{n(i)}^*\}_{i=1}^\infty)^\top)$$

is non-quasi-reflexive. ■

R e m a r k 1. There exist total nonnorming subspaces such that $\mathrm{cl}H_M(X)$ has finite codimension in M^* (see Proposition 2.3 in [9]).

R e m a r k 2. It may happen that X and X_M do not have common infinite dimensional subspaces. For example, in [7] it was proved that for some total subspace $M \subset l_\infty$ the space c_0 is isomorphic to $(l_1)_M$.

It is not clear whether X_M should contain a subspace isomorphic to a quotient space of X. At the moment I am able to show only that X_M itself is not necessarily a quotient of X and, in particular, that $(c_0)_M$ is not necessarily isomorphic to a quotient of c_0. To give the examples we need the following result.

Theorem 1. *Let X and Y be separable Banach spaces. Suppose that X^* and Y^* contain isomorphic copies of l_1. Then for arbitrary separable Banach space Z there exists a total nonnorming subspace $M \subset (X \oplus Z)^*$ such that $(X \oplus Z)_M$ is isomorphic to $X \oplus Y$.*

P r o o f. By Lemma 1 of [3] spaces X^* and Y^* contain norming subspaces isomorphic to l_1. We denote them by E and F respectively. Let $\{e_i^*\}_{i=1}^\infty \subset E$ and $\{f_j^*\}_{j=1}^\infty \subset F$ be equivalent to the unit vector basis of l_1. Without loss of generality we may suppose that $\{e_i^*\}_{i=1}^\infty$ contains a subset $\{g_{i,j}^*\}_{i,j=1}^\infty$ such that for each j the sequence $\{g_{i,j}^*\}_{i=1}^\infty$ is weak* null. Let $\{h_k^*\}_{k=1}^\infty = \{e_i^*\}_{i=1}^\infty \backslash \{g_{i,j}^*\}_{i,j=1}^\infty$. Let $M \subset (X \oplus Y)^*$ be the linear span of $\{h_k^*\}_{k=1}^\infty \cup \{g_{i,j}^* + f_j^*\}_{i,j=1}^\infty$ (we identify $(X \oplus Y)^*$ with $X^* \oplus Y^*$ and $x^* \in X^*$ with the pair $(x^*, 0) \in X^* \oplus Y^*$).

Standard verification (see [3, p. 56]) shows that $M \subset (X \oplus Y)^*$ is norming. Let $T : X \oplus Z \to X \oplus Y$ be defined by $T(x, z) = (x, Wz)$, where $W : Z \to Y$ is some injective operator with dense range.

The operator T satisfies the following conditions:

(a) T is an injective operator with dense range;

(b) $T^*|_M$ is an isomorphism.

The statement (a) is obvious. So we turn to statement (b).

Since sequences $\{e_i^*\}_{i=1}^\infty$ and $\{f_i^*\}_{i=1}^\infty$ are equivalent to the unit vector basis of l_1 then there exist $c_1, c_2 > 0$ such that for every

$$l^* = \sum_k a_k h_k^* + \sum_{i,j} b_{i,j}(g_{i,j}^* + f_j^*) \in M$$

there exists $x \in X,\ ||x|| = 1$ such that

$$l^*((x,0)) = \left(\sum_k a_k h_k^* + \sum_{i,j} b_{i,j} g_{i,j}^*\right)(x) \geq c_1 \left(\sum_k |a_k| + \sum_{i,j} |b_{i,j}|\right) \geq c_2 ||l^*||.$$

Hence

$$||T^* l^*|| = \sup_{(x,z) \in S(X \oplus Z)} |(T^* l^*)(x,z)| = \sup_{(x,z) \in S(X \oplus Z)} |(l^*)(x, Wz)| \geq c_2 ||l^*||,$$

where we choose $z = 0$.

Statements (a) and (b) immediately imply that $(X \oplus Z)_{T^*(M)}$ is isomorphic to $(X \oplus Y)$. ■

E x a m p l e. Let $X = Z = c_0$ and $Y = l_1$. By Theorem 1 there exists a subspace $M \subset (c_0)^*$ such that $(c_0)_M$ is isomorphic to $c_0 \oplus l_1$. On the other hand, it is well-known (and easy to verify) that $c_0 \oplus l_1$ is not isomorphic to a quotient space of c_0.

R e m a r k. If X_M is isomorphic to l_1 then X contains a complemented infinite dimensional subspace isomorphic to l_1.

In fact, the canonical embedding $C_M : X \to X_M$ is noncompact (since the restriction of its conjugate to M is an isomorphism). It is easy to see that $C_M(B(X))$ contains a sequence that is equivalent to the unit vector basis of l_1 and spans a complemented subspace isomorphic to l_1. We "lift" this subspace to X and get the required subspace.

Now we prove some special form of "universality" of the space l_1.

Theorem 2. *Let X be a separable Banach space. Then for arbitrary total nonnorming subspace $M \subset X^*$ there exists a total subspace $J \subset (l_1)^*$ such that X_M is isomorphic to $(l_1)_J$.*

P r o o f. Recall that if U and V are subspaces of a Banach space Z then the number $\delta(U, V)$ is defined by $\delta(U, V) = \inf\{||u - v|| : u \in S(U), v \in V\}$ and is called the *inclination* of U to V.

In [10, Corollary 2.3] it was proved:

Theorem A. *A Banach space Z is isomorphic to the completion of some Banach space with respect to a total nonnorming subspace if and only if Z^* contains subspaces K and N such that K is norming, N is weak* closed and infinite-dimensional, the quotient $Z/(N^\top)$ is separable and $\delta(K, N) > 0$.*

We apply Theorem A to the space X_M and denote by K and N the obtained subspaces. The subspace $N^\top \subset X_M$ is separable. Hence there exists a quotient mapping $F : l_1 \to N^\top$. Since N is weak* closed and is of infinite codimension in $(X_M)^*$, then the quotient $(X_M)/(N^\top)$ is infinite dimensional. Let $\{x_i\}_{i=1}^\infty$ be a normalized basic sequence in $(X_M)/(N^\top)$. Denote the quotient mapping $X_M \to (X_M)/(N^\top)$ by ψ. Let $z_i \in X_M$ $(i \in \mathbb{N})$ be such that $||z_i|| \le 2$ and $\psi(z_i) = x_i$.

Let $c = \delta(K, N)$. Introduce an operator $F_1 : l_1 \to X_M$ by

$$F_1(\{a_i\}_{i=1}^\infty) = \frac{c}{4} \sum_{i=1}^{\infty} a_i z_i + F(\{a_i\}_{i=1}^\infty).$$

We need the following definition. Let $a \geq 0, b \geq 0$. We shall say that subset $A \subset X^*$ is (a,b)-*norming* if the following conditions are satisfied:

$$(\forall x \in X)(\sup\{|x^*(x)| : x^* \in A\} \geq a||x||);$$

$$\sup\{||x^*|| : x^* \in A\} \leq b.$$

It is clear that F_1 is injective and that the canonical image of $F_1(B(l_1))$ in K^* is a $(\frac{c}{2},\ 1+\frac{c}{2})-$norming subset.

Let $L = \mathrm{cl}(F_1(l_1))$. Let $\{m_i\}$ be (finite or infinite) normalized complete minimal system in $(X_M)/L$. Denote the quotient mapping $X_M \to (X_M)/L$ by χ. Let $\{u_i\} \subset X_M$ be such that $||u_i|| \leq 2$ and $\chi(u_i) = m_i$.

Let $F_2 : l_1^d \oplus_1 l_1 \to X_M$ (where d is the cardinality of $\{m_i\}$) be defined by

$$F_2(\{a_i\}_{i=1}^d, \{b_i\}_{i=1}^\infty) = \frac{c}{4}\sum_{i=1}^{d} a_i u_i + F_1(\{b_i\}_{i=1}^\infty).$$

It is clear that the canonical image of $F_2(B(l_1^d \oplus l_1))$ in K^* is a $(\frac{c}{2},\ 1+\frac{c}{2})-$norming subset. Hence the restriction $F_2^*|_K$ is an isomorphism.

So F_2 satisfies the following conditions:

1. $F_2 : l_1(= l_1^d \oplus l_1) \to X_M$ is injective and has dense image.
2. $F_2^*|_K$ is an isomorphism.

Since K is a norming subspace of $(X_M)^*$ these two conditions imply that for $J := F_2^*(K)$ we have: $(l_1)_J$ is isomorphic to X_M. ■

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Пополнения относительно тотальных ненормирующих подпространств

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Изучается структура пополнений банахова пространства относительно тотальных ненормирующих подпространств сопряженного пространства. Из полученных результатов вытекает, в частности, что такие пополнения могут не быть изоморфны фактор-пространствам пополняемого пространства. Доказано, что в сепарабельном случае любое такое пополнение изоморфно некоторому пополнению пространства l_1.

Поповнення відносно тотальних ненормуючих підпросторів

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Вивчається структура поповнень банахова простору відносно тотальних ненормуючих підпросторів спряженого простору. З одержаних результатів випливає, зокрема, що такі поповнення можуть бути неізоморфними до фактор-просторів поповнюваного простору. Доведено, що у сепарабельному випадку будь-яке поповнення вказаного вигляду є ізоморфним до деякого поповнення простору l_1.